

RMP of $SL(2, \mathbb{R})$ in the continuum limit. II- Lyapunov exponent & fluctuations

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With :

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The group $\mathrm{SL}(2, \mathbb{R})$ and the Iwasawa representation

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

with

$$a, b, c, d \in \mathbb{R} \quad \& \quad ad - bc = 1$$

3 independent real parameters

Iwasawa :

$$M = \underbrace{\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}}_{K(\theta)} \underbrace{\begin{pmatrix} e^w & 0 \\ 0 & e^{-w} \end{pmatrix}}_{A(w)} \underbrace{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}}_{N(u)}$$

Physical models :

pointlike impurities and quantum localisation

A.Comtet, CT & Y.Tourigny, J.Stat.Phys. 140 (2010)

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Main question :

M_1, M_2, \dots : a set of i.i.d random matrices $\in \mathrm{SL}(2, \mathbb{R})$

$$\Pi_N = M_N \cdots M_2 M_1$$

Given a measure $\mu(dM)$ on $\mathrm{SL}(2, \mathbb{R})$,
what is the Lyapunov exponent

$$\gamma = \lim_{N \rightarrow \infty} \frac{\ln \|\Pi_N\|}{N} \quad ?$$

$$\Pi_N = M_N \cdots M_2 M_1 \quad \rightarrow \text{random walk in } \mathrm{SL}(2, \mathbb{R})$$

Diffusion in the group is complicated !

$\mathrm{SL}(2, \mathbb{R})$ is non compact \rightarrow No stationary distribution...



Work in *projective space* $\Rightarrow \exists$ limit law

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Projective space and Riccati variable

$$\mathbb{R}^2 \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \infty \text{ (projective space)}$$

vector $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow$ direction $z = x/y$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \mathcal{M}(z) = \frac{az + b}{cz + d}$$

Stationary distribution $f(z)$

$$\text{Random product : } \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \Pi_{n+1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = M_n \Pi_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

↑

$$\text{Random sequence : } z_{n+1} = \mathcal{M}_n(z_n)$$

$$\overbrace{\langle \delta(z - z_{n+1}) \rangle}_{z_{n+1}} = \langle \delta(z - \mathcal{M}_n(z_n)) \rangle_{z_n, M_n} = \left\langle f_n(\mathcal{M}^{-1}(z)) \frac{d\mathcal{M}^{-1}(z)}{dz} \right\rangle_M$$

Furstenberg theorem : $f_n(z) \xrightarrow{n \rightarrow \infty} f(z)$ (stationary distribution)

$$f(z) = \int_{\mathrm{SL}(2, \mathbb{R})} \mu(dM) f(\mathcal{M}^{-1}(z)) \frac{d\mathcal{M}^{-1}(z)}{dz} \quad \text{with } \mathcal{M}^{-1}(z) = \frac{dz - b}{-cz + a}$$

Furstenberg formula :

$$\gamma = \lim_{N \rightarrow \infty} \frac{\ln \|\prod_n M_n\|}{N} = \int dz f(z) \int_{\mathrm{SL}(2, \mathbb{R})} \mu(dM) \ln \frac{\left\| M \begin{pmatrix} z \\ 1 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} z \\ 1 \end{pmatrix} \right\|}$$

*** STILL DIFFICULT ! ***

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Continuum limit & Infinitesimal generators

$$\forall M_n \in \mathrm{SL}(2, \mathbb{R}), \quad M_n = \underbrace{K(\theta_n) A(w_n) N(u_n)}_{\text{Iwasawa decomposition}}$$

Continuum limit :

Consider $\Pi_N = M_N \cdots M_1$ with $\boxed{\theta_n, w_n, u_n \rightarrow 0}$

$$M_n \simeq \mathbf{1}_2 + \theta_n \Gamma_K + w_n \Gamma_A + u_n \Gamma_N$$

$$\Gamma_K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma_N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Randomness

$$\mu(dM) \leftrightarrow P(\theta, w, u) \xrightarrow[\text{limit}]{\text{continuum}} \text{Gaussian}$$

1) averages :

$$\mu = \begin{pmatrix} \langle \theta \rangle \\ \langle w \rangle \\ \langle u \rangle \end{pmatrix}$$

2) covariances :

$$D = \begin{pmatrix} D_{\theta\theta} & D_{\theta w} & D_{\theta u} \\ D_{\theta w} & D_{ww} & D_{wu} \\ D_{\theta u} & D_{wu} & D_{uu} \end{pmatrix}$$

$$\text{where } \left\{ \begin{array}{l} D_{ab} = \langle ab \rangle - \langle a \rangle \langle b \rangle \simeq \langle ab \rangle \\ a, b \in \{\theta, w, u\} \end{array} \right.$$

Disorder → 9 parameters

A remark about the continuum limit

$$\gamma = \lim_{N \rightarrow \infty} \frac{\ln \|\prod_n M_n\|}{N} = f(\bar{\theta}, \bar{w}, \dots, D_{\theta\theta}, D_{\theta w}, \dots, D_{wu}, \overbrace{\dots}^{\text{beyond cont. lim.}})$$

Scaling function G :

$$\gamma \underset{\epsilon \rightarrow 0}{\simeq} \epsilon \times G\left(\frac{\bar{\theta}}{\epsilon}, \dots, \frac{D_{\theta\theta}}{\epsilon}, \dots, \frac{D_{wu}}{\epsilon}\right)$$

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Three useful representations of $\mathrm{SL}(2, \mathbb{R})$

1) Action on the vector $\in \mathbb{R}^2$: matrix $M \simeq \mathbf{1}_2 + \theta \Gamma_K + w \Gamma_A + u \Gamma_N$

2) Action on the direction $z \in \overline{\mathbb{R}}$:

$$\mathcal{M}(z) \simeq z - [\theta g_K(z) + w g_A(z) + u g_N(z)]$$

Example of rotation $K(\theta)$:

$$\mathcal{K}_\theta(z) = \frac{z \cos \theta - \sin \theta}{z \sin \theta + \cos \theta} \simeq z - \theta \underbrace{(1 + z^2)}_{g_K(z)}$$
$$\mathbf{g}(z) \stackrel{\text{def}}{=} \begin{pmatrix} g_K(z) \\ g_A(z) \\ g_N(z) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 + z^2 \\ -2z \\ -1 \end{pmatrix}}_{\text{Iwasawa}}$$

3) Action on the distribution $f(z)$:

$$(\mathcal{T}_M f)(z) \stackrel{\text{def}}{=} f(\mathcal{M}^{-1}(z)) \simeq f(z) + [\theta g_K(z) + w g_A(z) + u g_N(z)] \frac{d}{dz} f(z)$$

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From the random recursion to the random process

Recurrence $z_{n+1} = \mathcal{M}_n(z_n)$:

$$\frac{\partial j_n(z)}{\partial z} \text{ with } j_n(z) = \left\langle \int_z^{\mathcal{M}^{-1}(z)} dt f_n(t) \right\rangle_M$$
$$f_{n+1}(z) - f_n(z) = \overbrace{\left\langle f_n(\mathcal{M}^{-1}(z)) \frac{d\mathcal{M}^{-1}(z)}{dz} \right\rangle_M} - f_n(z)$$

Continuum limit \downarrow

$$\begin{aligned} M_n &\rightarrow 1_2 \\ z_n &\xrightarrow{n \rightarrow x} z(x) \\ f_n(z) &\xrightarrow{} f(z; x) \end{aligned}$$

Fokker-Planck Equation :

$$\frac{\partial}{\partial x} f(z; x) = \mathcal{G}^\dagger f(z; x) = \frac{\partial}{\partial z} j(z; x)$$

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Generator \mathcal{G} of the diffusion

$$\text{Iwasawa } M = e^{\theta \Gamma_K} e^{w \Gamma_A} e^{u \Gamma_N} \longrightarrow \mathcal{T}_M = e^{\theta g_K(z) \frac{d}{dz}} e^{w g_A(z) \frac{d}{dz}} e^{u g_N(z) \frac{d}{dz}}$$

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Get

$$\frac{\partial}{\partial z} j(z; x) = \mathcal{G}^\dagger f(z; x) \quad \text{with} \quad \boxed{\mathcal{G}^\dagger = \frac{1}{2} \frac{d}{dz} \mathbf{g}(z) \cdot D \cdot \frac{d}{dz} \mathbf{g}(z) - \frac{d}{dz} \tilde{v}(z)}$$

$$\text{drift : } \tilde{v}(z) = -\mu \cdot \mathbf{g}(z) - \frac{1}{2} \mathbf{c} \cdot [\mathbf{g}(z) \times \mathbf{g}'(z)] \quad \text{with } \mathbf{c} = \begin{pmatrix} D_{wu} \\ -D_{\theta u} \\ D_{\theta w} \end{pmatrix}$$

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Stationary distribution

Furstenberg theorem : $f(z; x) \xrightarrow{x \rightarrow \infty} f(z)$ & $j(z; x) \xrightarrow{x \rightarrow \infty} j$

$$\left[\frac{1}{2} \frac{d}{dz} \mathcal{D}(z) - v(z) \right] f(z) = j$$

$\mathcal{D}(z) \stackrel{\text{def}}{=} \mathbf{g}(z) \cdot D \cdot \mathbf{g}(z)$: effective diffusion constant

$v(z) \stackrel{\text{def}}{=} \tilde{v}(z) - \frac{1}{2} \mathbf{g}'(z) \cdot D \cdot \mathbf{g}(z)$: drift

How to get γ rapidly ? Overview

Obtain γ without the knowledge of $f(z)$

crucial point : $\mathcal{D}(z)$ & $v(z)$ are *polynomials* (continuum limit)

$$\left[\frac{1}{2} \frac{d}{dz} \mathcal{D}(z) - v(z) \right] f(z) = j$$

Hilbert transform \downarrow $F(y) = \int_{-\infty}^{+\infty} dz \frac{f(z)}{y-z}$; $\text{Im } y > 0$

$$\left[\dots \right]_y F(y) = \Omega + \frac{D_{\theta\theta}}{2} (1 + y^2) + \bar{\theta} y - \bar{w}$$

where $\Omega = \gamma - i\pi j$ is the generalised Lyapunov exponent

Analyticity of $F(y)$ gives Ω

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Obtain γ without the knowledge of $f(z)$

crucial point : $\mathcal{D}(z)$ & $v(z)$ are *polynomials* (continuum limit)

$$\left[\frac{1}{2} \frac{d}{dz} \mathcal{D}(z) - v(z) \right] f(z) = j$$

Hilbert transform \downarrow $F(y) = \int_{-\infty}^{+\infty} dz \frac{f(z)}{y - z}$; $\text{Im } y > 0$

$$\left[\dots \right]_y F(y) = \Omega + \frac{D_{\theta\theta}}{2} (1 + y^2) + \bar{\theta} y - \bar{w}$$

where $\Omega = \gamma - i\pi j$ is the generalised Lyapunov exponent

Analyticity of $F(y)$ gives Ω

A simple example : only $D_{uu} \neq 0$ & $\langle \theta \rangle \neq 0$

$$\mathcal{D}(z) = D_{uu} \text{ & } v(z) = \tilde{v}(z) = -\bar{\theta}(1 + z^2)$$

$$\left[\frac{1}{2} D_{uu} \frac{d}{dy} + \bar{\theta}(1 + y^2) \right] F(y) = \Omega + \bar{\theta} y \quad ; \operatorname{Im} y > 0$$

J.-M. Luck, J. Phys. A (2004)

$$F(y) = \frac{2}{D_{uu}} e^{-\frac{2\bar{\theta}}{D_{uu}}(y + \frac{1}{3}y^3)} \int_{-\infty}^y dy' (\Omega + \bar{\theta} y') e^{+\frac{2\bar{\theta}}{D_{uu}}(y' + \frac{1}{3}y'^3)}$$

- F analytic in the upper half plane
- decays for $|y| \rightarrow \infty$

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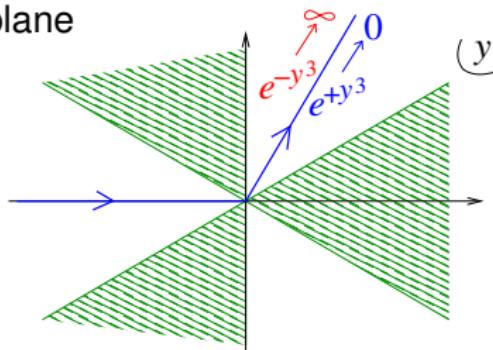
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Lyapunov exponent → Airy functions

$$\int_{-\infty}^{\infty} e^{i\pi/3} dy (\Omega + \bar{\theta} y) e^{+\frac{2\bar{\theta}}{D_{uu}}(y + \frac{1}{3}y^3)} = 0 \Rightarrow \int_{-\infty}^{\infty} e^{i\pi/3} dt (\Omega + \beta t) e^{+\frac{1}{3}t^3 - \xi t} = 0$$

where $\beta = (\bar{\theta}^2 D_{uu}/2)^{1/3}$ and $\xi = -\bar{\theta}^2/\beta^2 = -(2\bar{\theta}/D_{uu})^{2/3}$

i.e.

$$\left(-\beta \frac{d}{d\xi} + \Omega \right) \text{Ai}(e^{2i\pi/3}\xi) = 0 \text{ where } \text{Ai}(x) = \int_{\infty e^{-i\pi/3}}^{\infty} \frac{dt}{2i\pi} e^{\frac{1}{3}t^3 - xt}$$

$$\boxed{\Omega = \beta e^{2i\pi/3} \frac{\text{Ai}'(e^{2i\pi/3}\xi)}{\text{Ai}(e^{2i\pi/3}\xi)} = \beta \frac{\text{Ai}'(\xi) - i\text{Bi}'(\xi)}{\text{Ai}(\xi) - i\text{Bi}(\xi)}}$$

Physical pb : lattice of δ -scatterers, $M_n = K(\bar{\theta})N(u_n)$

$$\left[-\frac{d^2}{dx^2} + \overbrace{\sum_n v_n \delta(x - n\ell)}^{V(x)} \right] \psi(x) = k^2 \psi(x) \quad \text{with} \quad \begin{cases} \bar{\theta} = k\ell \\ u_n = v_n/k \end{cases}$$

Continuum limit : $V(x) \xrightarrow{\ell \rightarrow 0 \text{ & } v_n \rightarrow 0}$ Gaussian white noise

$$\langle V(x) V(x') \rangle = \underbrace{\frac{1}{\ell}}_{\stackrel{\text{def}}{=} \sigma} D_{vv} \delta(x - x')$$

Relation to the problem studied by Halperin (1965)

$$\bar{\theta} = k\ell \text{ & } D_{uu} = \frac{\ell\sigma}{k^2} \quad \Rightarrow \quad \beta = \ell (\sigma/2)^{1/3} \text{ & } \xi = -\frac{k^2}{(\sigma/2)^{2/3}}$$

$$\frac{1}{\ell} \Omega = \left(\frac{\sigma}{2} \right)^{1/3} \frac{\text{Ai}'(\xi) - i \text{Bi}'(\xi)}{\text{Ai}(\xi) - i \text{Bi}(\xi)} \quad \text{with } \xi = -(2/\sigma)^{2/3} E$$

Halperin, Phys. Rev. (1965)

Step 1 : Relation between $f(z)$ & γ

- In the simple case ($D_{\theta\theta} = 0$ & $w = 0$) : $\gamma = \bar{\theta} \int dz z f(z)$
- In the general case : Furstenberg formula gives

$$\begin{aligned}\gamma = & -\bar{w} + D_{\theta u} + (\bar{\theta} + 2D_{\theta w}) \int dz z f(z) \\ & + \frac{1}{2} D_{\theta\theta} \int dz z \frac{d}{dz} [(1+z^2)f(z)]\end{aligned}$$

Step 2 : Hilbert transform of the distribution

$$\left[\frac{1}{2} \frac{d}{dz} \mathcal{D}(z) - v(z) \right] f(z) = j$$

$$F(y) = \int_{-\infty}^{+\infty} dz \frac{f(z)}{y - z} \quad \text{Im } y > 0$$

$\mathcal{D}(z) = \mathbf{g}(z) \cdot D \cdot \mathbf{g}(z)$ & $v(z)$ polynomials (\leftarrow continuum limit)

$\Rightarrow F$ obeys :

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Equation involving the Lyapunov γ

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Equation involving the Lyapunov γ

Step 3 : Obtain $\Omega = \gamma - i\pi j$ from analytic properties of $F(y)$

$$\left[\frac{1}{2} \frac{d}{dy} \mathcal{D}(y) - v(y) \right] F(y) = \Omega + \frac{D_{\theta\theta}}{2} (1 + y^2) + \bar{\theta} y - \bar{w}$$

has solution :

$$F(y) = \frac{2}{\mathcal{D}(y)} e^{\int dy \frac{2v(y)}{\mathcal{D}(y)}} \int_{-\infty}^y dy' \left[\Omega + \frac{D_{\theta\theta}}{2} (1 + y'^2) + \bar{\theta} y' - \bar{w} \right] e^{-\int dy' \frac{2v(y')}{\mathcal{D}(y')}}$$

- ① $F(y)$ analytic in the upper half plane ($\text{Im } y > 0$)
- ② $F(y) \sim 1/y$ for $|y| \rightarrow \infty$

$$\exists \varphi, \quad \int_{-\infty}^{\infty e^{i\varphi}} dy \left[\Omega + \frac{D_{\theta\theta}}{2} (1 + y^2) + \bar{\theta} y - \bar{w} \right] e^{-\int dy \frac{2v(y)}{\mathcal{D}(y)}} = 0$$

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Conclusion (1) : Classification of solutions

$$\mathcal{D}(z) = \mathbf{g}(z) \cdot D \cdot \mathbf{g}(z) \rightarrow 4 \text{ zeros}$$

Generalised Lyapunov exponent :

$$\Omega = \gamma - i\pi j = \frac{d}{d(\cdot)} \ln(\text{special fct})$$

Disorder	Multiplicity of zeros	Special function	Related work
u	1 quadruple	Airy	Halperin '65
w	2 double	Bessel K_ν	Bouchaud, Comtet, Georges & LeDoussal '90 (↔ Derrida, Hilhorst '83)
θ	2 double	Bessel $I_{i\lambda}$	
$w \& u$	1 double & 2 simple	Whittaker	Hagendorf & CT '08
$\theta, w \& u$ $\bar{\theta} = 0, \dots$ $D_{\theta w} = 0, \dots$	4 simple	elliptic	
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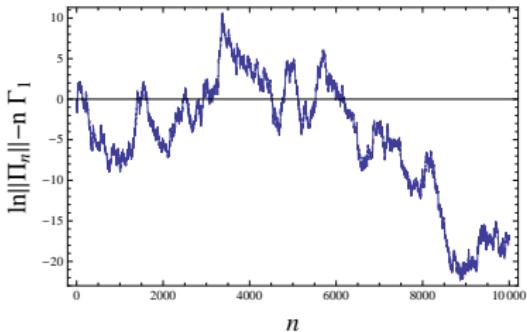
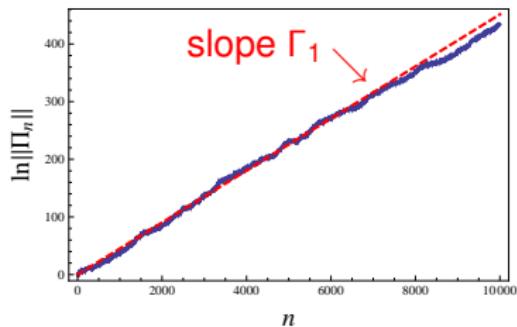
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- 1 Introduction
- 2 Lyapunov exponent in the continuum limit
- 3 Fluctuations (specific cases)

Fluctuations of $\ln \|\Pi_n\|$



Generalised central limit theorem :

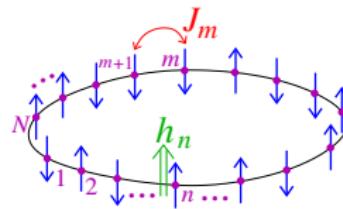
Bougerol & Lacroix, (the book, chapter V) 1985

$$\Pi_n = M_n \cdots M_2 M_1 \quad (\text{non commuting})$$

$$\Gamma_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left\| \Pi_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| \quad \& \quad \Gamma_2 = \lim_{n \rightarrow \infty} \frac{1}{n} \text{Var} \left(\ln \left\| \Pi_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\| \right)$$

Transfer matrices in statistical physics

Random Ising chain :



$$\mathcal{H}(\{\sigma_n\}) = - \sum_{n=1}^N (\textcolor{red}{J}_n \sigma_n \sigma_{n+1} + \textcolor{green}{h}_n \sigma_n)$$
$$\sigma_n = \pm 1$$

Transfer matrix approach :

$$\begin{aligned}\mathcal{Z}_N &= \sum_{\{\sigma_n\}} e^{-\mathcal{H}(\{\sigma_n\})} \\ &= \text{Tr} \{ T_N \cdots T_1 \}\end{aligned}$$

with $T_n \stackrel{\text{def}}{=} \underbrace{\begin{pmatrix} e^{+\textcolor{green}{h}_n} & 0 \\ 0 & e^{-\textcolor{green}{h}_n} \end{pmatrix} \begin{pmatrix} e^{+J_n} & e^{-J_n} \\ e^{-J_n} & e^{+J_n} \end{pmatrix}}_{M_n = \frac{T_n}{\sqrt{2 \sinh 2J_n}} \in \text{subgroup of } \text{SL}(2, \mathbb{R})}$

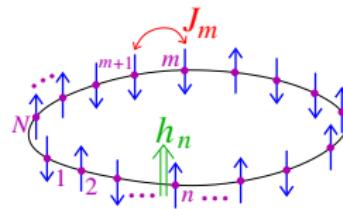
Free energy :

$$\mathcal{F}_N = -\ln \mathcal{Z}_N = -\ln \|T_N \cdots T_2 T_1\|$$

mean : Γ_1
variance : Γ_2

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Transfer matrices in quantum mechanics

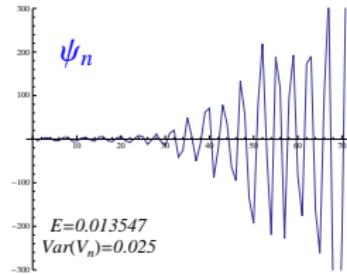
$$-\psi''(x) + V(x) \psi(x) = k^2 \psi(x)$$

$$\begin{pmatrix} \psi'(x_R) \\ \psi(x_R) \end{pmatrix} = M \begin{pmatrix} \psi'(x_L) \\ \psi(x_L) \end{pmatrix} \quad \begin{pmatrix} \Psi_L \\ \Psi_L' \end{pmatrix} \xrightarrow{\qquad V(x) \qquad} \begin{pmatrix} \Psi_R \\ \Psi_R' \end{pmatrix}$$

Current conservation $\Rightarrow M^\dagger \sigma_y M = \sigma_y \Rightarrow e^{i\theta} M \in \text{SL}(2, \mathbb{R})$

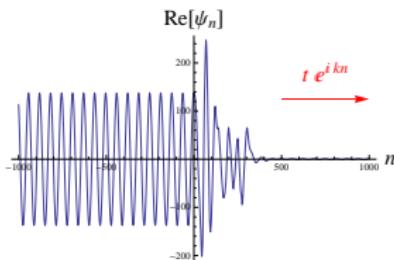
$$\begin{pmatrix} \psi'(x) \\ \psi(x) \end{pmatrix} = \Pi_N \begin{pmatrix} \psi'(0) \\ \psi(0) \end{pmatrix}$$

$$\boxed{\ln ||\Pi_N|| \leftrightarrow \ln |\psi(x)|}$$



Importance of fluctuations (1) : transmission

Fluctuations of the transmission probability \mathcal{T}

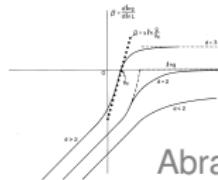


$\psi(x)$ the solution with $\begin{cases} \psi(0) = 0 \\ \psi'(0) = 1 \end{cases}$

For long sample : $\mathcal{T} = |t|^2 \sim |\psi(L)|^{-2}$

$$P(\mathcal{T}; L) \simeq \frac{1}{\mathcal{T} \sqrt{8\pi\gamma_2 L}} \exp - \frac{(\ln \mathcal{T} + 2\gamma_1 L)^2}{8\gamma_2 L}$$

Single Parameter Scaling (SPS) hypothesis (weak disorder)



$$\gamma_2 = \lim_{x \rightarrow \infty} \frac{\text{Var}(\ln |\psi(x)|)}{x} \simeq \gamma_1 = \lim_{x \rightarrow \infty} \frac{\ln |\psi(x)|}{x}$$

Abrahams, Anderson, Licciardello & Ramakrishnan, PRL 42 (1979)

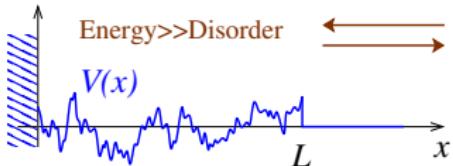
Anderson, Thouless, Abraham & Fisher, PRB 22 (1980)

Shapiro, PRB 34 (1986)

etc.

Importance of fluctuations (2) : LDoS

Stationary scattering state :



$$\psi(x; k^2) \underset{x>L}{=} \frac{1}{\sqrt{2\pi\nu_k}} \left(e^{-ik(x-L)} + e^{+ik(x-L)+i\eta(k^2)} \right)$$

$$v_k = dE/dk = 2k : \text{group velocity}$$

Distribution of the LDoS $\rho(x; E) = \langle x | \delta(E - H) | x \rangle = |\psi(x; E)|^2$

$$P(\tilde{\rho}; x) \underset{\substack{\text{weak} \\ \text{disorder}}}{\simeq} \frac{1}{\tilde{\rho} \sqrt{8\pi\gamma_1(L-x)}} \exp - \frac{[\ln(\tilde{\rho}/\rho_0) + 2\gamma_1(L-x)]^2}{8\gamma_1(L-x)}$$

$$\text{where } \rho_0 = 1/(2\pi\nu_k)$$

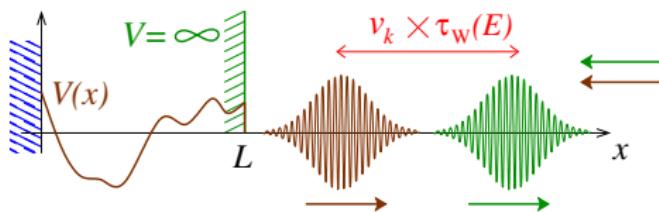
Berezinskii blocks method → Altshuler & Prigodin, Sov. Phys. JETP **68** (1989)

Importance of fluctuations (3) : Wigner time delay

$\eta(E)$: reflection phase shift

Wigner time delay

$$\tau_w(E) \stackrel{\text{def}}{=} \frac{d\eta(E)}{dE}$$



Exponential functional of the BM

$$\tau_w(E) \simeq 2\pi \int_0^L dx \underbrace{|\psi(x; E)|^2}_{\rho(x; E)} \xrightarrow[\text{disorder}]{\text{weak}} \frac{1}{k} \int_0^L dx e^{-2\gamma_1 x + 2\sqrt{\gamma_1} B(x)}$$

CT & A.Comtet, PRL 82 (1999)

Fluctuations : two specific cases

1) K and N

$$M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} 1 & u_n \\ 0 & 1 \end{pmatrix}$$

2) K and A

$$M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix}$$

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The related localisation model

Dirac equation with random mass

$$[\sigma_2 i \partial_x + \sigma_1 m(x)] \Psi(x) = \varepsilon \Psi(x) \quad \text{where } \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

with

$$\langle m(x)m(x') \rangle_c = g \delta(x - x')$$

Motivations

- relation to
- Supersymmetric quantum mechanics
Review : A. Comtet & CT '97
 - Organic molecules : Takayama, Lin-Liu & Maki '80
 - Sinai diffusion (classical)
Bouchaud, Comtet, Georges & Le Doussal '90
 - Quantum spin chain models
Fisher '94, Le Doussal, Monthus '99, ...

Delocalisation :

$$\langle m(x) \rangle = 0 \quad \Rightarrow \quad \gamma_1 \underset{\varepsilon \rightarrow 0}{\simeq} \frac{g}{\ln(g/\varepsilon)} \rightarrow 0$$

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$$M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix} \quad \text{where } \theta_n = \varepsilon \ell_n$$

$\varepsilon \in i\mathbb{R}$ – Transfer matrices for $(\psi, -i\chi)$

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\leftrightarrow Ising chain

Continuum limit $w_n \rightarrow 0$ & $\ell_n \rightarrow 0$

non Gaussian white noise \longrightarrow Gaussian white noise with $g = \frac{\langle w_n^2 \rangle}{\langle \ell_n \rangle}$

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$\varepsilon \in i\mathbb{R}$ – Transfer matrices for $(\psi, -i\chi)$

$$M_n = \begin{pmatrix} \cosh \tilde{\theta}_n & \sinh \tilde{\theta}_n \\ \sinh \tilde{\theta}_n & \cosh \tilde{\theta}_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix} \quad \text{where } \tilde{\theta}_n = -i\varepsilon \ell_n$$

\leftrightarrow Ising chain

Continuum limit $w_n \rightarrow 0$ & $\ell_n \rightarrow 0$

non Gaussian white noise \longrightarrow Gaussian white noise with $g = \frac{\langle w_n^2 \rangle}{\langle \ell_n \rangle}$

From random matrix product to random process

$$\text{Spinor } \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$$

$$\Psi(x_{N+1}^-) = \Pi_N \Psi(x_1^-) \quad \Rightarrow \quad \begin{cases} x & \leftrightarrow N/\rho \\ \ln |\psi(x)| & \leftrightarrow \ln \|\Pi_N\| \\ \gamma_{1,2} & \leftrightarrow \rho \Gamma_{1,2} \end{cases}$$

$\rho = 1/\langle \ell_n \rangle$: density of "mass impurities"

$$\text{Riccati variable } z(x) \stackrel{\text{def}}{=} -\varepsilon \chi(x)/\psi(x) = \psi'(x)/\psi(x) - m(x)$$

$$\ln |\psi(x)| = \int_0^x dt [z(t) + m(t)]$$

$$\begin{cases} \chi' = -m\chi + \varepsilon\psi \\ \psi' = m\psi - \varepsilon\chi \end{cases} \quad \Rightarrow \quad \frac{d}{dx} z(x) = -\varepsilon^2 - z(x)^2 - 2z(x)m(x)$$

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Generator

$$\mathcal{G} = 2g z \partial_z z \partial_z - (\varepsilon^2 + z^2) \partial_z$$

Result

$$\gamma_2 = g - \langle z \ln |z/\varepsilon| \rangle + \int dz dz' z G(z|z') \left(z' - \frac{\varepsilon^2}{z'} \right) f(z')$$

$$\mathcal{G}^\dagger f(z) = 0 \quad (\text{stationary distribution})$$

$$\mathcal{G}^\dagger G(z|z') = f(z) - \delta(z - z')$$

$$\gamma_2 = \lim_{x \rightarrow \infty} \frac{\text{Var}(\ln |\psi(x)|)}{x} = g + 2 \lim_{x \rightarrow \infty} \langle z(x) \int_0^x dt [z(t) + m(t)] \rangle_c$$

Useful relation

$$\begin{aligned} 2 \ln |\psi(x)/\psi(x_0)| &= 2 \int_{x_0}^x dt [z(t) + m(t)] \\ &= -\ln \left| \frac{z(x)}{z(x_0)} \right| + \int_{x_0}^x dt \left(z(t) - \frac{\varepsilon^2}{z(t)} \right) \end{aligned}$$

Propagator $\partial_x P_x(z|z') = \mathcal{G}^\dagger P_x(z|z')$

$$\begin{aligned} &\lim_{x \rightarrow \infty} \left\langle z(x) \int_{x_0}^x dt \left(z(t) - \frac{\varepsilon^2}{z(t)} \right) \right\rangle_c \\ &= \int dz z \underbrace{\int_0^\infty d\tau [P_\tau(z|z') - f(z)] \left(z' - \frac{\varepsilon^2}{z'} \right) f(z')}_{=G(z|z')} \end{aligned}$$

Limiting behaviours

- High energy, $\varepsilon \gg g$

$$\gamma_2 \simeq \gamma_1 \simeq g/2 \quad (\text{SPS})$$

- Small energy, $|\varepsilon| \ll g$

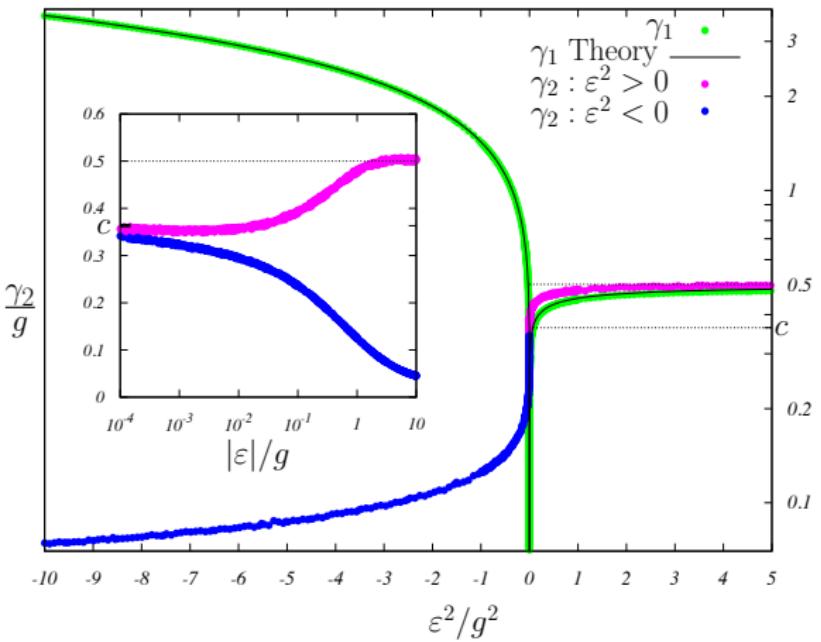
$$\left. \begin{array}{l} \gamma_2 \underset{\varepsilon \rightarrow 0}{\simeq} g \left[\frac{1}{3} + \frac{1}{2 \ln(2g/|\varepsilon|)} \right] \\ \gamma_2 \underset{\varepsilon \rightarrow i0}{\simeq} g \left[\frac{1}{3} - \frac{1}{2 \ln(2g/|\varepsilon|)} \right] \end{array} \right\} \gg \gamma_1 \simeq \frac{g}{\ln(g/|\varepsilon|)}$$

saturation of the fluctuations for $\varepsilon = 0$

- Large imaginary energy \rightarrow perturbative solution of the SDE

$$\gamma_2 \underset{\varepsilon \rightarrow i\infty}{\simeq} \frac{g^2}{4|\varepsilon|} \ll \gamma_1 \simeq \sqrt{|\varepsilon|} + \frac{g}{2}$$

Numerics



Conclusion (2)

$$\mathcal{G}^\dagger f(z) = 0$$

$$\mathcal{G}^\dagger G(z|z') = f(z) - \delta(z - z')$$

1) Matrices $M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} 1 & u_n \\ 0 & 1 \end{pmatrix}$
 $\rightarrow \mathcal{G} = (\sigma/2) \partial_z^2 - (\varepsilon^2 + z^2) \partial_z$

$$\gamma_2 = 2 \int dz dz' z G(z|z') z' f(z')$$

A different representation was obtained in : Schomerus & Titov, PRE '02

2) Matrices $M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix}$
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Thank you.

- A. Comtet, J.-M. Luck, CT & Y. Tourigny, J. Stat. Phys. **150**, 13 (2013)
K. Ramola & CT, J. Stat. Phys. **157**, 497 (2014)

Appendices

Appendix A : Lie algebra of $\mathrm{SL}(2, \mathbb{R})$

Iwasawa decomposition :

$$M = \underbrace{\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}}_{K(\theta)} \underbrace{\begin{pmatrix} e^w & 0 \\ 0 & e^{-w} \end{pmatrix}}_{A(w)} \underbrace{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}}_{N(u)}$$

Infinitesimal generators

$$\Gamma_K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \Gamma_N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

Lie algebra :

$$[\Gamma_K, \Gamma_A] = 2\Gamma_K + 4\Gamma_N$$

$$[\Gamma_K, \Gamma_N] = -\Gamma_A$$

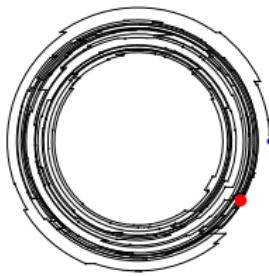
$$[\Gamma_A, \Gamma_N] = 2\Gamma_N$$

Appendix B : Examples of matrix products (1)

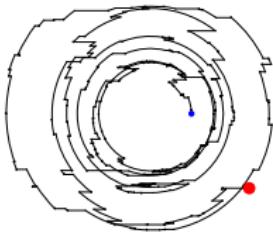
$$\Pi_N = M_N \cdots M_1 \quad \text{with} \quad M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} 1 & u_n \\ 0 & 1 \end{pmatrix}$$

Motion of $\begin{pmatrix} x_N \\ y_N \end{pmatrix} = \Pi_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

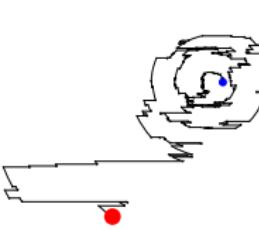
Scale $\theta_n \propto k$ and $u_n \propto 1/k$



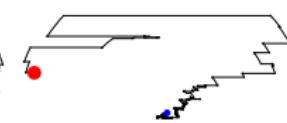
$k = 5$



$k = 2$



$k = 1$

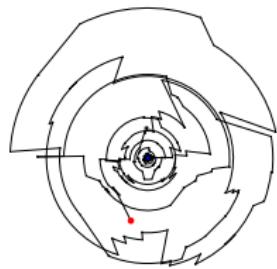


$k = 0.5$

Appendix B : Examples of matrix products (2)

$$\Pi_N = M_N \cdots M_1 \quad \text{with} \quad M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix}$$

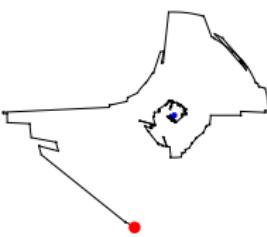
Scale $\theta_n \propto k$ and $w_n \propto k^0$



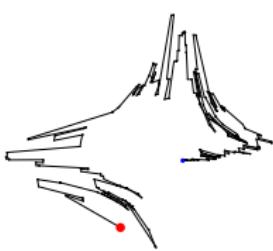
$k = 5$



$k = 2$



$k = 1$



$k = 0.2$

Appendix C : From multiplicative to additive noise

$$\frac{d}{dx} z(x) = -\varepsilon^2 - z(x)^2 - 2z(x) m(x)$$
$$\downarrow \quad \zeta = \mp \ln(\pm z/|\varepsilon|)/2 \quad \text{for } z \in \mathbb{R}_\pm$$

$$\frac{d}{dx} \zeta(x) = -U'(\zeta(x)) + m(x)$$

where the potential is

$$U(\zeta) = \begin{cases} \frac{|\varepsilon|}{2} \cosh 2\zeta & \text{for } \varepsilon \in i\mathbb{R} \\ -\frac{|\varepsilon|}{2} \sinh 2\zeta & \text{for } \varepsilon \in \mathbb{R} \end{cases}$$

Lyapunov exponent :

$$\gamma_1 = \lim_{x \rightarrow \infty} \frac{\langle \ln |\psi(x)| \rangle}{x}$$

Generalised Lyapunov exponent :

$$\Lambda(q) = \lim_{x \rightarrow \infty} \frac{\ln \langle |\psi(x)|^q \rangle}{x} = \sum_{n=1}^{\infty} \frac{q^n}{n!} \gamma_n$$

Paladin & Vulpiani, Phys. Rep. **156** (1987)

Generator

$$\mathcal{G}^\dagger = \frac{g}{2} \partial_\zeta^2 + \partial_\zeta \mathcal{U}'(\zeta)$$

Generalised Lyapunov exponent $\Lambda(q) = q\gamma_1 + \frac{q^2}{2}\gamma_2 + \dots$

$$\gamma_1 = 2 \langle \mathcal{U}(\zeta) \rangle$$

$$\gamma_2 = g - 2 \langle \zeta \mathcal{U}'(\zeta) \rangle + 8 \int d\zeta d\zeta' \mathcal{U}(\zeta) \mathcal{G}(\zeta|\zeta') \mathcal{U}(\zeta') \mathcal{P}(\zeta')$$

where

$$\mathcal{G}^\dagger \mathcal{P}(\zeta) = 0$$

$$\mathcal{G}^\dagger \mathcal{G}(\zeta|\zeta') = \mathcal{P}(\zeta) - \delta(\zeta - \zeta')$$

Appendix D : Lyapunov exponent for $\varepsilon = 0$

The norm

$$\|\Pi_N\| \stackrel{\text{def}}{=} \int_{|\Psi_0|=1} d\Psi_0 |\Pi_N \Psi_0| \quad \text{with } \Psi_0 = \begin{pmatrix} \sin \Theta_0 \\ -\cos \Theta_0 \end{pmatrix}$$

Case $\varepsilon = 0$

$$\Pi_N = \begin{pmatrix} e^W & 0 \\ 0 & e^{-W} \end{pmatrix} \quad \text{with } W = \sum_{n=1}^N w_n$$

hence

$$|\Pi_N \Psi_0| = \sqrt{\cosh 2W - \cos 2\Theta_0 \sinh 2W}$$

$$|\Pi_N \Psi_0| = \sqrt{\cosh 2W - \cos 2\Theta_0 \sinh 2W}$$

Remark :

- $\Theta_0 = 0 \Rightarrow |\Pi_N \Psi_0| = e^{-W}$
- $\Theta_0 = \pi/2 \Rightarrow |\Pi_N \Psi_0| = e^{+W}$

Reflects the behaviour of the solutions of $[\sigma_2 i \partial_x + \sigma_1 m(x)] \Psi(x) = 0$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\int^x dx' m(x')} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\int^x dx' m(x')}$$

The norm

$$||\Pi_N|| = \int_0^\pi \frac{d\Theta_0}{\pi} \sqrt{\cosh 2W - \cos 2\Theta_0 \sinh 2W} = \frac{2e^{|W|}}{\pi} \mathbf{E} \left(\sqrt{1 - e^{-4|W|}} \right)$$

$$\ln ||\Pi_N|| \simeq |W| - \ln(\pi/2) \quad \text{for } |W| \gg 1$$

Moments

$W = \sum_n w_n \Rightarrow P_x(W) = \frac{1}{\sqrt{2\pi gx}} e^{-W^2/(2gx)}$ with $g = \rho \langle w_n^2 \rangle$, hence

$$\langle \ln ||\Pi_N|| \rangle \simeq \sqrt{\frac{2gx}{\pi}} - \ln(\pi/2)$$

$$\text{Var}(\ln ||\Pi_N||) \simeq gx \left(1 - \frac{2}{\pi} \right)$$

Lyapunov

$$\gamma_1 = 0$$

&

$$\gamma_2 = g \left(1 - \frac{2}{\pi} \right) = g \times 0.363380\dots$$

Appendix E : Localisation for $\varepsilon \rightarrow 0$ (Dirac)

- Usual measure of localisation is $\tilde{\xi}_\varepsilon = 1/\gamma_1 \simeq (1/g) \ln(g/|\varepsilon|)$ (**wrong**)
- Another length scale

$$\sqrt{\gamma_2 x} \gtrsim \gamma_1 x \quad \text{for} \quad x \lesssim \frac{\gamma_2}{\gamma_1^2} \sim \xi_\varepsilon = (1/g) \ln^2(g/|\varepsilon|)$$

ξ_ε : D. S. Fisher, *Random AF quantum spin chains*, PRB **50** (1994)

Localisation is dominated by the fluctuations for $\varepsilon \rightarrow 0$

Relation to boundary condition sensitivity (Thouless criterion)

$$\psi(0) = \psi(L) = 0 \Rightarrow \text{spectrum } \{\varepsilon_n\}$$

$$\int_0^L \frac{dx}{L} \underbrace{\langle \rho(x; \varepsilon) \rangle}_{\text{LDoS}} \simeq \overbrace{\varrho(\varepsilon)}^{\text{bulk}} \mathcal{D}(L/\xi_\varepsilon) \quad \text{where } 1/\xi_\varepsilon = \int_0^\varepsilon d\varepsilon' \varrho(\varepsilon') \sim \frac{g}{\ln^2(g/\varepsilon)}$$

CT & C.Hagendorf, J.Phys.A **43** (2010)

Appendix F : Band center anomaly (Anderson model)

$$-\varphi_{n+1} + V_n \varphi_n - \varphi_{n-1} = \varepsilon \varphi_n$$

Weak disorder (perturbative) expansion breaks down at band center :

- Weak disorder expansion :

$$\gamma_1 \simeq \frac{\langle V_n^2 \rangle}{4\sqrt{4 - \varepsilon^2}} \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{8} \langle V_n^2 \rangle$$

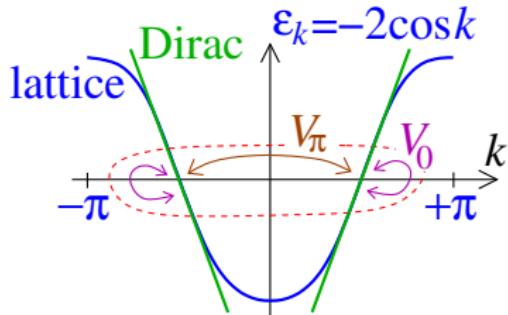
Derrida & Gardner (1984); J.-M. Luck, the book, (1992)

- Band center anomaly of the Lyapunov

$$\gamma_1 = \underbrace{[\Gamma(3/4)/\Gamma(1/4)]^2}_{\simeq 0.114} \langle V_n^2 \rangle \text{ at } \varepsilon = 0$$

Kappus & Wegner, Z. Phys. **45** (1981)
Derrida & Gardner, J. Physique **45** (1984)

Continuum limit of the lattice model



Continuum limit for $\varepsilon \simeq 0$:

$$[-i\sigma_3 \partial_x + V_0(x) + \sigma_1 V_\pi(x)] \tilde{\Psi}(x) = \varepsilon \tilde{\Psi}(x)$$

- Forward scattering $V_0 \rightarrow$ strength g_0
- Backward scattering $V_\pi \rightarrow$ strength g

Anomaly

$$\text{deviation from SPS} \frac{\gamma_2}{\gamma_1} \xrightarrow{\text{disorder} \rightarrow 0} 1$$

Transfer matrices

Choose

$$V_0(x) = \sum_n v_n \delta(x - x_n) \text{ and } V_\pi(x) = \sum_n w_n \delta(x - x_n)$$

$$M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} e^{w_n} & 0 \\ 0 & e^{-w_n} \end{pmatrix}$$

$$\text{where } \theta_n = \varepsilon (x_{n+1} - x_n) - v_n$$

Lyapunov in the continuum limit $\varepsilon = 0$, $v_n \rightarrow 0$ & $w_n \rightarrow 0$

→ A.Comtet, J.-M.Luck, CT & Y.Tourigny, J.Stat.Phys. 150 (2013); § 6

$$\gamma_1 = g \left[\frac{1}{k^2} \left(\frac{\mathbf{E}(k)}{\mathbf{K}(k)} - 1 \right) + 1 \right] \quad \text{with } k = \frac{1}{\sqrt{1 + g_0/g}}$$

Check : $(g_0 = g)$

$$\gamma_1 = g \left[2 \frac{\mathbf{E}(1/\sqrt{2})}{\mathbf{K}(1/\sqrt{2})} - 1 \right] = g \left(\frac{2\Gamma(3/4)}{\Gamma(1/4)} \right)^2$$

Tune the BC anomaly at $\varepsilon = 0$

- $g_0 = g$:

$$\frac{\gamma_2}{\gamma_1} \simeq g \times 1.047$$

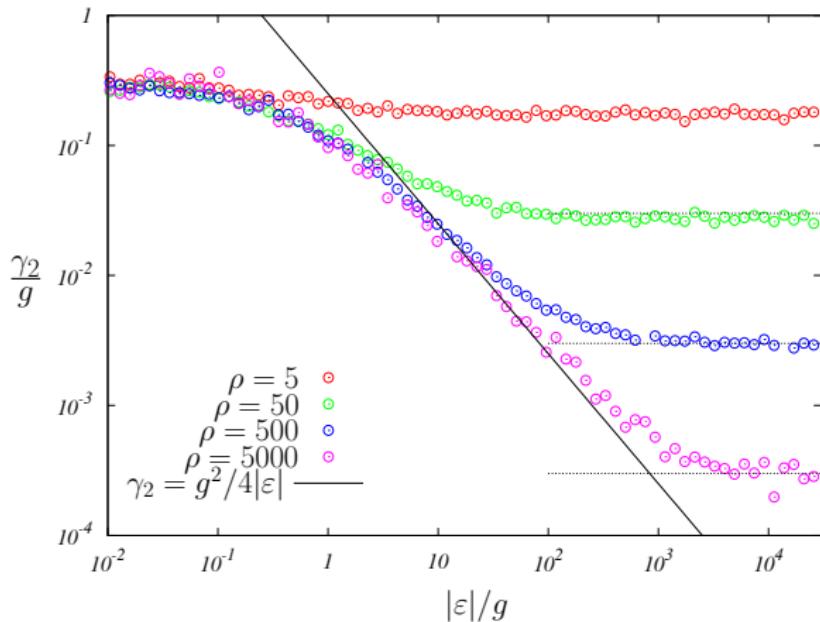
Schomerus & Titov, PRB **67** (2003)

- $g_0 \ll g$:

$$\frac{\gamma_2}{\gamma_1} \sim \begin{cases} \ln(g/g_0) & \text{at } \varepsilon = 0 \\ \ln(g/|\varepsilon|) & \text{for } g_0 \ll |\varepsilon| \ll g \end{cases}$$

Appendix G : Non Gaussian vs Gaussian white noise $m(x)$

$\varepsilon \rightarrow i\infty$ with $m(x) = \sum_n w_n \delta(x - x_n)$
saturation of the fluctuations for $\varepsilon \rightarrow i\infty$



Appendix H : Random Schrödinger operator

$$\left[-\frac{d^2}{dx^2} + \sum_n v_n \delta(x - x_n) \right] \psi(x) = E \psi(x)$$

Transfer matrices

$$M_n = \begin{pmatrix} \cos \theta_n & -\sin \theta_n \\ \sin \theta_n & \cos \theta_n \end{pmatrix} \begin{pmatrix} 1 & u_n \\ 0 & 1 \end{pmatrix} \text{ for } E = k^2$$

$$M_n = \begin{pmatrix} \cosh \theta_n & \sinh \theta_n \\ \sinh \theta_n & \cosh \theta_n \end{pmatrix} \begin{pmatrix} 1 & u_n \\ 0 & 1 \end{pmatrix} \text{ for } E = -k^2$$

$$\theta_n = k(x_{n+1} - x_n) \text{ and } u_n = v_n/k$$

Riccati $z = \psi'/\psi$

$$\frac{dz(x)}{dx} = -E - z(x)^2 + V(x)$$

Continuum limit ($\theta_n \rightarrow 0$ and $u_n \rightarrow 0$) :

$V(x)$ a Gaussian white noise : $\langle V(x)V(x') \rangle = \sigma \delta(x - x')$

$$\Rightarrow \text{generator } \mathcal{G} = (\sigma/2)\partial_z^2 - (E + z^2)\partial_z$$

Lyapunov

$$\gamma_1 = \langle z \rangle = \int dz z f(z)$$

$$\gamma_2 = 2 \int dz dz' z G(z|z') z' f(z')$$

where

$$\mathcal{G}^\dagger f(z) = 0$$

$$\mathcal{G}^\dagger G(z|z') = f(z) - \delta(z - z')$$

Effective potential $\mathcal{U}(z) = Ez + z^3/3$

$$\mathcal{G} = (\sigma/2)\partial_z^2 - \mathcal{U}'(z)\partial_z \quad \& \quad \mathcal{G}^\dagger = (\sigma/2)\partial_z^2 + \partial_z\mathcal{U}'(z)$$

Stationary distribution $\mathcal{G}^\dagger f(z) = 0$

$$f(z) = \frac{2N}{\sigma} f_0(z) \int_{-\infty}^z \frac{dt}{f_0(t)} \quad \text{with } f_0(z) = e^{-\frac{2}{\sigma}\mathcal{U}(z)}$$

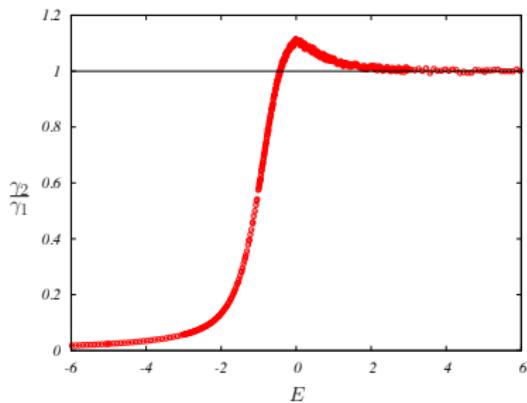
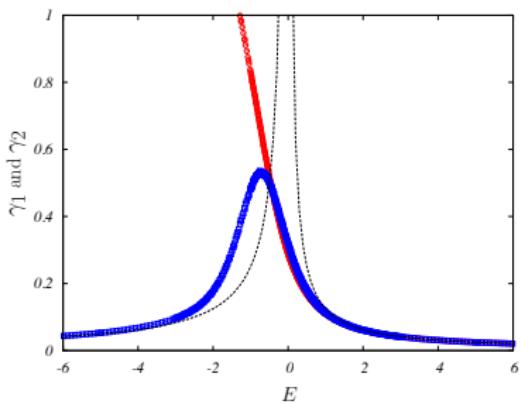
Solution of $\mathcal{G}^\dagger G(z|z') = f(z) - \delta(z - z')$

$$G(z|z') = \frac{1}{N} \left\{ f(z) \left[c(z') + \int_{-\infty}^z dt f(t) \right] - f_0(z) \int_{-\infty}^z dt \frac{f(t)^2}{f_0(t)} + \frac{f_0(z_>)f(z_<)}{f_0(z')} \right\}$$
$$c(z') + \frac{1}{2} = \frac{\sigma}{2N} \left[\int_{-\infty}^{+\infty} dz'' f(z'')^2 f(-z'') - f(-z') f(z') \right] - \int_{-\infty}^{z'} dz'' f(z'')$$

Limiting values

$$\gamma_2 \underset{E \rightarrow \infty}{\simeq} \frac{\sigma}{8E} \simeq \gamma_1 \text{ at leading order (SPS)}$$

$$\gamma_2 \underset{E \rightarrow -\infty}{\simeq} \frac{\sigma}{4(-E)} \ll \gamma_1 \simeq \sqrt{-E}$$



Generalised Lyapunov

$$\Lambda(q) = \lim_{x \rightarrow \infty} \frac{\ln \langle |\psi(x)|^q \rangle}{x} = \sum_{n=1}^{\infty} \frac{q^n}{n!} \gamma_n$$

Paladin & Vulpiani, Phys. Rep. **156** (1987)

$$\langle |\psi(x)|^q \rangle = \left\langle e^{q \int_0^x dt z(t)} \right\rangle = \int dz \langle z | e^{x(\mathcal{G}^\dagger + qz)} | z_0 \rangle \\ \underset{x \rightarrow \infty}{\sim} e^{x\Lambda(q)}$$

where

$$[\mathcal{G}^\dagger + qz] \Phi_0^R(z; q) = \Lambda(q) \Phi_0^R(z; q)$$

Perturbative analysis of $[\mathcal{G}^\dagger + q z] \Phi_0^R(z; q) = \Lambda(q) \Phi_0^R(z; q)$

Schomerus & Titov, PRE **66** (2002)

$$\Lambda(q) = q \gamma_1 + \frac{q^2}{2!} \gamma_2 + \dots$$

$$\Phi_0^R(z; q) = f(z) + q \varphi_1(z) + q^2 \varphi_2(z) + \dots$$

We get

$$\mathcal{G}^\dagger \varphi_1(z) = (\gamma_1 - z) f(z) \xrightarrow{\int dz} 0 = \gamma_1 - \int dz z f(z)$$

$$\mathcal{G}^\dagger \varphi_2(z) = (\gamma_1 - z) \varphi_1(z) + \frac{1}{2} \gamma_2 f(z)$$

\vdots

We deduce Schomerus & Titov's result

$$\gamma_2 = 2 \int dz (z - \gamma_1) \varphi_1(z)$$

$$\varphi_1(z) = N \left(\frac{2}{\sigma} \right)^2 f_0(z) \int_{-\infty}^z \frac{dz'}{f_0(z')} \int_{-\infty}^{z'} dz'' (\gamma_1 - z'') f_0(z'') \int_{-\infty}^{z''} \frac{dz'''}{f_0(z''')}$$

$$\text{where } f_0(z) = e^{-\frac{2}{\sigma} \mathcal{U}(z)}$$

This is a different integral representation from

$$\gamma_2 = 2 \int dz dz' z G(z|z') z' f(z')$$