Six-vertex Model with Partial Domain Wall Boundary Conditions. Exact Solution

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Brunel–Bielefeld Workshop

University of Bielefeld

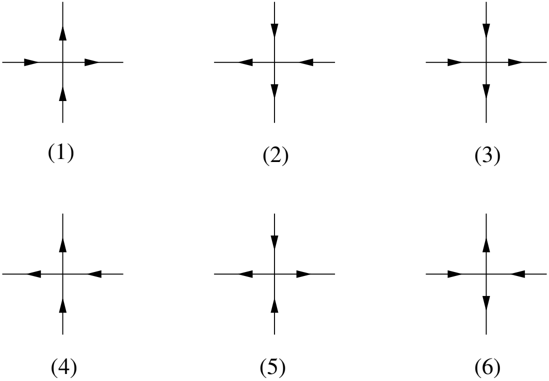
December 11–12, 2015

Six-Vertex Model

We consider the six-vertex model on a rectangular lattice of size $(n-m) \times n$ where $0 \le m < n$. The states of the model are realized by placing arrows on edges of the lattice obeying the *ice rule*, meaning that at each vertex there are exactly two arrows pointing in and two arrows pointing out.

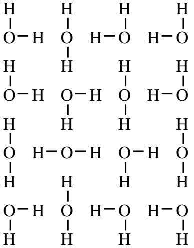
Vertex Configurations

There are *six possible configurations of arrows at each vertex* (this explains the name of the model), and we label the six vertex types as shown:



Square Ice Model

The name of the square ice model comes from the two-dimensional arrangement of water molecules, H_2O, with oxygen atoms at the vertices of a lattice and one hydrogen atom between each pair of adjacent oxygen atoms. We place an arrow in the direction from a hydrogen atom toward an oxygen atom if there is a bond between them.



The corresponding square ice model.

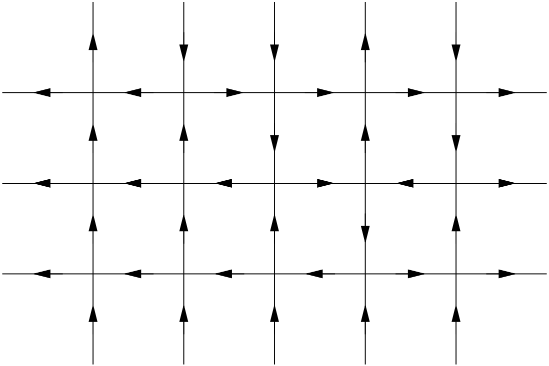
Six-Vertex Model

The six-vertex (ice) model has been introduced by Pauling and then by Slater in 1930s, and it was solved exactly in the thermodynamic limit by Lieb and Sutherland, for periodic boundary conditions. As we shall see, the six-vertex model differs from other standard models of statistical mechanics, such as the Ising model, in that it is very *sensitive to boundary conditions*. The model has been studied with free boundary conditions, anti-periodic boundary conditions, boundary loop conditions, etc. In this talk, we will discuss an exact solution of the six-vertex model with *partial domain wall boundary conditions* (pDWBC).

Partial Domain Wall Boundary Conditions

The *partial domain wall boundary conditions* (pDWBC) are defined in the following way. On the left and right boundaries all arrows point *out* of the lattice, and on the bottom boundary all arrows point *in*. The top boundary is *free*, and the ice-rule implies that there are exactly m arrows pointing out on this boundary, and the remaining $(n - m)$ arrows point in.

An example of a configuration with pDWBC



An example of the arrow configuration with the partial domain wall boundary conditions on the 3×5 lattice.

Configuration weights

For each of the six vertex types we assign a weight w_i, $i = 1, \ldots, 6$, and define the *weight of an arrow configuration* as the product of the weights of the vertices in the configuration:

$$w(\sigma) = \prod_{x \in V_{n-m,n}} w_{t(x;\sigma)} = \prod_{j=1}^{6} w_i^{N_i(\sigma)},$$

where $V_{n-m,n}$ is the set of vertices in the lattice, $t(x;\sigma)$ is the type of vertex at the vertex $x \in V_{n-m,n}$ in the configuration σ, and $N_i(\sigma)$ is the number of vertices of type i in the configuration σ.

Partition function and the Gibbs measure

The *partition function* of the model is given by

$$Z_{n-m,n} = \sum_{\sigma} w(\sigma),$$

where the sum runs over all possible configurations σ with pDWBC, and the *Gibbs probability measure* is given by

$$p_{n-m,n}(\sigma) = \frac{w(\sigma)}{Z_{n-m,n}}.$$

Our main goal will be to evaluate the *asymptotic behavior of the partition function* $Z_{n-m,n}$ as $n \to \infty$.

Domain Wall Boundary Conditions

When $m = 0$, the pDWBC reduces to the *domain wall boundary conditions* on the $n \times n$ lattice, and the asymptotic expansion of the partition function $Z_{n,n}$ as $n \to \infty$ has been studied in detail in a series of papers by Bleher and coauthors: Fokin, Liechty, and Bothner. For a complete description, see the monograph

P. Bleher and K. Liechty, *Random Matrices and the Six-Vertex Model,*(CRM Monograph), American Mathematical Society, 2014

Invariants and Reduction of Parameters

A priori, the six-vertex model has six parameters: the weights $w_1, \dots, w_6$. By observing some invariants, we can reduce the number of parameters to three. Namely we have the following proposition.

Invariants

Proposition 1. *In the six vertex model on the $(n-m)\times n$ lattice, the following equations hold for every state σ satisfying pDWBC:*

$$\begin{aligned} N_1(\sigma)+N_2(\sigma)+N_3(\sigma)+N_4(\sigma)+N_5(\sigma)+N_6(\sigma) &= n(n-m),\\ N_5(\sigma)-N_6(\sigma) &= n-m,\\ N_1(\sigma)-N_2(\sigma)+N_4(\sigma)-N_3(\sigma) &= m(n-m). \end{aligned}$$

The first equation is trivial and simply counts the total number of vertices. The second equation follows from the fact that in each row there is one more type-5 vertex than type-6 vertex, which is a direct consequence of the ice rule in each row. The third equation is less obvious and it follows from some equations for the height function.

Reduction of Parameters

Let us now discuss how to use Proposition 1 to reduce the number of parameters. Let us write

$$w_1 = ae^{-\alpha}, \quad w_2 = ae^{\alpha}, \quad w_3 = be^{-\beta},$$
$$w_4 = be^{\beta}, \quad w_5 = ce^{-\xi}, \quad w_6 = ce^{\xi},$$

where

$$a = \sqrt{w_1 w_2}, \quad e^{\alpha} = \sqrt{\frac{w_2}{w_1}}, \quad b = \sqrt{w_3 w_4}, \quad e^{\beta} = \sqrt{\frac{w_4}{w_3}},$$
$$c = \sqrt{w_5 w_6}, \quad e^{\xi} = \sqrt{\frac{w_6}{w_5}}.$$

Reduction of Parameters

Then Proposition 1 implies that

$$
\begin{aligned}
& w_1^{N_1} w_2^{N_2} w_3^{N_3} w_4^{N_4} w_5^{N_5} w_6^{N_6} \\
& = (ae^{-\eta})^{N_1} (ae^{\eta})^{N_2} (be^{-\eta})^{N_3} (be^{\eta})^{N_4} c^{N_5} c^{N_6} \\
& \times \left(\frac{w_1 w_4}{w_2 w_3} \right)^{m(n-m)/4} \left(\frac{w_5}{w_6} \right)^{(n-m)/2},
\end{aligned}
$$

where

$$\eta = \frac{\alpha + \beta}{2}.$$

Reduction of Parameters

This implies the relation between partition functions,

$$\begin{aligned} Z_{n-m,n}(w_1, w_2, w_3, w_4, w_5, w_6) \\ = Z_{n-m,n}(ae^{-\eta}, ae^{\eta}, be^{-\eta}, be^{\eta}, c, c), \\ \times \left(\frac{w_1 w_4}{w_2 w_3}\right)^{m(n-m)/4} \left(\frac{w_5}{w_6}\right)^{(n-m)/2}, \end{aligned}$$

hence the general case reduces to the weights

$$\begin{aligned} &w_1 = ae^{-\eta}, \quad w_2 = ae^{\eta}, \quad w_3 = be^{-\eta}, \quad w_4 = be^{\eta}, \\ &w_5 = c, \quad w_6 = c. \end{aligned}$$

with the *four parameters*, a, b, c, and η.

Reduction of Parameters

Furthermore,

$$Z_{n-m,n}(ae^{-\eta}, ae^{\eta}, be^{-\eta}, be^{\eta}, c, c)$$
$$= c^{n(n-m)} Z_{n-m,n}\left(\frac{ae^{-\eta}}{c}, \frac{ae^{\eta}}{c}, \frac{be^{-\eta}}{c}, \frac{be^{\eta}}{c}, 1, 1\right),$$

and so the model reduces to the *three parameters*, $\frac{a}{c}$, $\frac{b}{c}$, and η, where

$$\frac{a}{c} > 0, \quad \frac{b}{c} > 0, \quad \eta \in \mathbb{R}.$$

Invariants for DWBC

Remark. In the six vertex model on the $n \times n$ lattice with *domain wall boundary conditions* (DWBC), the following equations hold for every state σ:

$$N_1(\sigma) + N_2(\sigma) + N_3(\sigma) + N_4(\sigma) + N_5(\sigma) + N_6(\sigma) = n^2,$$
$$N_5(\sigma) - N_6(\sigma) = n,$$
$$N_1(\sigma) = N_2(\sigma), \quad N_3(\sigma) = N_4(\sigma),$$

and we can reduce the weights $w_1, \dots, w_6$ to the *two parameters*,

$$\frac{a}{c} > 0, \quad \frac{b}{c} > 0,$$

where

$$w_1 = w_2 = a, \quad w_3 = w_4 = b, \quad w_5 = w_6 = c.$$

Phase Diagram

- **Phase Diagram of the Six-Vertex Model**

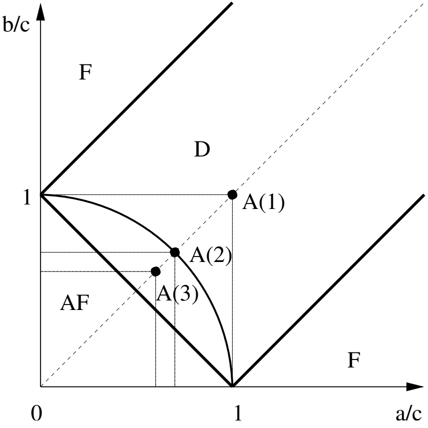
Introduce the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}.$$

The phase diagram of the six-vertex model consists of the three phase regions: *ferroelectric phase region*, $\Delta > 1$; the *antiferroelectric phase region*, $\Delta < -1$; and, the *disordered phase region*, $-1 < \Delta < 1$.

Phase Diagram

- **Phase Diagram of the Six-Vertex Model**



Ferroelectric Phase Region: Parameterization

In this talk we will consider the *ferroelectric phase region*.
The ferroelectric phase region on the phase diagram is described as

$$\mathcal{F} = \{a > b + c\} \cup \{b > a + c\}.$$

Fix two real parameters t and γ, with $0 < |\gamma| < t$, and introduce the parameterization of a, b, c as

$$a = \sinh(t - \gamma)\,, \quad b = \sinh(t + \gamma)\,, \quad c = \sinh(2|\gamma|).$$

We will assume that $\gamma > 0$, which corresponds to $b > a + c$.

Ferroelectric Phase Region: Parameterization

Respectively, the general weights in the *ferroelectric phase region with pDWBC* are parameterized by the *three real parameters* t, γ, η as

$$w_1 = \sinh(t-\gamma)e^{-\eta}, \quad w_2 = \sinh(t-\gamma)e^{\eta},$$
$$w_3 = \sinh(t+\gamma)e^{-\eta}, \quad w_4 = \sinh(t+\gamma)e^{\eta},$$
$$w_5 = w_6 = \sinh(2|\gamma|), \quad 0 < |\gamma| < t.$$

Ferroelectric Phase Region: Specialization

In what follows, we consider a *specialization of the general weights* to the case when $\eta = \gamma$, so that

$$w_1 = \sinh(t-\gamma)e^{-\gamma}, \quad w_2 = \sinh(t-\gamma)e^{\gamma},$$
$$w_3 = \sinh(t+\gamma)e^{-\gamma}, \quad w_4 = \sinh(t+\gamma)e^{\gamma},$$
$$w_5 = \sinh(2|\gamma|), \quad w_6 = \sinh(2|\gamma|), \quad 0 < |\gamma| < t.$$

Thus, we consider *a two-parameter family of weights* in the three-dimensional space of general ferroelectric weights with pDWBC.

Main Result: Asymptotics of the Partition Function

Theorem 2. *For any* $\varepsilon > 0$ *there is a constant* $n_0 > 0$ *such that for any* $n \geq n_0$ *and any* m *in the interval* $0 \leq m < n$ *we have that*

$$Z_{n-m,n} = C(m)F^{n(n-m)}G^{m(n-m)}H^{n-m}\left(1+\xi_{nm}\right),$$

where

$$F = \sinh(t+\gamma), \quad G = e^{\gamma}, \quad H = e^{\gamma - t},$$
$$C(m) = 1 - e^{-4\gamma(m+1)},$$

and

$$|\xi_{nm}| \leq \rho^m e^{-n^{1-\varepsilon}}, \quad \rho = e^{-2\gamma} < 1.$$

Limiting Free Energy

Remarks. 1. It is interesting to notice that the *limiting free energy per site f depends on the aspect ratio $r = \frac{n-m}{n}$* of the rectangular lattice of size $(n-m) \times n$. Namely, it follows from Theorem 2 that

$$f = \lim_{n,m\to\infty;\ \frac{n-m}{n}\to r} \frac{1}{(n-m)n} \ln Z_{n-m,n}$$
$$= \ln[\sinh(t+\gamma)] + (1-r)\gamma.$$

2. Observe that the error term is exponentially small as $n, m \to \infty$.

Sketch of the Proof of Theorem 2

The Major Steps of the Proof of Theorem 2

1. Determinantal Formula for the Partition Function.
2. Partition Function in Terms of Orthogonal Polynomials.
3. Asymptotic Analysis of the Orthogonal Polynomials.
4. Asymptotics of the Partition Function.
5. Evaluation of the Constant Factor.

Determinantal Formula for the Partition Function

The starting point for our analysis is the following determinantal formula for the partition function. Introduce the notations

$$\varphi(t) := \sinh(t-\gamma)\sinh(t+\gamma) = ab\,,$$
$$\phi(t) := \frac{\sinh(2\gamma)}{\sinh(t-\gamma)\sinh(t+\gamma)} = \frac{c}{ab}\,.$$

Determinantal Formula for the Partition Function

Theorem 3. *We have that*

$$Z_{n-m,n} = \frac{(-1)^{m(m+1)/2-nm}\varphi(t)^{n(n-m)}e^{m(n-m)t}}{2^{m(m-1)/2}\prod_{j=0}^{n-m-1} j! \prod_{j=0}^{n-1} j!}\,\tau_{n-m,n},$$

where $\tau_{n-m,n}$ is a determinant of mixed Vandermonde/Hankel type: $\tau_{n-m,n} =$

$$\det\begin{pmatrix} 1 & (-2) & (-2)^2 & \dots & (-2)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (-2m) & (-2m)^2 & \dots & (-2m)^{n-1} \\ \phi(t) & \phi'(t) & \phi''(t) & \dots & \phi^{(n-1)}(t) \\ \phi'(t) & \phi''(t) & \phi'''(t) & \dots & \phi^{(n)}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi^{(n-m-1)}(t) & \phi^{(n-m)}(t) & \phi^{(n-m+1)}(t) & \dots & \phi^{(2n-m-2)}(t) \end{pmatrix}.$$

Determinantal Formula for the Partition Function

Remarks. 1. The existence of a determinantal formula for the partition function of the six-vertex model with pDWBC $Z_{n-m,n}$ is far from obvious. No such formula is known for periodic or free boundary conditions.

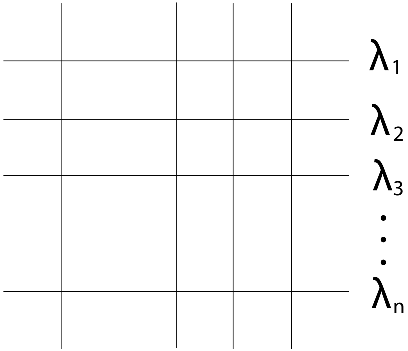
2. We can prove the determinantal formula only for the *specialization of the general weights* to the case when $\eta = \gamma$, so that

$$w_1 = \sinh(t-\gamma)e^{-\gamma}, \quad w_2 = \sinh(t-\gamma)e^{\gamma},$$
$$w_3 = \sinh(t+\gamma)e^{-\gamma}, \quad w_4 = \sinh(t+\gamma)e^{\gamma},$$
$$w_5 = w_6 = \sinh(2|\gamma|), \quad 0 < |\gamma| < t.$$

The proof of the determinantal formula *breaks down* when $\eta \neq \gamma$.

Partially Inhomogeneous Six-Vertex Model

The proof of the determinantal formula is based on the (partially) inhomogeneous Izergin–Korepin formula. We begin with a partially inhomogeneous six vertex model with DWBC. That is, consider the $n \times n$ square lattice with parameters $(\lambda_1, \ldots, \lambda_n)$ assigned to horizontal lines from top to bottom.



Partially Inhomogeneous Weights

Introduce the weights,

$$w_j = \begin{cases} a_-(\lambda_j) := e^{-\gamma} a(\lambda_j), & \text{if vertex in row } j \text{ is of type } 1 \\ a_+(\lambda_j) := e^{\gamma} a(\lambda_j), & \text{if vertex in row } j \text{ is of type } 2 \\ b_-(\lambda_j) := e^{-\gamma} b(\lambda_j), & \text{if vertex in row } j \text{ is of type } 3 \\ b_+(\lambda_j) := e^{\gamma} b(\lambda_j), & \text{if vertex in row } j \text{ is of type } 4 \\ c(\lambda_j) := \sinh(2\gamma), & \text{if vertex in row } j \text{ is of type } 5 \text{ or } 6\,, \end{cases}$$

where

$$a(\lambda) = \sinh(\lambda - \gamma)\,, \quad b(\lambda) = \sinh(\lambda + \gamma)\,, \quad c(\lambda) \equiv c = \sinh(2\gamma).$$

The Izergin-Korepin formula

The Izergin-Korepin formula for the partially inhomogeneous partition function is

$$Z_n^{\mathrm{inh}} = \frac{(-1)^{n(n-1)/2} \prod_{j=1}^{n} \varphi(\lambda_j)^n}{\prod_{j=0}^{n-1} j! \prod_{1 \le j < k \le n} \sinh(\lambda_j - \lambda_k)} \det \left(\phi^{(k-1)}(\lambda_j)\right)_{j,k=1}^{n},$$

Now we take the limit as $\lambda_1, \ldots, \lambda_m \to \infty$, then the limit as $\lambda_{m+1}, \ldots, \lambda_n \to \lambda$, apply many times the l'Hôpital rule, set $t = \lambda - \mu$, and this gives the determinant of the mixed Vandermonde/Hankel type.

Next step is an expression of the partition function $Z_{n-m,n}$ in terms of discrete orthogonal polynomials.

Discrete Orthogonal Polynomials

Introduce the *weight*

$$w(x) = \left[e^{-2(t-\gamma)(x+m+1)} - e^{-2(t+\gamma)(x+m+1)} \right] \prod_{k=1}^{m} (x+k).$$

Notice that we assume that $t > \gamma > 0$, hence $w(x) > 0$ for $x \geq 0$, and therefore we can introduce monic polynomials *orthogonal on the lattice* $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ with respect to the weight $w(x)$:

$$\sum_{x=0}^{\infty} p_j(x) p_k(x) w(x) = h_k \delta_{jk} ,$$

where $h_k = h_k(m) > 0$ are normalizing constants.

Partition Function in Terms of Orthogonal Polynomials

We have

Theorem 4. *The partition function* $Z_{n-m,n}$ *is given by the following formula:*

$$Z_{n-m,n} = (2ab)^{n(n-m)} e^{m(n-m)t} \prod_{j=0}^{n-m-1} \frac{h_j}{j!\,(j+m)!}\,.$$

The Riemann–Hilbert approach

It is natural to apply now the Riemann–Hilbert approach to discrete orthogonal polynomials in order to obtain the large j asymptotics of the normalizing coefficients h_j. This is what we tried to do for a long time, but there were some serious problems in this approach (I'll tell about them later), and we changed the strategy.

Approximation by the Meixner Polynomials

Let us rewrite $w(x)$ as

$$w(x) = \left[e^{-2(t-\gamma)(x+m+1)} - e^{-2(t+\gamma)(x+m+1)}\right] \frac{m!(m+1)_x}{x!},$$

where

$$(\beta)_x = \beta(\beta+1)\dots(\beta+x-1)$$

is the Pochhammer symbol. Observe that since $\gamma > 0$, the first exponent in $w(x)$, $e^{-2(t-\gamma)(x+m+1)}$, is *dominating* as $(x+m+1) \to \infty$.

Approximation by the Meixner Polynomials

As *an approximation* to $w(x)$, let us drop the second exponent, $e^{-2(t+\gamma)(x+m+1)}$, in $w(x)$ and consider the auxiliary weight

$$w^{\mathrm{M}}(x) = e^{-2(t-\gamma)(x+m+1)} \frac{m!(m+1)_x}{x!}$$
$$= C_m e^{-2(t-\gamma)x} \frac{(m+1)_x}{x!},$$

where

$$C_m = m! e^{-2(t-\gamma)(m+1)},$$

so that

$$w(x) = w^{\mathrm{M}}(x) \left[1 - e^{-4\gamma(x+m+1)}\right].$$

The orthogonal polynomials with respect to the weight $w^{\mathrm{M}}(x)$ are the *Meixner polynomials*.

Meixner Partition Function

As an approximation to the partition function $Z_{n-m,n}$, we introduce the *Meixner partition function*,

$$Z^{\mathrm{M}}_{n-m,n} = (2ab)^{n(n-m)} e^{m(n-m)t} \prod_{k=0}^{n-m-1} \frac{h_k^{\mathrm{M}}}{k!\,(k+m)!}\,.$$

The normalizing constants for the Meixner polynomials h_k^{M} are known explicitly, and in this way we obtain that

$$Z^{\mathrm{M}}_{n-m,n} = b^{n(n-m)} e^{m(n-m)\gamma}\, e^{-(n-m)(t-\gamma)}\,.$$

Ratio of the Partition Functions

Now we would like to estimate the ratio,

$$\frac{Z_{n-m,n}}{Z^{\mathrm{M}}_{n-m,n}} = \prod_{k=0}^{n-m-1} \frac{h_k}{h_k^{\mathrm{M}}}\,.$$

This will be done, by showing that h_k/h_k^{M} is *exponentially close* to 1 as $k \to \infty$.

Difference of the Normalizing Constants

We have the following *identity*:

$$h_k - h_k^{\mathrm{M}} = \sum_{l=0}^{\infty} p_k(l) p_k^{\mathrm{M}}(l) \left[w(l) - w^{\mathrm{M}}(l) \right],$$

where $p_k(l)$ and $p_k^{\mathrm{M}}(l)$ are monic orthogonal polynomials with respect to the weights $w(l)$ and $w^{\mathrm{M}}(l)$, respectively. We use this identity to estimate $|h_k - h_k^{\mathrm{M}}|$.

Estimation of the Ratio of the Normalizing Constants

We prove the following result:

Theorem 5. *Fix any* $1 > \varepsilon > 0$. *Then there is a constant* $C_\varepsilon > 0$ *such that*

$$h_k = h_k^{\mathrm{M}} e^{r_k},$$

where

$$|r_k| \le C_\varepsilon e^{-2\gamma m - k^{1-\varepsilon}},$$

for all m *in the interval* $0 \le m < n$ *and* $k \ge 0$.

This is the central technical result of the work.

Estimation of the Partition Function

From Theorem 5 we obtain that for any fixed $1 > \varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$Z_{n-m,m} = C(m)\, Z^{\mathrm{M}}_{n-m,m} e^{\xi_{n-m,m}},$$

where

$$|\xi_{n-m,m}| \le C_\varepsilon e^{-2\gamma m} e^{-n^{1-\varepsilon}},$$

and

$$C(m) = \prod_{k=0}^{\infty} \frac{h_k}{h_k^{\mathrm{M}}}.$$

Evaluation of the Constant Factor

Our next goal will be to *calculate the constant factor* $C(m)$. This will be done in *two steps*: first, with the help of the Toda equation, we will find the form of the dependence of $C(m)$ on the parameter t of the six-vertex model, and second, we will find the large t asymptotics of $C(m)$. By combining these two steps, we will obtain the exact value of $C(m)$.

The Toda Equation

The weight $w(x)$ can be written as

$$w(x) = e^{-2t(x+m+1)} u(x)\,,$$
$$u(x) = 2\sinh[2\gamma(x+m+1)]\,\frac{(x+m)!}{x!}\,.$$

Since the dependence of $w(x)$ on t is a *linear exponent*, we have *the Toda equation*:

$$\left(\ln \prod_{k=0}^{n-m-1} h_k\right)'' = \frac{4h_{n-m}}{h_{n-m-1}}\,, \qquad (\;)' = \frac{\partial}{\partial t}\,.$$

The form of the Dependence of $C(m)$ on t

From the Toda equation and the estimates of the normalizing constants h_k we obtain that

$$\ln C(m) = C_0 + C_1 t + \mathcal{O}\left(\rho^m e^{-n^{1-\varepsilon}}\right),$$

where C_0, C_1 are independent of t (but they may depend on m, n). However, $\ln C(m)$ does not depend on n and according to the latter equation, as $n \to \infty$ it is a limit of linear functions of the argument t. This implies that $\ln C(m)$ is a *linear function* of t as well, so that

$$\ln C(m) = d_0(m) + d_1(m)t\,.$$

Next we calculate $d_0(m)$ and $d_1(m)$.

Calculation of $C(m)$

Consider the following regime:

$$\gamma \text{ is fixed}, \quad m \text{ is fixed}, \qquad t \to \infty,$$

and let us evaluate the asymptotics of $C(m)$ in this regime.

Calculation of $C(m)$

Some calculations show that as $t \to \infty$,

$$\frac{h_0}{h_0^{\mathrm{M}}} = 1 - e^{-4\gamma(m+1)} \left(\frac{1 - e^{-2t+2\gamma}}{1 - e^{-2t-2\gamma}} \right)^{m+1}$$
$$= 1 - e^{-4\gamma(m+1)} + \mathcal{O}(e^{-2t}).$$

and there exists $t_0 > 0$ such that

$$\frac{h_k}{h_k^{\mathrm{M}}} = e^{r_k}, \quad |r_k| \le e^{-ct-k^{1/2}},$$

for all $t \ge t_0$ and $k \ge 1$. This implies the following estimate:

Calculation of $C(m)$

$$\ln C(m) = \ln \left[1 - e^{-4\gamma(m+1)}\right] + \mathcal{O}(e^{-2t}).$$

Comparing this with the linear form of $C(m)$ as a function of t, we conclude that

$$C(m) = 1 - e^{-4\gamma(m+1)}.$$

This finishes the proof of Theorem 2.

Final Remarks

An interpolation problem and the *Riemann–Hilbert approach* can be used as well, to obtain an asymptotic formula for the discrete orthogonal polynomials $p_k(z)$ with respect to the weight $w(x) = w(x; m)$. We consider here a *scaling* regime, when $m, k \to \infty$ in such a way that $m = k\xi$ where $0 \le \xi \le A$ for some $A > 0$.

Energy Functional

To describe the corresponding *equilibrium measure*, introduce the potential function

$$V(x) = 2(t-\gamma)x + x\ln x - x\ln(x+\xi) - \xi\ln(x+\xi) + \xi,$$

and the *energy functional*

$$I_V(\nu) = -\iint_{x\neq y} \log|x-y|d\nu(x)d\nu(y) + \int V(x)d\nu(x).$$

Equilibrium Measure

The equilibrium measure $\nu_{\rm eq}$ minimizes $I_V(\nu)$ over the space of probability measures ν on the line with the *constraint*

$$\nu E \le mE,$$

for any measurable set E, where mE is the Lebesgue measure. An analysis of the minimization problem reveals a *phase transition* at $\xi = \xi_c$, where

$$\xi_c = e^{2t-2\gamma} - 1.$$

Phase Transition: $\xi < \xi_c$

Namely, for $0 \le \xi < \xi_c$ there are numbers $0 < a < b$ such that the equilibrium measure $\nu_{\rm eq}$ is *saturated* on the interval $[0, a]$ so that

$$\frac{d\nu_{\rm eq}(x)}{dx} = 1, \quad 0 \le x \le a,$$

and $\nu_{\rm eq}$ has a *band* on the interval (a, b), so that

$$0 < \frac{d\nu_{\rm eq}(x)}{dx} < 1, \quad a < x < b.$$

Finally, the interval $[b, \infty)$ is a *void* one, so that

$$\frac{d\nu_{\rm eq}(x)}{dx} = 0, \quad x \ge b.$$

Phase Transition: $\xi > \xi_c$

For $\xi > \xi_c$, there is no saturated interval, and the equilibrium measure is supported by a band (a, b), where $0 < a < b$.

It is interesting to notice that the *phase transition in the equilibrium problem has no effect* on the asymptotic behavior of the partition function $Z_{n-m,m}$ in Theorem 2, and the free energy is an *analytic* function of the parameters γ, t, r of the problem.

The End

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Thank you!