FAMILIES OF L-FUNCTIONS AND 1-LEVEL DENSITIES

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Abstract. In these notes we will describe some of the families of L-functions with (conjectured) symmetry types unitary, orthogonal, and symplectic. Then we will indicate a general method to compute the 1-level density functions and compare the results for our families with the 1-level density functions that can be computed for the scaled limits of $U(N)$, $USp(2N)$, $SO(2N)$, and $SO(2N+1)$.

1. The Selberg Class of L-functions

All of the L-functions in the families we will discuss belong to the Selberg class (at least conjecturally; see [S], [CG], or [KP] for more details about this section) whose definition we now give. For detailed information about the L-functions in these families see [IK].

Let $s = \sigma + it$ with σ and t real. An *L-function* is a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda_n}{n^s},$$

satisfying the Ramanujan bound $\lambda_n \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$, which has three additional properties.

- Analytic continuation: $L(s)$ continues to a meromorphic function of finite order, with at most finitely many poles, and all poles are located on the $\sigma = 1$ line.
- Functional equation: There is a number ε with $|\varepsilon| = 1$, and a function $\gamma_L(s)$ of the form

  $$\gamma_L(s) = P(s) Q^s \prod_{j=1}^{k} \Gamma(w_j s + \mu_j)$$

  where P is a polynomial whose only zeros in $\sigma > 0$ are at the poles of $L(s)$, $Q > 0$, $w_j > 0$, and $\Re\mu_j \geq 0$, such that

  $$\xi_L(s) := \gamma_L(s) L(s)$$

  is entire, and

  $$\xi_L(s) = \varepsilon \overline{\xi_L}(1-s),$$

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where $\overline{\xi_L}(s) = \overline{\xi_L(\overline{s})}$.

ϵ is often called the "sign" of the functional equation (especially when it is ± 1) or the root number. It is sometimes convenient to write the functional equation in asymmetric form:

$$L(s) = \varepsilon X_L(s)\overline{L}(1-s),$$

where $X_L(s) = \dfrac{\overline{\gamma_L}(1-s)}{\gamma_L(s)}$.

- Euler product: For $\sigma > 1$ we have

$$L(s) = \prod_p L_p(1/p^s),$$

where the product is over the primes p, and

$$L_p(1/p^s) = \sum_{k=0}^{\infty} \frac{\lambda_{p^k}}{p^{ks}} = \exp\left(\sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}\right),$$

where $b_n \ll n^\theta$ with $\theta < \frac{1}{2}$.

Note that $L(s) \equiv 1$ is the only constant L-function, the set of L-functions is closed under products, and if $L(s)$ is an L-function then so is $L(s+iy)$ for any real y. An L-function is called *primitive* if it cannot be written as a nontrivial product of L-functions, and it can be shown, assuming Selberg's orthonormality conjectures, that any L-function has a unique representation as a product of primitive L-functions.

The *degree* of the L-function is the sum of the w_j. In all known examples, one may take $w_j = 1/2$ for all j; this may require using the duplication formula for the Gamma-function. This is a useful thing to do, because then $\gamma_L(s)$ is uniquely determined. Also, in all known examples the Euler product is expressible as the reciprocal of a polynomial whose degree is equal to d for all primes that do not divide the level q which (in the formulation with all $w_j = 1/2$) is given by $q = \pi^d Q^2$; for the primes dividing q it is the case in all known examples that the Euler factor is the reciprocal of a polynomial of degree at most d.

For each L-function we define its *log-conductor* $c(L,t)$ at the point $1/2+it$ by

$$c(L,t) = \frac{X_L'}{X_L}(1/2+it). \tag{1}$$

The density of zeros near the point $1/2+it$ is given by

$$\frac{2\pi}{c(L,t)}.$$

2. Degree 1 L-functions

2.1. The Riemann zeta-function. The Riemann zeta-function is given by

$$\zeta(s) := 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The series converges in the half-plane where the real part of s is larger than 1. Riemann proved that $\zeta(s)$ has an analytic continuation to the whole plane apart from a simple pole at $s = 1$. Moreover, he proved that $\zeta(s)$ satisfies a *functional equation* which in its symmetric form is given by

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s)$$

where $\Gamma(s)$ is the usual Gamma-function. Euler had earlier proved that

$$\begin{aligned}\zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right)\left(1 + \frac{1}{5^s} + \dots\right)\dots \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}\end{aligned}$$

where the infinite product (called the *Euler product*) is over all the prime numbers. The product converges when the real part of s is greater than 1. The Euler product implies that there are no zeros of $\zeta(s)$ with real part greater than 1; the functional equation implies that there are no zeros with real parts less than 0, apart from the *trivial zeros* at $s = -2, -4, -6, \dots$. Thus, all of the complex zeros are in the *critical strip* $0 \le \Re s \le 1$. The functional equation shows that the complex zeros are symmetric with respect to the line $\Re s = \frac{1}{2}$. Riemann calculated the first few complex zeros $\frac{1}{2} + i14.134\dots$, $\frac{1}{2} + i21.022\dots$ and proved that the number $N(T)$ of zeros with imaginary parts between 0 and T is

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(1/T)$$

where $S(T) = \frac{1}{\pi} \arg \zeta(1/2+iT)$ is computed by continuous variation starting from $\arg \zeta(2) = 0$ and proceeding along straight lines, first up to $2 + iT$ and then to $1/2 + iT$. Riemann also proved that $S(T) = O(\log T)$. Note for future reference that at a height T the average gap between zero heights is $\sim 2\pi/\log T$. Riemann suggested that the number $N_0(T)$ of zeros of $\zeta(1/2 + it)$ with $0 < t \le T$ seemed to be about

$$\frac{T}{2\pi} \log T;$$

and then made his conjecture that all of the zeros of $\zeta(s)$ in fact lie on the 1/2-line; this is the Riemann Hypothesis.

The family $\{\zeta(1/2 + it) : t \in \mathcal{R}\}$ parametrized by t is a unitary family.

2.2. **Dirichlet L-functions.** The simplest L-function after the ζ-function is the Dirichlet L-function for the non-trivial character of conductor 3:

$$L(s,\chi_{-3}) = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \frac{1}{7^s} - \frac{1}{8^s} + - \dots .$$

This can be written as an Euler product

$$L(s,\chi_{-3}) = \prod_{p\equiv 1 \bmod 3} (1-p^{-s})^{-1} \prod_{p\equiv 2 \bmod 3} (1+p^{-s})^{-1},$$

satisfies the functional equation

$$\xi(s,\chi_{-3}) := \left(\tfrac{\pi}{3}\right)^{-\frac{s}{2}} \Gamma(\tfrac{s+1}{2}) L(s,\chi_3) = \xi(1-s,\chi_{-3}),$$

and is expected to have all of its non-trivial zeros on the 1/2-line. In general, a Dirichlet character is a completely multiplicative periodic function $\chi : \mathbb{N} \to \mathbb{C}$; i.e. $\chi(mn) = \chi(m)\chi(n)$ for all m, n and $\chi(m+q) = \chi(m)$ for some integer q. It is the *primitive* characters which lead to the L-functions in the Selberg class. For each $q \ge 1$ there are precisely

$$\psi(q) = \sum_{d|q} \mu(d)\phi(q/d) \tag{2}$$

primitive characters to the modulus q. (Here ϕ is Euler's phi-function and μ is Möbius' mu-function. If q has the factorization $q = p_1^{e_1} \dots p_r^{e_r}$, then any primitive character χ mod q has a unique representation as a product $\chi = \chi_1 \dots \chi_r$ where χ_j is a primitive character modulo $p_j^{e_j}$. We now describe how to construct the primitive characters modulo p^e. If p is odd, then the number of integers less than or equal to p^e and relatively prime to p^e is given by $\phi(p^e) = p^e - p^{e-1}$. These reduced residues modulo p^e form a multiplicative group which is cyclic; let g be a generator of this group (i.e. a *primitive root* of p^e.) We can specify any character χ modulo p^e by saying what the value of $\chi(g)$ is (clearly this value must be a $\phi(p^e)$ root of unity). The primitive characters are those for which $\chi(g) = \exp(2\pi i a/\phi(p^e))$ where $(a, \phi(p^e)) = 1$. For $p = 2$, the reduced residues modulo 2^e do not form a cyclic group unless $e = 1$ or 2. If $e \ge 3$ then the reduced residues are given by $\pm 5^j$ with $j = 0, 1, \dots 2^{e-2}$. The primitive characters χ modulo 2^e are determined by the value of $\chi(5) = \exp(2\pi i a/2^{e-2})$ with $1 \le a \le 2^{e-2}$ odd and by the value of $\chi(-1) = \pm 1$. This describes all primitive characters.

For each primitive character χ mod q the *Gauss sum* is given by

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) e(n/q). \tag{3}$$

where $e(x) = \exp(2\pi i x)$. It satisfies $|\tau(\chi)| = \sqrt{q}$; we write $\tau(\chi) = \varepsilon_\chi \sqrt{q}$. The Dirichlet L-function is given by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

for $\sigma > 1$. Odd characters are those for which $\chi(-1) = -1$; even characters have $\chi(-1) = 1$. The functional equation for an even character is

$$\xi(s,\chi) := (\pi/\sqrt{q})^{-s/2}\Gamma(s/2)L(s,\chi) = \varepsilon_\chi \xi(1-s,\overline{\chi}). \tag{4}$$

For an odd character, the functional equation is

$$\xi(s,\chi) := (\pi/\sqrt{q})^{-s/2}\Gamma((s+1)/2)L(s,\chi) = \varepsilon_\chi \xi(1-s,\overline{\chi}). \tag{5}$$

We have described the primitive characters above. Imprimitive characters arise in two ways. First, the principal character χ_0 modulo p defined by $\chi_0(n) = 0$ if $p \mid n$ and $= +1$ if $p \nmid n$ is an imprimitive character. Second, a primitive character modulo p^e regarded as a character modulo p^f where $f > e$ is an imprimitive character. Finally, the product of a primitive character with an imprimitive character is an imprimitive character. Any character χ (primitive or imprimitive) which satisfies $\chi(m+q) = \chi(m)$ is called a character modulo q. There are $\phi(q)$ characters modulo q.

2.2.1. *Orthogonality relations.* The basic orthogonality relation is expressed by: if $(mn,q) = 1$, then

$$\sum_{\chi \bmod q} \chi(m)\overline{\chi(n)} = \begin{cases} \phi(q) \text{ if } m = n \bmod q \\ 0 \text{ if } m \neq n \bmod q \end{cases} \tag{6}$$

For primitive characters, this takes the shape: if $(mn,q) = 1$, then

$$\sideset{}{^*}\sum_{\chi \bmod q} \chi(m)\overline{\chi(n)} = \sum_{d \mid (q,m-n)} \phi(d)\mu(q/d). \tag{7}$$

The Polya-Vinogradov inequality asserts that

$$|\sum_{n=1}^{N} \chi(n)| \ll q^{1/2}\log q \tag{8}$$

for any non-principal character χ mod q.

The family $\{L(1/2,\chi_q) : q \text{ is primitive}\}$ is also a unitary family.

2.3. **Real primitive characters.** A special role is played by the real or quadratic Dirichlet characters. These we denote by χ_d where d is a fundamental discriminant: d can be positive or negative, is either odd, squarefree, and congruent to 1 modulo 4, or is 4 times a squarefree integer congruent to 2 or 3 modulo 4. Thus, the sequence of positive fundamental discriminants begins $d = 1, 5, 8, 12, 13, 17, 21, 24, 28, \ldots$ and the sequence of negative fundamental discriminants begins d=-3, -4, -7, -8, -11, -15, -19, -20, -23, -24, $\ldots$. The character χ_d only takes on the values $+1, 0, -1$; it is primitive with the modulus $|d|$. If $d > 0$, then χ_d is an even character and if $d < 0$ it is an odd character. The character χ_d is the character associated with the quadratic field $Q(\sqrt{d})$. In particular, the prime p *splits* or factors in this field if $\chi_d(p) = +1$; it remains prime if $\chi_d(p) = -1$; and it *ramifies* (has a square factor) if $\chi_d(p) = 0$. The real characters χ_d can be decomposed into a product of characters $\chi_{-4}, \chi_8, \chi_{-8}$, χ_p ($p \equiv 1 \bmod 4$), and χ_{-p}, ($p \equiv 3 \bmod 4$) for odd primes p where $\chi_{\pm p}(n) = \left(\frac{n}{p}\right)$ is the Legendre symbol ($= 1$ if n is a non-zero square modulo p, and $= -1$ if n is a non-zero non-square modulo p, $= 0$ if $p \mid n$). The character $\chi_{-4}(n)$ is 0 for even n, is $+1$ for n congruent to 1 modulo 4, and is -1 for n congruent to 3 modulo 4. The character $\chi_8(n)$ is 0 for even n, is $+1$ for n congruent to ± 1 modulo 8, and is -1 for n congruent to ± 3 modulo 8. Finally, $\chi_{-8}(n)$ is 0 for even n, is $+1$ for n congruent to 1 or 3 modulo 8, and is -1 for n congruent to 5 or 7 modulo 8.

2.3.1. *Orthogonality.* First of all, the number $N_q^+(x)$ of fundamental discriminants d with $0 < d \le x$ and $(d, q) = 1$ satisfies $N_q^+(x) \sim \frac{3}{\pi^2}\frac{\phi(q)}{q}x$ and similarly the number $N_q^-(x)$ of negative fundamental discriminants d with $0 < -d < x$ and $(d, q) = 1$ satisfies $N_q^-(x) \sim \frac{3}{\pi^2}\frac{\phi(q)}{q}x$.

By the Polya-Vinogradov inequality,

$$\sum_{0<d\le x} \chi_d(n) = \begin{cases} N_n^+(x) \text{ if } n \text{ is a square} \\ O(n^{1/2+\epsilon}) \text{ if } n \text{ is not a square} \end{cases} \tag{9}$$

The family $\{L(1/2, \chi_d) : d \text{ is a fundamental discriminant }\}$ is a symplectic family.

3. Degree two L-functions

3.1. Modular L-functions.

A first example of a degree 2 L-function arises from Ramanujan's tau-function, defined implicitly by

$$x\prod_{n=1}^{\infty}(1 - x^n)^{24} = \sum_{n=1}^{\infty} \tau(n)x^n.$$

The Fourier series

$$\Delta(z) := \sum_{n=1}^{\infty} \tau(n)e(nz),$$

where $e(z) = \exp(2\pi i z)$, satisfies

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for all integers a, b, c, d with $ad - bc = 1$. A function satisfying these equations is called a *modular form* of weight 12. The associated L-function is

$$L_\Delta(s) := \sum_{n=1}^{\infty} \frac{\tau(n)/n^{11/2}}{n^s} = \prod_p \left(1 - \frac{\tau(p)/p^{11/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1};$$

it satisfies the functional equation

$$\xi_\Delta(s) := (2\pi)^{-s}\Gamma(s + 11/2)L_\Delta(s) = \xi_\Delta(1 - s).$$

Note that, by the duplication formula, this can be written in the form

$$\xi_\Delta(s) = \pi^{-s}\Gamma\big(\frac{s}{2} + \frac{11}{4}\big)\Gamma\big(\frac{s}{2} + \frac{13}{4}\big)L_\Delta(s).$$

It is expected that all of the complex zeros of $L_\Delta(s)$ are on the 1/2-line. In general a cusp form of weight k for the full modular group is a holomorphic function f on the upper half-plane which satisfies

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for all integers a, b, c, d with $ad - bc = 1$ and also has the property that $\lim_{y\to\infty} f(iy) = 0$. Cusp forms for the whole modular group exist only for even integers $k = 12$ and $k \geq 16$. The cusp forms of a given weight k of this form make a complex vector space S_k of dimension $[k/12]$ if $k \neq 2 \bmod 12$ and of dimension $[k/12] - 1$ if $k = 2 \bmod 12$. Each such vector space has a special basis H_k of Hecke eigenforms which consist of functions $f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz)$ for which

$$a_f(m)a_f(n) = \sum_{d|(m,n)} d^{k-1}a_f(mn/d^2). \qquad (10)$$

The Fourier coefficients $a_f(n)$ are real algebraic integers of degree at most the dimension $\#H_k$ of the vector space. Thus, when $k = 12, 16, 18, 20, 22, 26$ the spaces are one dimensional and the coefficients are ordinary integers. We can express these explicitly in terms of the Eisenstein series

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)e(nz)$$

and

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)e(nz)$$

where $\sigma_r(n)$ is the sum of the rth powers of the positive divisors of n:

$$\sigma_r(n) = \sum_{d|n} d^r.$$

Then, $\Delta(z)E_4(z)$ gives the unique Hecke form of weight 16; $\Delta(z)E_6(z)$ gives the unique Hecke form of weight 18; $\Delta(z)E_4(z)^2$ is the Hecke form of weight 20; $\Delta(z)E_4(z)E_6(z)$ is the Hecke form of weight 22; and $\Delta(z)E_4(z)^2E_6(z)$ is the Hecke form of weight 26. The two Hecke forms of weight 24 are given by

$$\Delta(z)E_4(z)^3 + x\Delta(z)^2$$

where $x = -156 \pm 12\sqrt{144169}$. To define the L-function, we scale the coefficients and write

$$\lambda_f(n) = \frac{a_f(n)}{n^{(k-1)/2}}.$$

Then the L-function associated with a Hecke form f of weight k is given by

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^s = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}. \tag{11}$$

By Deligne's theorem $\lambda_f(p) = 2\cos\theta_f(p)$ for a real $\theta_f(p)$. It is conjectured (Sato-Tate) that for each f the $\{\theta_f(p) : p \text{ prime}\}$ is uniformly distributed on $[0, \pi)$ with respect to the measure $\frac{2}{\pi}\sin^2\theta \, d\theta$. We write $2\cos\theta_f(p) = \alpha_f(p) + \overline{\alpha_f(p)}$ where $\alpha_f(p) = e^{i\theta_f(p)}$; then

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha_f(p)}}{p^s}\right)^{-1}. \tag{12}$$

The functional equation satisfied by $L_f(s)$ is

$$\xi_f(s) = (2\pi)^{-s}\Gamma(s + (k-1)/2) L_f(s) = (-1)^{k/2}\xi_f(1-s). \tag{13}$$

Note that the *sign* $(-1)^{k/2}$ of the functional equation is $+1$ when $k \equiv 0 \bmod 4$ and is -1 when $k \equiv 2 \bmod 4$.

3.1.1. *Orthogonality relations.* The Petersson inner product on the space S_k is defined by

$$\langle f, g\rangle = \iint_{\mathcal{D}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}. \tag{14}$$

Here the integration is over the *fundamental domain*

$$\mathcal{D} := \{(x, y) : -1/2 \le x \le 1/2, y \ge \sqrt{1 - x^2}\}.$$

Let $\mathcal{F}$ be an orthogonal basis of S_k with respect to this inner product. The Petersson formula tells us that

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f \in \mathcal{F}} \frac{a_f(m)\overline{a_f(n)}}{\langle f, f\rangle} = \delta_{m,n} + 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{S(m, n, c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{15}$$

where J_{k-1} is the Bessel function of index $k-1$ and $S(m, n, c)$ is the Kloosterman sum

$$S(m, n, c) = \sum_{(x,c)=1} e((mx + n\overline{x})/c) \tag{16}$$

where the sum is over a set of reduced residue classes modulo c and where $\overline{x}$ satisfies $x\overline{x} \equiv 1 \bmod c$. By a theorem of Weil, $|S(m, n, c)| \le (m, n, c)^{1/2} d(c)\sqrt{c}$ where $d(c)$ is the number of positive divisors of c.

The family $\{L_f(1/2) : f \text{ is a primitive form of weight } k\}$ is an orthogonal family. If we restrict to $k \equiv 0 \bmod 4$ then it is an even orthogonal family and if we restrict to $k \equiv 2 \bmod 4$ then it is an odd orthogonal family.

3.2. **Higher level modular forms.** An example of a higher level modular form is the modular form $\sum_{n=1}^{\infty} a(n)e(nz)$ associated to an elliptic curve $E : y^2 = x^3 + Ax + B$ where A, B are integers. The associated L-function, called the Hasse-Weil L-function, is

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a(n)/n^{1/2}}{n^s} = \prod_{p\nmid q} \left(1 - \frac{a(p)/p^{1/2}}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \prod_{p|q} \left(1 - \frac{a(p)/p^{1/2}}{p^s}\right)^{-1}$$

where q is the conductor of the curve. The coefficients $a(n)$ are constructed easily from $a(p)$ for prime p; in turn the $a(p)$ are given by $a(p) = p - N_p$ where N_p is the number of solutions of E when considered modulo p. The work of Wiles and others proved that these L-functions are associated to modular forms of weight 2. This modularity implies the functional equation

$$\xi_E(s) := (2\pi/\sqrt{q})^{-s}\Gamma(s + 1/2)L_E(s) = w_E\xi_E(1 - s) \tag{17}$$

where $w_E = \pm 1$ is the sign of the functional equation. It is believed that all of the complex zeros of $L_E(s)$ are on the 1/2-line.

3.2.1. *Level q cusp forms.* We let $\Gamma_0(q)$ denote the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integers a, b, c, d satisfying $ad - bc = 1$ and $q \mid c$. This group is called the *Hecke congruence group of level q*. A function f holomorphic on the upper half plane satisfying

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z) \tag{18}$$

for all matrices in $\Gamma_0(q)$ and $\lim_{y\to 0} f(\frac{a}{q} + iy) = 0$ for all rational numbers a/q is called a cusp form for $\Gamma_0(q)$; the space of these is a finite dimensional vector space $S_k(q)$. The space S_k above is the same as $S_k(1)$. Again, these spaces are empty unless k is an even integer. If k is an even integer, then

$$\dim S_k(q) = \frac{(k-1)}{12}\nu(q) + \left(\left[\frac{k}{4}\right] - \frac{k-1}{4}\right)\nu_2(q) + \left(\left[\frac{k}{3}\right] - \frac{k-1}{3}\right)\nu_3(q) - \frac{\nu_\infty(q)}{2}$$

where $\nu(q)$ is the index of the subgroup $\Gamma_0(q)$ in the full modular group $\Gamma_0(1)$:

$$\nu(q) = q \prod_{p|q} \left(1 + \frac{1}{p}\right);$$

$\nu_\infty(q)$ is the number of *cusps* of $\Gamma_0(q)$:

$$\nu_\infty(q) = \sum_{d|q} \phi((d, q/d));$$

$\nu_2(q)$ is the number of inequivalent *elliptic points* of order 2:

$$\nu_2(q) = \begin{cases} 0 \text{ if } 4 \mid q \\ \prod_{p|q}(1 + \chi_{-4}(p)) \text{ otherwise} \end{cases}$$

and $\nu_3(q)$ is the number of inequivalent *elliptic points* of order 3:

$$\nu_3(q) = \begin{cases} 0 \text{ if } 9 \mid q \\ \prod_{p|q}(1+\chi_{-3}(p)) \text{ otherwise} \end{cases}.$$

It is clear from this formula that the dimension of $S_k(q)$ grows approximately linearly with q and k. For the spaces $S_k(q)$ the issue of primitive forms and imprimitive forms arise, much as the situation with characters. In fact, one should think of the Fourier coefficients of cusp forms as being a generalization of characters. They are not periodic, but they act as harmonic detectors, much as characters do, through their orthogonality relations (below). Imprimitive cusp forms arise in two ways. Firstly , if $f(z) \in S_k(q)$, then $f(z) \in S_k(dq)$ for any integer $d > 1$. Secondly, if $f(z) \in S_k(q)$, then $f(dz) \in S_k(\Gamma_0(dq))$ for any $d > 1$. The dimension of the space $S_k^{\text{new}}(q)$ generated by primitive forms is given by

$$\dim S_k^{\text{new}}(q) = \sum_{d|q} \mu_2(d) \dim S_k(q/d)$$

where $\mu_2(n)$ is the multiplicative function defined for prime powers by $\mu_2(p^e) = -2$ if $e = 1$, $= 1$ if $e = 2$, and $= 0$ if $e > 2$. The set of primitive forms (or Hecke forms) which generate this space is denoted $H_k(q)$. The elements f of this set have a Fourier series

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e(nz)$$

where the $a_f(n) = \lambda_f(n)n^{(k-1)/2}$ have the property that the associated L-function has an Euler product

$$\begin{aligned} L_f(s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \\ &= \prod_{p\nmid q} \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \prod_{p|q} \left(1 - \frac{\lambda_f(p)}{p^s}\right)^{-1}. \end{aligned}$$

We can express this as

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f'(p)}{p^s}\right)^{-1}$$

where if $p \nmid q$ then $\alpha_f'(p) = \overline{\alpha_f(p)}$ whereas if $p \mid q$ then $\alpha_f'(p) = 0$. The Hecke relations, equivalent to the Euler product, are given by

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,q)=1}} \lambda_f(mn/d^2).$$

The functional equation of the L-function is

$$\xi_f(s) := (2\pi/\sqrt{q})^{-s}\Gamma(s + (k-1)/2)L_f(s) = \pm\xi_f(1-s).$$

Now the $\pm$ depends on more than the weight k.

The family $\{L_f(1/2) : f \text{ is a primitive form of weight 2 and level } q\}$ is an orthogonal family. If we restrict to those f with a plus in the functional equation, then it is an even orthogonal family and if we restrict to those f with a minus in the functional equation then it is an odd orthogonal family.

3.3. **Twists of modular L-functions by real characters.** If $L_f(s) = \sum \lambda_f(n) n^{-s}$ is an L-function associated with a primitive form f, then we can form the twisted L-function

$$L_f(s, \chi) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}$$

where $\chi(n)$ is a primitive character with modulus q. In general, if L_f has level N and $(N, q) = 1$, then $L_f(s, \chi)$ will have level Nq^2. If the functional equation of L_f is

$$\xi_f(s) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s + (k-1)/2) L_f(s) = w_f \xi_f(1-s)$$

then the functional equation of the twist by a real quadratic character $L_f(s, \chi_d)$ is

$$\xi_f(s, \chi_d) = \left(\frac{\sqrt{N}|d|}{2\pi}\right)^s \Gamma(s + (k-1)/2) L_f(s, \chi_d) = w_f \chi_d(-N) \xi_f(1-s, \chi_d). \tag{19}$$

3.4. **Maass forms.** There is another kind of cusp form associated with the group $\Gamma_0(q)$. This is a function $f(z)$ which is real analytic on the upper half-plane. It transforms like a weight 0 cusp form and is an eigenfunction of the Laplace operator:

$$\Delta := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

It has a Fourier expansion as a linear combination of terms $e(nx)$ in which the dependence on y is expressed through K-Bessel functions. The prototype for these is given by the Eisenstein series (for the full modular group)

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(1)} y(\gamma z)^s = \sum_{(c,d)=1} \frac{y^s}{|cz + d|^{2s}}$$

where $y(z)$ denotes the imaginary part of z and where Γ_∞ is the group which fixes ∞, i.e. the group of matrices $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for integer b. This is not a cusp form (because it doesn't vanish at iy as $y \to \infty$.) However, its Fourier expansion is similar to that of the Maass cusp forms for which no explicit construction is known (apart from some forms with eigenvalue 1/4). Let

$$\theta(s) := \pi^{-s} \Gamma(s) \zeta(2s) = \theta(1-s).$$

Then $\theta(s)E(z,s) =$

$$\theta(s)y^s + \theta(1-s)y^{1-s} + 4y^{1/2}\sum_{n=1}^\infty \sum_{ab=n}(a/b)^{s-1/2}K_{s-1/2}(2\pi ny)\cos(2\pi nx).$$

Since $\theta(s)$, $\sum_{ab=n}(a/b)^{s-1/2}$ and $K_{s-1/2}(2\pi ny)$ are all invariant under $s\to 1-s$, we see that $\theta(s)E(z,s)=\theta(1-s)E(z,1-s)$.

A Maass form f with eigenvalue $\lambda = 1/4+\kappa^2$ satisfies $(\Delta+\lambda)f=0$ and has Fourier expansion

$$f(z) = y^{1/2}\sum_{n=1}^\infty \lambda_f(n)K_{i\kappa}(2\pi ny)\cos(2\pi nx)$$

for an even Maass form and

$$f(z) = y^{1/2}\sum_{n=1}^\infty \lambda_f(n)K_{i\kappa}(2\pi ny)\sin(2\pi nx)$$

for an odd Maass form.

The L-function $L_f(s) = \sum_{n=1}^\infty \lambda_f(n)N^{-s}$ associated with a Maass form is entire, has an Euler product, and satisfies the functional equation

$$\xi_f(s) := \pi^{-s}\Gamma((s+i\kappa)/2)\Gamma((s-i\kappa)/2)L_f(s) = \xi_f(1-s)$$

for even Maass forms and

$$\xi_f(s) := \pi^{-s}\Gamma((s+1+i\kappa)/2)\Gamma((s+1-i\kappa)/2)L_f(s) = \xi_f(1-s)$$

for odd Maass forms.

Selberg's trace formula provides us with a kind of Weyl law for the number of Maass forms with eigenvalue less than a given quantity.

Ramanujan's conjecture for Maass forms is that $|\lambda_f(p)|\le 2$. However, this has not yet been proven. The best result is $\lambda_f(p)\ll p^{1/9}$. Thus, we don't know for sure that the Maass-form L-functions are in the Selberg class.

4. Higher degree L-functions

4.1. **Symmetric square L-functions.** Recall that the Euler product for a level q modular form has the shape

$$L_f(s) = \prod_p \left(1-\frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1-\frac{\alpha'_f(p)}{p^s}\right)^{-1}.$$

We can form the symmetric square L-function associated to f as

$$L_f(\mathrm{sym}^2, s) = \prod_p \left(1 - \frac{\alpha_f^2(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)\alpha'_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_f(p)^2}{p^s}\right)^{-1}.$$

Note that this L-function has a degree three Euler product associated with it. Shimura proved that this is an entire function which satisfies the functional equation

$$\begin{aligned} \xi_f(\mathrm{sym}^2, s) :&= \pi^{-3s/2} q^s \Gamma(s/2)\Gamma((s+k-1)/2)\Gamma((s+k)/2) L_f(\mathrm{sym}^2, s) \\ &= \xi_f(\mathrm{sym}^2, 1-s). \end{aligned} \tag{20}$$

The family $\{L_f(\mathrm{sym}^2, 1/2) : f \text{ is a primitive form of weight } k\}$ is a symplectic family. Similarly for $\{L_f(\mathrm{sym}^2, 1/2) : f \text{ is a primitive form of weight 2 and level } q\}$.

More generally, for an L-function with Euler product

$$L(s) = \prod_p \prod_{j=1}^d \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}$$

one can form its symmetric square L-function

$$L(\mathrm{sym}^2, s) = \prod_p \prod_{1 \le i \le j \le d} \left(1 - \frac{\alpha_i(p)\alpha_j(p)}{p^s}\right)^{-1};$$

this is expected to be a degree $k(k+1)/2$ L-function in the Selberg class which may or may not be primitive. One can also form the exterior square L-function

$$L(\mathrm{ext}^2, s) = \prod_p \prod_{1 \le i < j \le d} \left(1 - \frac{\alpha_i(p)\alpha_j(p)}{p^s}\right)^{-1};$$

this is expected to be an L-function in the Selberg class of degree $k(k-1)/2$, which is not necessarily primitive. See the nice survey of Bump [B] for more information about these L-functions.

4.2. **Convolution L-functions.** Given two L-functions

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha'_f(p)}{p^s}\right)^{-1}$$

where $f \in H_k(q_1)$ and

$$L_g(s) = \prod_p \left(1 - \frac{\beta_g(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta'_g(p)}{p^s}\right)^{-1}$$

where $g \in H_\ell(q_2)$ with $(q_1,q_2)=1$ we form the convolution L-function

$$\begin{aligned} L_{f\times g}(s) &= \prod_p \left(1-\frac{\alpha_f(p)\beta_g(p)}{p^s}\right)^{-1}\left(1-\frac{\alpha_f(p)\beta_g'(p)}{p^s}\right)^{-1}\times \\ &\qquad \left(1-\frac{\alpha_f'(p)\beta_g(p)}{p^s}\right)^{-1}\left(1-\frac{\alpha_f'(p)\beta_g'(p)}{p^s}\right)^{-1}. \end{aligned}$$

If $f\ne g$, then this L-function is entire – an Euler product of degree 4 – and satisfies the functional equation

$$\begin{aligned} \xi_{f\times g}(s): &= (2\pi)^{-2s}(q_1q_2)^s\Gamma(s+(|k-\ell|)/2)\Gamma(s-1+(k+\ell-1)/2)L_{f\times g}(s) \\ &= \pm\xi_{f\times g}(1-s). \end{aligned}$$

One can form a convolution between any two L-functions

$$L_1(s)=\prod_p\prod_{i=1}^{d_1}\left(1-\frac{\alpha_i(p)}{p^s}\right)^{-1}$$

and

$$L_2(s)=\prod_p\prod_{j=1}^{d_2}\left(1-\frac{\beta_j(p)}{p^s}\right)^{-1};$$

it is given by

$$L(s)=\prod_p\prod_{i=1}^{d_1}\prod_{j=1}^{d_2}\left(1-\frac{\alpha_i(p)\beta_j(p)}{p^s}\right)^{-1}.$$

In general, this convolution L-function is expected to be an L-function in the Selberg class of degree d_1d_2.

5. One-level densities

Now we indicate how to compute the one level density functions for these families. Since one-level densities have to do with low-lying zeros, we will focus attention on zeros near the point 1/2. Suppose that we have a family $\mathcal{F}$ ordered by log-conductor $c(L)=c(L,0)$. We assume the Riemann Hypothesis for L for convenience, and let γ_L denote a generic ordinate of a zero of L. We want to consider

$$D(f,\mathcal{F},Y) := \sum_{\substack{L\in\mathcal{F}\\ c(L)\le\log Y}}\sum_{\gamma_L} f(c(L)\gamma_L). \tag{21}$$

The density conjecture is that

$$\lim_{Y\to\infty}\frac{D(f,\mathcal{F},Y)}{\sum_{\substack{L\in\mathcal{F}\\ c(L)\le\log Y}}1}=\int_{-\infty}^{\infty} f(x)W_{\mathcal{F}}(x)\,dx$$

where $W_{\mathcal{F}}(x) = W_G(x)$ is the one-level density function for the (scaled) limit of $U(N)$,$O(N)$, $USp(2N)$, $SO(2N)$, or $SO(2N+1)$.

Recall that

$$\begin{aligned} W_U(x) &= 1, \\ W_{SO^+}(x) &= 1 + \frac{\sin 2\pi x}{2\pi x}, \\ W_{SO^-}(x) &= \delta_0(x) + 1 - \frac{\sin 2\pi x}{2\pi x}, \\ W_O(x) &= 1 + \frac{1}{2}\delta_0(x), \\ W_{USp}(x) &= 1 - \frac{\sin 2\pi x}{2\pi x}. \end{aligned}$$

By Plancherel's formula (and because f is even),

$$\int_{-\infty}^{\infty} f(x) W_G(x)\ dx = \int_{-\infty}^{\infty} \hat{f}(x) \hat{W}_G(x)\ dx.$$

So, it is useful to record that

$$\begin{aligned} \hat{W}_U(x) &= \delta_0(x), \\ \hat{W}_{SO^+}(x) &= \delta_0(x) + \frac{1}{2}\chi_{[-1,1]}(x), \\ \hat{W}_{SO^-}(x) &= \delta_0(x) - \frac{1}{2}\chi_{[-1,1]}(x) + 1, \\ \hat{W}_O(x) &= \delta_0(x) + \frac{1}{2}, \\ \hat{W}_{USp}(x) &= \delta_0(x) - \frac{1}{2}\chi_{[-1,1]}(x). \end{aligned}$$

The fundamental tool for beginning any calculations is the explicit formula.

5.1. **Explicit formulae.** We describe an explicit formula of the type initially found by Riemann, and later studied especially by by Guinand and Weil. We suppose that L is entire and in the Selberg class. Let $\Lambda_L(n)$ be defined as the Dirichlet series coefficients of $-L'/L$:

$$\frac{L'}{L}(s) = -\sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^s}. \tag{22}$$

We further assume that the function $\phi(t)$ is even and decays rapidly and that the Fourier transform

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{-2\pi i x t}\ dt$$

has compact support. Then

$$\sum_{\gamma_L} \phi(\gamma_L) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_L'}{X_L}(1/2+it)\phi(t)\ dt - \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{\sqrt{n}}\hat{\phi}\bigg(\frac{\log n}{2\pi}\bigg) \tag{23}$$

Idea of proof. If we pretend that $F(s)$ is a holomorphic function with $F(1/2+it)=\phi(t)$ then we can easily see the terms of the explicit formula emerging. We consider the integral

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(s)F(s)\ ds.$$

Expanding L'/L into its Dirichlet series and integrating term-by term we find one-half of the sum over n on the right side of the explicit formula. Moving the path of integration to $\Re s = 1/4$, we obtain the sum over the γ_L on the left-side from the residues of the poles of L'/L at its zeros. Then, change variables $s \to 1-s$ and use the functional equation $L'/L(1-s) = X'/X(s) - L'/L(s)$ and consider these terms separately. The integral-term in the explicit formula follows by moving the path of integration in the X'/X term to $\Re s = 1/2$. For the other term, we move the path back into a region where the Dirichlet series converges absolutely and then integrate term-by-term. Because of the evenness of $\hat{\phi}$ we obtain the other half of the sum over n on the right side of the explicit formula.

5.2. **The X'/X term.** We now want to substitute the explicit formula (23) into the D formula (21). To scale things we let

$$\phi(t) = f\bigg(\frac{t\log Y}{2\pi}\bigg);$$

then

$$\hat{\phi}(t) = \frac{2\pi}{\log Y}\hat{f}\bigg(\frac{2\pi t}{\log Y}\bigg).$$

The explicit formula then becomes

$$\sum_{\gamma_L} f\bigg(\frac{\gamma\log Y}{2\pi}\bigg) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X_L'}{X_L}(1/2+it) f\big(\frac{t\log Y}{2\pi}\big)\ dt - \frac{4\pi}{\log Y}\sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{\sqrt{n}}\hat{f}\bigg(\frac{\log n}{\log Y}\bigg)$$

If we assume that all of the $w_j = 1/2$, then we have

$$X_L(s) = \frac{Q^{1-s}\prod_{j=1}^d \Gamma\big(\frac{1-s}{2}+\overline{\mu_j}\big)}{Q^s\prod_{j=1}^d \Gamma\big(\frac{s}{2}+\mu_j\big)}$$

and

$$-\frac{X_L'}{X_L}(1/2+it) = 2\log Q + \Re\sum_{j=1}^{d} \frac{\Gamma'}{\Gamma}\ \big(\frac{1}{4}+\frac{it}{2}+\mu_j\big)$$

Therefore, in the situation that $|\mu_j| \ll 1$, so that Q is the main parameter,

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{X_L'}{X_L}(1/2+it)f\Big(\frac{t\log Y}{2\pi}\Big)\,dt = \frac{2\log Q}{\log Y}\hat{f}(0)+O\Big(\frac{\log\log Y}{\log Y}\Big) = \frac{\log q}{\log Y}\hat{f}(0)+O\Big(\frac{\log\log Y}{\log Y}\Big).$$

In the family of L-functions associated with primitive forms of level 1 and large weight k, the parameter k appears in the shifts μ_j in the gamma factors of the functional equation. In this case

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{X_L'}{X_L}(1/2+it)f\Big(\frac{t\log Y}{2\pi}\Big)\,dt = \frac{\log k}{\log Y}\hat{f}(0)+O\Big(\frac{\log\log Y}{\log Y}\Big).$$

A similar phenomenon happens with Maass forms in which case the μ_j are complex numbers with large imaginary part. Also, if we wanted to consider the one level density of zeros of $\zeta(s+iT)$ with a large T then we would have a $\log T$ term in place of $2\log Q$.

To handle the sum with the $\Lambda_L(n)$ we assume that the support of $\hat{f}$ is contained in the interval $[-a,a]$. Then our sum over n is truncated at $n=Y^a$. The idea is to have a as large as possible.

The $\Lambda_L(n)$ are 0 unless n is a power of a prime. The terms with $n=p^k$ for some $k\geq 3$ clearly converge. Thus,

$$\sum_{n=1}^{\infty}\frac{\Lambda_L(n)}{\sqrt{n}}\hat{f}\left(\frac{\log n}{\log Y}\right) = \sum_{p\leq Y^a}\frac{\Lambda_L(p)}{\sqrt{p}}\hat{f}\left(\frac{\log p}{\log Y}\right)+\sum_{p\leq Y^{a/2}}\frac{\Lambda_L(p^2)}{p}\hat{f}\left(\frac{2\log p}{\log Y}\right)+O(1).$$

Let's obtain a formula for $\Lambda_L(p^k)$. Suppose that

$$L(s)=\prod_p L_p(1/p^s)=\prod_p\prod_{j=1}^{d}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1}.$$

and suppose also (for convenience and as is generally believed) that $|\alpha_j(p)|=1$ or 0. Then a brief calculation using the power series expansion for $\log(1-x)$ yields the formula

$$\Lambda_L(p^k)=\log p\sum_{j=1}^{k}\alpha_{j,L}(p)^k.$$

On the other hand, we have

$$L_p(x)=1+\lambda(p)x+\lambda(p^2)x^2+\cdots=\prod_{j=1}^{d}(1+\alpha_j(p)x+\alpha_j(p)^2x^2+\dots)$$

so that

$$\sum_{j=1}^{d} \alpha_j(p) = \lambda(p)$$

and

$$\sum_{j=1}^{d} \alpha_j(p)^2 \quad = \quad \lambda(p^2) - \sum_{1\le i<j\le d} \alpha_i(p)\alpha_j(p)$$

In a typical situation there will be a number δ such that

$$\sum_{j=1}^{d} \alpha_j(p)^2 - \delta$$

is small on average; the value of δ depends on the behavior of

$$\prod_p (1 + \frac{1}{p^s} \sum_{j=1}^{d} \alpha_j(p)^2)$$

near $s = 1$; if it is analytic, then $\delta = 0$; if it has a simple pole, then $\delta = 1$, and if it has a simple zero, then $\delta = -1$. It should be the case that δ is constant throughout $L \in \mathcal{F}$. Then it is not hard to show that

$$\begin{aligned} \sum_{p\le Y^{a/2}} \frac{\Lambda_L(p^2)}{p} \hat{f}\left(\frac{2\log p}{\log Y}\right) &\sim \delta \sum_{p\le Y^{a/2}} \frac{\log p}{p} \hat{f}\left(\frac{2\log p}{\log Y}\right) \\ &\sim \delta \int_1^{Y^{a/2}} \hat{f}(2(\log u)/\log Y) \frac{du}{u} \\ &= \frac{\delta}{2} \int_{-a}^{a} \hat{f}(v)\ dv. \end{aligned}$$

This should be valid for any bounded a. Thus, we arrive at

$$\sum_{\gamma_L} f\left(\frac{\gamma \log Y}{2\pi}\right) \quad = \quad \frac{\log q}{\log Y}\hat{f}(0) - \frac{\delta}{2}\int_{-a}^{a} \hat{f}(v)\ dv - \frac{4\pi}{\log Y} \sum_{p\le Y^a} \frac{a_{p,L}\log p}{\sqrt{p}} \hat{f}\left(\frac{\log p}{\log Y}\right) + o(1)$$

This leads to

$$\begin{aligned} \frac{D(f,\mathcal{F},Y)}{\sum_{\substack{L\in\mathcal{F} \\ c(L)\le \log Y}} 1} &= \hat{f}(0) - \frac{\delta}{2}\int_{-a}^{a} \hat{f}(v)\ dv - \frac{4\pi}{\log Y} \frac{\sum_{\substack{L\in\mathcal{F} \\ c(L)\le \log Y}} \sum_{p\le Y^{a/2}} \frac{a_L(p)\log p}{\sqrt{p}} \hat{f}\left(\frac{\log p}{\log Y}\right)}{\sum_{\substack{L\in\mathcal{F} \\ c(L)\le \log Y}} 1} + o(1) \\ &= \int_{-a}^{a} \hat{f}(x)(-\frac{\delta}{2} + \delta_0(x))\ dx - \frac{4\pi}{\log Y} \frac{\sum_{\substack{L\in\mathcal{F} \\ c(L)\le \log Y}} \sum_{p\le Y^{a/2}} \frac{a_L(p)\log p}{\sqrt{p}} \hat{f}\left(\frac{\log p}{\log Y}\right)}{\sum_{\substack{L\in\mathcal{F} \\ c(L)\le \log Y}} 1} + o(1). \end{aligned}$$

What remains is to calculate the contribution from the primes. This is the suble part. From comparing the formula above with what is predicted we see that in the unitary case, we expect $\delta = 0$ and the sum over primes should never contribute to the main term for any a. In the symplectic case, we expect $\delta = 1$ and that the sum over primes is small when $a < 1$ but for $a \geq 1$ the sum over primes should contribute

$$\int_1^a \hat{f}(x)\, dx.$$

For the case of symmetry type SO^+, we expect $\delta = -1$ and that the sum over primes contributes $-\int_1^a \hat{f}(x)\, dx$ when $a > 1$. For the case of SO^- we expect $\delta = -1$ and that the sum over primes should give $\int_1^a \hat{f}(x)\, dx$ for $a > 1$. Finally, in the case of symmetry type O, we expect $\delta = -1$ and that the primes never contribute to the main term.

5.3. **Sample calculations.**

5.3.1. *Zeta.* The simplest example is one level density of zeros of $\{\zeta(1/2 + it) : T \leq t \leq 2T\}$. (Our explicit formula assumed that our L-function was entire, and so needs to be modified for $\zeta(s)$; the following is intended to capture the spirit of the calculation.) In this case, we have that $\Lambda_L(p) = \log p\ p^{-it}$ and the "sum" over L is an integral with respect to t over $[T, 2T]$. Since $\int_T^{2T} p^{-it}\, dt \ll 1/\log p$, we have

$$\frac{1}{T}\int_T^{2T} \sum_{p \leq T^a} \frac{\log p}{p^{1/2+it}} \ll \frac{1}{T}\sum_{p \leq T^a} p^{-1/2} \ll T^{a/2-1}/\log T = o(1)$$

as long as $a \leq 2$. Moreover, $\Lambda_L(p^2) = \log p\ p^{-2it}$ so

$$\frac{1}{T}\int_T^{2T} \sum_{p \leq T^{a/2}} \frac{\log p}{p^{1+2it}} \ll a(\log T)/T = o(1)$$

for any fixed a. Thus, we obtain the one-level density for $\zeta(s + it)$ for support of $\hat{f}$ in $[-2, 2]$, and we see that it agrees with the predicted one-level density for a unitary symmetry type.

Being more careful in this analysis, we note that the Riemann Hypothesis implies that

$$\sum_{p \leq T^a} \frac{p^{-2iT}}{p^{1/2}} \ll_\epsilon T^{a\epsilon}$$

so that in fact we obtain the one level density for this family for any finite range $[-a, a]$.

5.3.2. *All Dirichlet L-functions* $L(s, \chi)$. A similar argument can be carried out for the unitary family of all Dirichlet characters modulo q; Hughes and Rudnick [HR] have obtained one-level density for this family for any test function whose Fourier transform has support in $[-a, a] = [-2, 2]$.

To extend this range, it seems that one would need a result of the sort

$$\sum_{\substack{n\equiv 1 \bmod q\\ n\le X}} \frac{\Lambda(n)}{\sqrt{n}} = \frac{X}{\phi(q)} + O_\epsilon\left(\frac{X^{1/2+\epsilon}}{q^\theta}\right)$$

for some positive θ. Such error terms for primes in arithmetic progressions (with θ as large as 1/2) have been conjectured by various people, including Montgomery.

5.3.3. *Dirichlet L-functions with real character $L(s,\chi_d)$.* For the family of real quadratic characters we see, by the Polya-Vinogradov inequality, that

$$\frac{1}{X^*}\sum_{d\le X}\sum_{p\le X^a}\frac{\log p\ \chi_d(p)}{\sqrt{p}} \ll X^{-1}\sum_{p\le X^a}\log^2 p \ll X^{a-1}\log X = o(1)$$

provided that $a<1$; here X^* denotes $\sum_{d\le X} 1$. The term involving squares of primes is

$$\begin{aligned}\frac{1}{X^*}\sum_{d\le X}\sum_{p\le X^{a/2}}\frac{\log p\ \chi_d(p^2)}{p}\hat f\left(\frac{2\log p}{\log d}\right) &\sim \sum_{p\le X^{a/2}}\frac{\log p}{p}\hat f\left(\frac{2\log p}{\log X}\right)\\ &\sim \int_1^{X^{a/2}} \hat f(2(\log u)/\log X)\frac{du}{u}\\ &= \frac{1}{2}\int_{-a}^{a}\hat f(v)\ dv.\end{aligned}$$

This is in agreement with the prediction for one-level density for a symplectic family.

Ozluk and Snyder [OS], assuming GRH, have proven one-level density for the family of real Dirichlet L-functions $L(s,\chi_d)$ for $\hat f$ supported in $[-2,2]$. To detect the contributions from the prime sum when $a>1$, they use a transformation property of sums

$$\sum_d \chi_d(p)g(d)$$

for smooth g, where the point is that $\chi_d(p)=\left(\frac{d}{p}\right)$ (the Legendre symbol for p) is periodic in the d variable with period p. This transformation is just an application of the Poisson summation formula ($\sum_d g(d)=\sum_d \hat g(d)$) to each residue class modulo p separately; if one thinks of the above sum as being over all d (and not just fundamental discriminants d), then then one has

$$\sum_d \left(\frac{d}{p}\right)g(d) = \frac{1}{\sqrt{p}}\sum_d\left(\frac{d}{p}\right)\hat g(d/p).$$

Now we bring in the sum over p and get a contribution from the d which are squares (for which $\left(\frac{d}{p}\right)=1$). This techniques allows them to take $a=2$.

6. Statements of further results

Iwaniec, Luo, and Sarnak [ILS] have obtained one-level density theorems for the orthogonal families of L-functions of newforms of level 1 and large weight k, when $\hat{f}$ has support in $[-2, 2]$; similarly for L-functions of newforms of weight 2 and large (prime) level q and also for the symmetric squares of these families, all for $\hat{f}$ supported in $[-2, 2]$. The first examples can be separated into even orthogonal and odd orthogonal or combined to have orthogonal. The symmetric square examples have a symplectic symmetry type. With an additional hypothesis about the behavior of a certain exponential sum over primes, they can obtain the larger support $[-7/3, 7/3]$. The extra main terms from their prime sums arise from extracting main terms out of the Kloosterman sums that arise in the Petersson formula (15).

Matthew Young [Y] has looked at families of elliptic curve L-functions. Let

$$\mathcal{F}_1 = \{L_E(s) \text{ where } E : y^2 = x^3 + ax + b^2\}$$

be a family of rank (at least) one elliptic curves. Young can do one-level density for this family for $\hat{f}$ supported in $[-7/9, 7/9]$. Let

$$\mathcal{F}_2 = \{L_E(s) \text{ where } E : y^2 = x^3 + ax + b\}$$

be the family of all elliptic curves. Young can do one-level density for $\hat{f}$ supported in $[-23/48, 23/48]$. Note, that in this case, it is expected that, $\hat{W}_{\mathcal{F}_2}(x) = 1 + \frac{3}{2}\delta_0(x)$ (see [Sn]).

Royer[Ro] has proven one-level density results for families of L-functions associated with (fixed) weight k and (large) level N newforms where the space is restricted to N with a fixed number ℓ of prime divisors and such that the newforms have prescribed behavior under the *Atkin -Lehner* operators. These results are for test functions with support of $\hat{f}$ in $[-2, 2]$.

Finally, we mention the interesting work of Fouvry and Iwaniec [FI] on low lying zeros of dihedral L-functions. These are L-functions of characters of the class group of $Q(\sqrt{d})$ for a fundamental discriminant d. The L-functions are associated with modular forms of weight 1 but instead of having an orthogonal symmetry type, it is expected that they have a symplectic symmetry type, because the symmetric square L-functions have a pole at $s = 1$. The authors obtain the one-level density for $\hat{f}$ supported in $[-1, 1]$.

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