Universiteit Utrecht

Large deviation statistics of the cosmic density field

Cora Uhlemann

Institute for Theoretical Physics, University Utrecht

Delta Institute for Theoretical Physics, The Netherlands

in collaboration with:
Sandrine Codis (CITA), Francis Bernardeau, Christophe Pichon & Paulo Reimberg (IAP)

Kosmologietag Bielefeld, April 2016

Outline

- From the CMB to the Cosmic Web
- Dark matter dynamics
- Large deviation principle
- Probability distribution of density in cells
- Interesting first applications:
  clustering & constraining cosmology

Outline

- From the CMB to the Cosmic Web
- Dark matter dynamics
- Large deviation principle
- Probability distribution of density in cells
- Interesting first applications: clustering & constraining cosmology

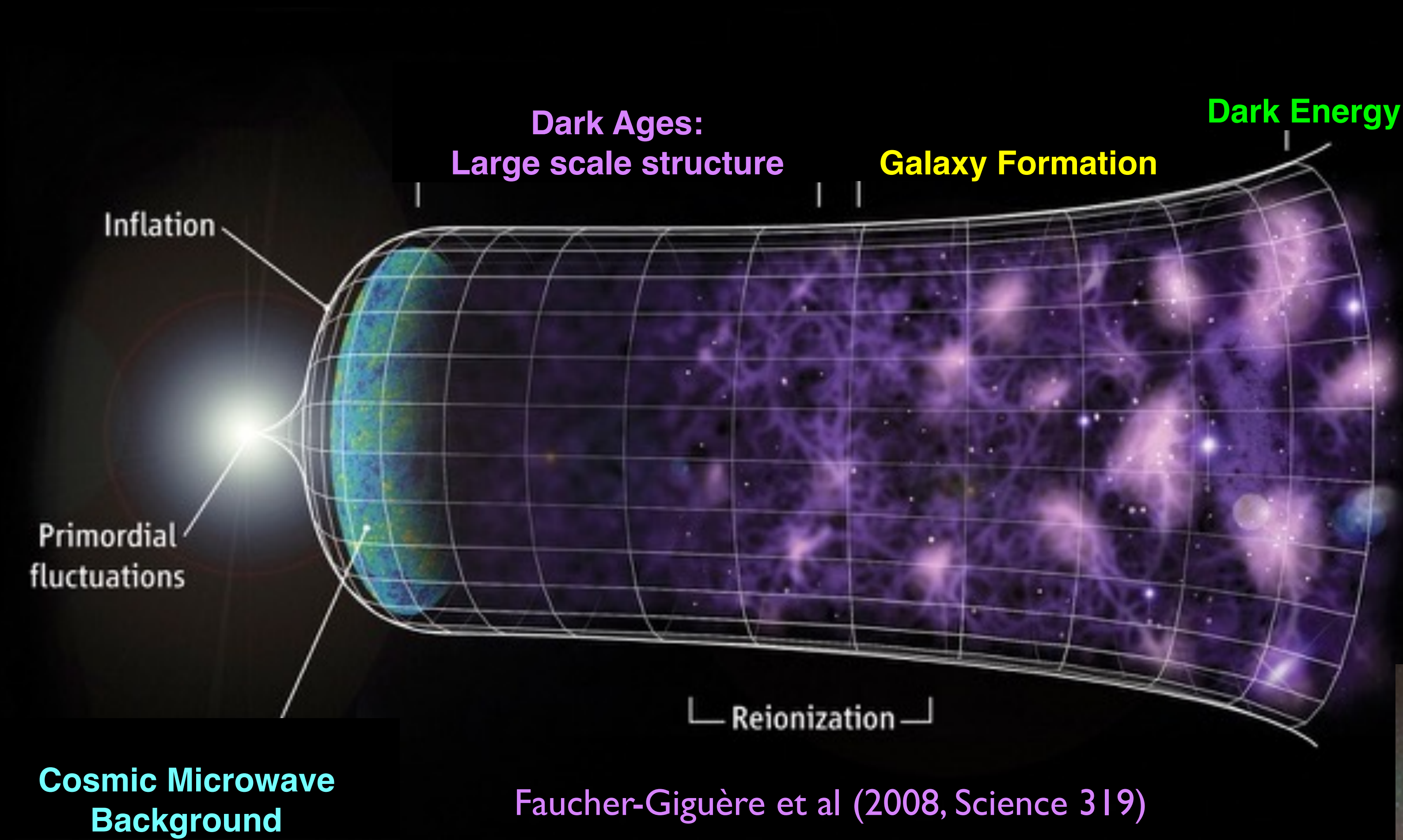
Cosmological Evolution

-13.7 billion years: nearly uniform initial state

today: cosmic web structure on all scales

Cosmological Evolution

-13.7 billion years: nearly uniform initial state



Faucher-Giguère et al (2008, Science 319)

PLANCK

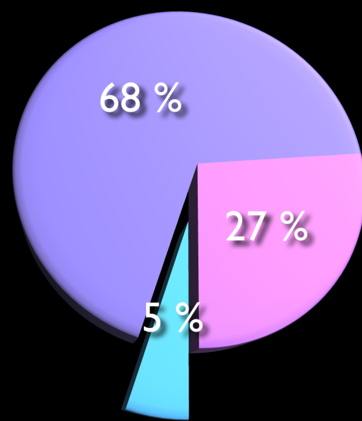
DES

2MASS

today: cosmic web structure on all scales

Cosmological Standard Model

ΛCDM



Dark Energy

cosmological constant

or dynamical?

late-time

accelerated expansion

baryons

standard model
of particle physics

Cold Dark Matter

collisionless

only gravitational
interactions

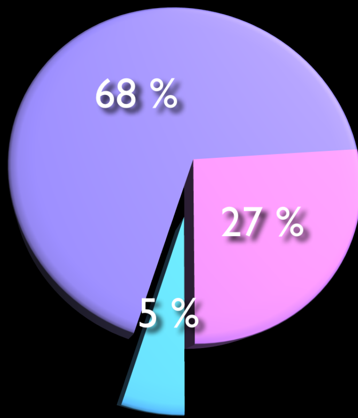
cold

initially small velocities

Large Scale Structure

Cosmological Standard Model

ΛCDM



Dark Energy

cosmological constant

or dynamical?

late-time

accelerated expansion

baryons

standard model
of particle physics

Cold Dark Matter

collisionless

only gravitational
interactions

cold

initially small velocities

Large Scale Structure

constrain ⟷ observe ⟷ understand

Outline

- From the CMB to the Cosmic Web
- Dark matter dynamics
- Large deviation principle
- Probability distribution of density in cells
- Interesting first applications: clustering & constraining cosmology

phase space distribution function f(t,x,p)

$$f_N \to f$$

- **N-body:** non-relativistic, purely gravitational
- **continuous**: ensemble average, no collisions

Vlasov - Poisson equation

$$\partial_\tau f(\boldsymbol{x},\boldsymbol{p},\tau) = -\frac{\boldsymbol{p}}{am}\boldsymbol{\nabla}_x f + am\boldsymbol{\nabla}_x V \boldsymbol{\nabla}_p f$$

gravitational potential

$$\Delta V(\boldsymbol{x},\tau) = \frac{4\pi G}{a}\rho_0 \left[\rho(\boldsymbol{x},\tau) - 1\right]$$

phase space distribution function f(t,x,p)

f_N

f

- **N-body:** non-relativistic, purely gravitational
- **continuous**: ensemble average, no collisions

Vlasov - Poisson equation

$$\partial_\tau f(\boldsymbol{x},\boldsymbol{p},\tau) = -\frac{\boldsymbol{p}}{am}\boldsymbol{\nabla}_x f + am\boldsymbol{\nabla}_x V \boldsymbol{\nabla}_p f$$

gravitational potential

$$\Delta V(\boldsymbol{x},\tau) = \frac{4\pi G}{a}\rho_0\left[\rho(\boldsymbol{x},\tau) - 1\right]$$

partial

nonlinear

integro

density

$$\rho = \int d^3p\, f$$

velocity

$$\rho\boldsymbol{v} = \int d^3p\, f\boldsymbol{p}$$

Solving is hard!

phase space distribution function f(t,x,p)

- **N-body:** non-relativistic, purely gravitational
- **continuous**: ensemble average, no collisions

$f_N \to f$

Vlasov - Poisson equation

gravitational potential

$$\partial_\tau f(\boldsymbol{x},\boldsymbol{p},\tau) = -\frac{\boldsymbol{p}}{am}\boldsymbol{\nabla}_x f + am\boldsymbol{\nabla}_x V \boldsymbol{\nabla}_p f$$

$$\Delta V(\boldsymbol{x},\tau) = \frac{4\pi G}{a}\rho_0\left[\rho(\boldsymbol{x},\tau)-1\right]$$

partial

nonlinear

integro

density

$$\rho = \int d^3p\, f$$

velocity

$$\rho\boldsymbol{v} = \int d^3p\, f\boldsymbol{p}$$

Solving is hard!

- **extensive computing:**
  - solve gravitational evolution of N-particles numerically
- **extensive thinking:**
  - choose a special ansatz for f: density & velocity
  - solve analytically / perturbatively, approximations, …

Spherical collapse

- matter conservation during time evolution
- Gauss theorem: shell evolution by enclosed mass

$$\rho_R(t) R^3(t) = \bar{\rho}(t_0) r^3 \qquad \ddot{R} = -\frac{G M_R}{R^2}$$

- spherical overdensity like „separate universe"

- EdS universe: parametric solution

$$\rho(\theta), \tau(\theta)$$

can be used to predict

- when halo's form
- mass function of halos

just from initial density field!

Spherical collapse

- matter conservation during time evolution
- Gauss theorem: shell evolution by enclosed mass

$$\rho_R(t)R^3(t) = \bar{\rho}(t_0)r^3 \qquad \ddot{R} = -\frac{GM_R}{R^2}$$

- spherical overdensity like „separate universe"

can be used to predict
- when halo's form
- mass function of halos

just from initial density field!

- EdS universe: parametric solution

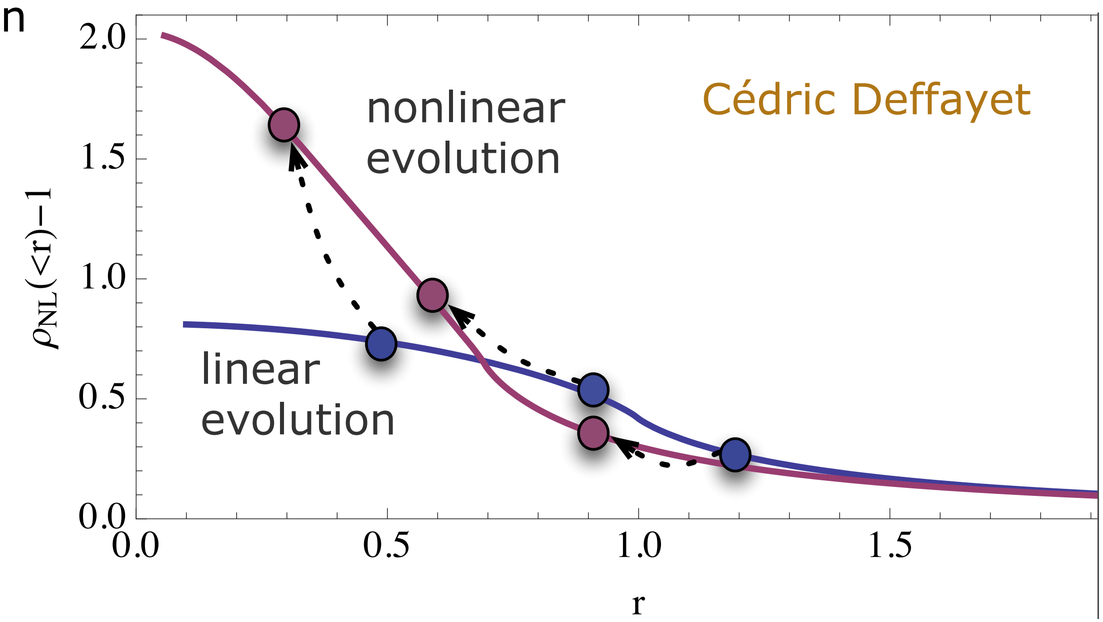
$$\rho(\theta), \tau(\theta)$$

simple approximation

$$\tau \to \rho = (1 + \tau/\nu)^{-\nu}$$

$$r \to R = r\rho^{-1/3}$$

Bernardeau `92



Outline

- From the CMB to the Cosmic Web
- Dark matter dynamics
- Large deviation principle
- Probability distribution of density in cells
- Interesting first applications: clustering & constraining cosmology

Large Deviation Principle

What is the most likely way for an unlikely event to happen?

- Central Limit Theorem
  - convergence of the PDF to Gaussian

- Large Deviation Principle: beyond
  - What about rare event tail?
  - exponential decay of the PDF

review paper
Touchette '09

Large Deviation Principle

What is the most likely way for an unlikely event to happen?

- Central Limit Theorem
  - convergence of the PDF to Gaussian

- Large Deviation Principle: beyond
  - What about rare event tail?
  - exponential decay of the PDF

review paper Touchette '09

Let's toss coins!

- n tosses: average number of heads
- heads (1) or tails (0) with equal probability
- compute via Bernoulli sequence

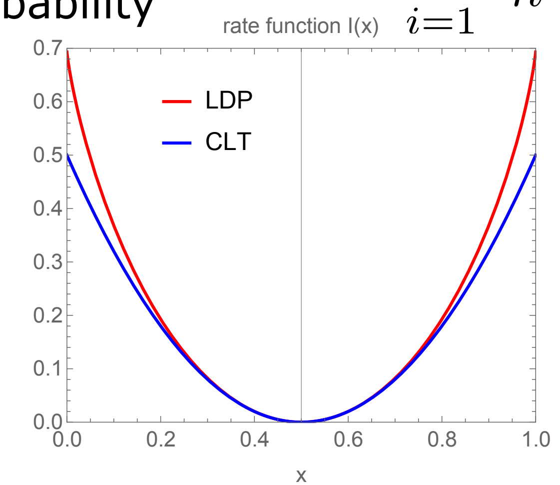
$$X_n = \sum_{i=1}^{n} \frac{w_i}{n}$$

$$P_{X_n}(x) \approx \exp(-nI(x))$$

- rate function of exponential decay

$$I(x) = x\ln[2x] + (1-x)\ln[2(1-x)]$$

$$I(x) \simeq 2(x-1/2)^2$$



The guiding principles for large deviations

Gärtner-Ellis Theorem

Cramer `38, Varadhan `84, Gärtner `77 & Ellis `84

- rate function is the Legendre transformation of the scaled cumulant generating function
- heuristic: **steepest decent method**

$$I(\rho) = \sup_{\lambda} \left(\lambda\rho - \varphi(\lambda)\right)$$

$$\exp[n\varphi(\lambda)] = \int P_n(\rho)\exp(n\lambda\rho)d\rho \approx \int \exp(-n[I(\rho) - \lambda\rho])d\rho$$

validity: large n = small variance σ^2

The guiding principles for large deviations

Gärtner-Ellis Theorem

Cramer `38, Varadhan `84, Gärtner `77 & Ellis `84

- rate function is the Legendre transformation of the scaled cumulant generating function
- heuristic: **steepest decent method**

$$I(\rho) = \sup_{\lambda} \left(\lambda\rho - \varphi(\lambda)\right)$$

$$\exp[n\varphi(\lambda)] = \int P_n(\rho)\exp(n\lambda\rho)d\rho \approx \int \exp(-n[I(\rho) - \lambda\rho])d\rho$$

Contraction principle

validity: large n = small variance σ^2

- based again on steepest decent
- obtain rate function after any mapping
- We will apply this in two ways
  - mapping between initial & final densities
  - mapping to optimise analytical approximation

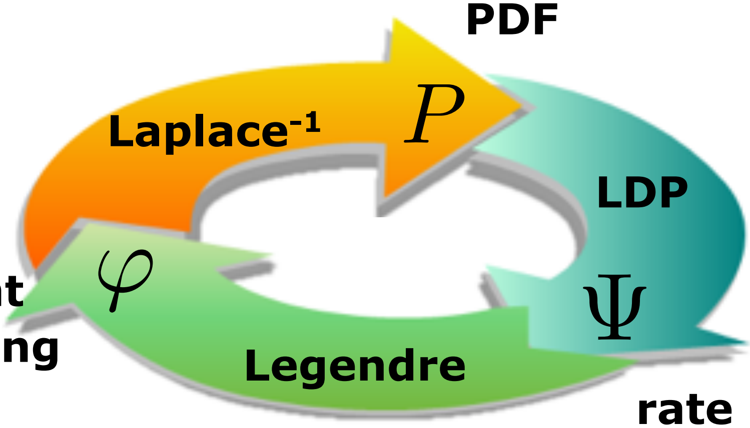
$$I(\rho) = \inf_{\{\tau\}:\rho(\{\tau\})} I_\tau(\tau)$$

an unlikely fluctuation is brought about by the least unlikely of all unlikely paths

Our desire: PDF of final densities

Going around in circles

PDF



LDP

cumulant generating function

rate function

Ascending by contracting

$$\Psi(\tau) \rightarrow \Psi(\rho)$$

- mapping: initial -> final rate
- 1:1 relation requires symmetry
- spherical configurations

Extrapolating to finite variance

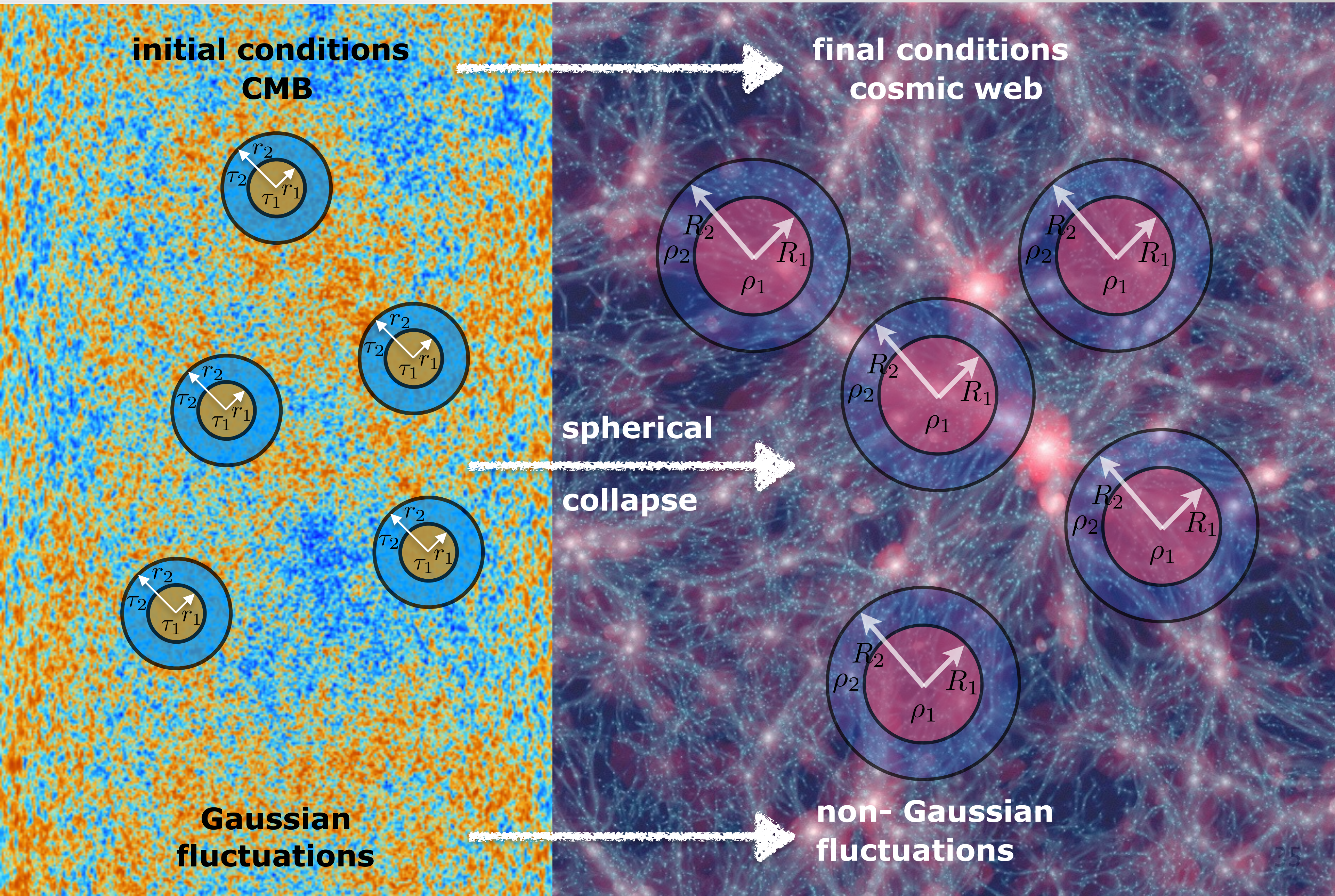
- theorems hold for limit $\sigma \rightarrow 0$
- conjectured validity for finite σ
- requires weak dependence on σ for cumulant generating function

Bernardeau & Reimberg `15

Our knowledge: PDF of initial densities

LDP @ Large Scale Structure

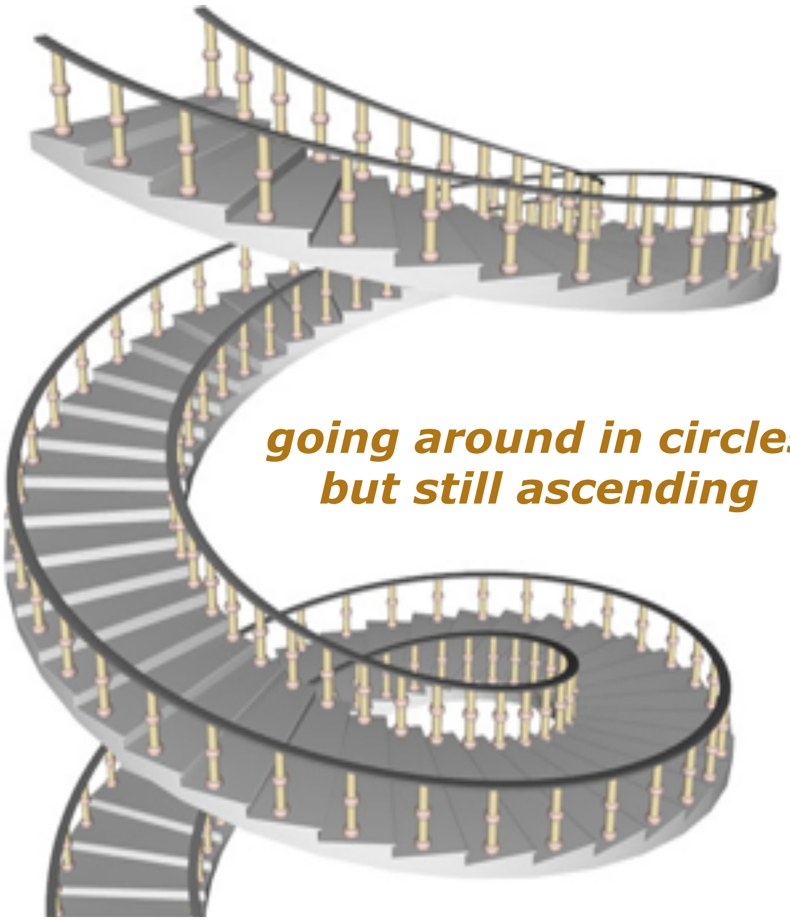
Large Deviation Statistics of the cosmic density field
Universiteit Utrecht
Cora Uhlemann



Outline

- From the CMB to the Cosmic Web
- Dark matter dynamics
- Large deviation principle
- Probability distribution of density in cells
- Interesting first applications: clustering & constraining cosmology

Our desire: PDF of final densities



going around in circles
but still ascending

Knowledge: PDF of initial densities $\Psi(\tau) = \dfrac{\tau^2}{2\sigma_\tau^2}$

Our desire: PDF of final densities

$$\Psi(\rho)$$

1. Contraction principle

- want 1:1 mapping initial -> final
- consider density in sphere
- use symmetry: spherical collapse

$$\rho(R) = \rho(\tau(r))$$

Knowledge: PDF of initial densities

$$\Psi(\tau) = \frac{\tau^2}{2\sigma_\tau^2}$$

Our desire: PDF of final densities

$$\varphi(\lambda)$$

2. Large deviation principle
- do Legendre transform
- get cumulant generating function

$$\varphi(\lambda) = \sup_{\rho}[\lambda\rho - \Psi(\rho)]$$

$$\Psi(\rho)$$

going around in circles but still ascending

1. Contraction principle
- want 1:1 mapping initial -> final
- consider density in sphere
- use symmetry: spherical collapse

$$\rho(R) = \rho(\tau(r))$$

Knowledge: PDF of initial densities

$$\Psi(\tau) = \frac{\tau^2}{2\sigma_\tau^2}$$

Our desire: PDF of final densities

3. Inverse Laplace transform

$$\exp \varphi(\lambda) = \int P(\rho) \exp(\lambda\rho)\, d\rho$$

$$P(\rho)$$

2. Large deviation principle

- do Legendre transform
- get cumulant generating function

$$\varphi(\lambda) = \sup_\rho [\lambda\rho - \Psi(\rho)]$$

$$\varphi(\lambda)$$

going around in circles
but still ascending

1. Contraction principle

- want 1:1 mapping initial -> final
- consider density in sphere
- use symmetry: spherical collapse

$$\rho(R) = \rho(\tau(r))$$

$$\Psi(\rho)$$

Knowledge: PDF of initial densities

$$\Psi(\tau) = \frac{\tau^2}{2\sigma_\tau^2}$$

Our desire: PDF of final densities

$P(\rho)$

Inverse Laplace transform hard
- path integration in complex plane
- saddle-point approximation not good

$$P(\rho) \simeq \sqrt{\frac{\Psi''(\rho)}{2\pi}} \exp[-\Psi(\rho)]$$

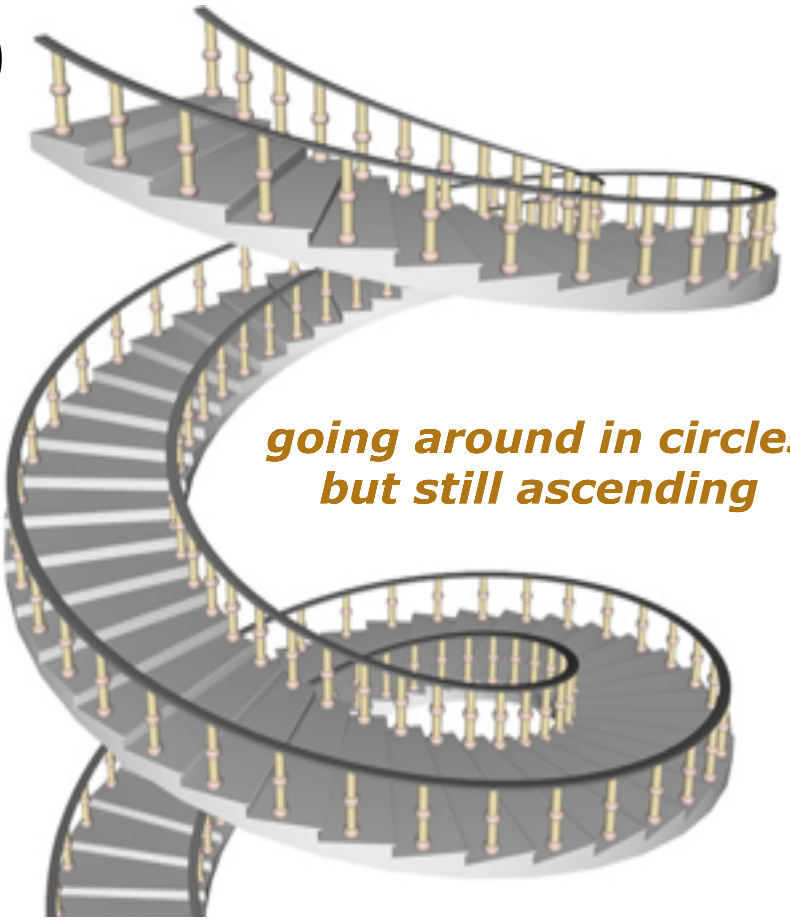
$\varphi(\lambda)$

Legendre transform straightforward
- rate function not everywhere convex
- not valid everywhere $\lambda = \Psi'$

$\Psi(\rho)$

Spherical collapse piece of cake

$$\rho(\tau) = (1 - \tau/\nu)^{-\nu}\,,\ r = R\rho^{1/3}$$

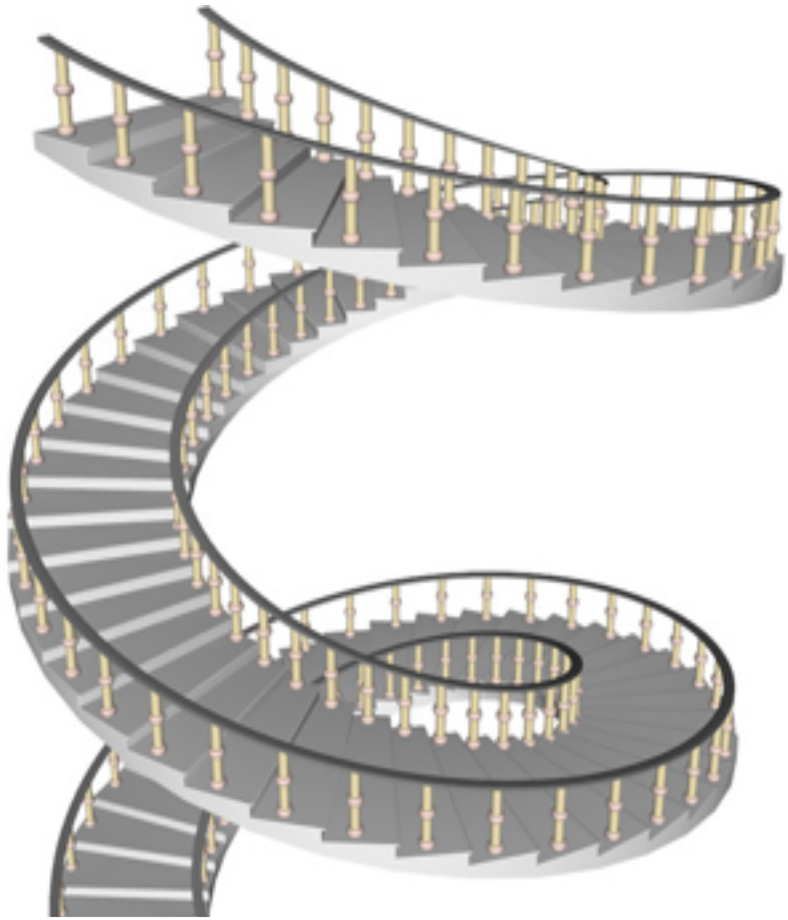


going around in circles but still ascending

Knowledge: PDF of initial densities

$$\Psi(\tau) = \frac{\tau^2}{2\sigma_\tau^2}$$

Our desire: PDF of final densities

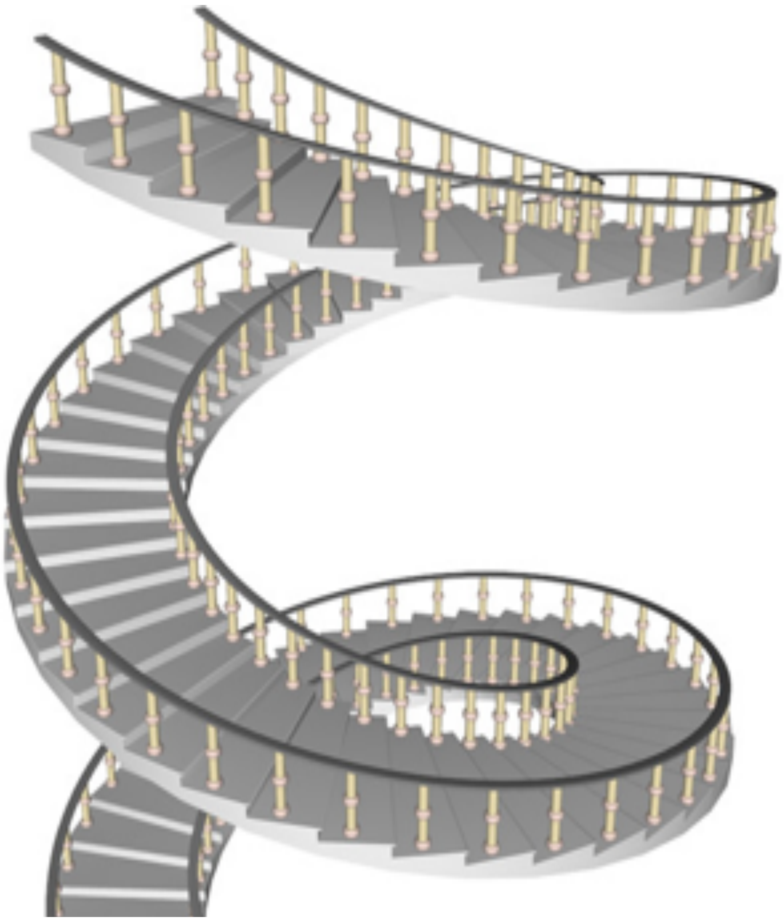


Legendre transform straightforward
- rate function not everywhere convex
- not valid everywhere $\lambda = \Psi'$

$\Psi(\rho)$

Knowledge: PDF of initial densities $\Psi(\tau) = \dfrac{\tau^2}{2\sigma_\tau^2}$

Our desire: PDF of final densities



Legendre transform straightforward

- rate function everywhere convex
- valid everywhere $\lambda = \Psi'$

$$\Psi(\mu)$$

Recall: Contraction principle

- can take any mapping $\mu = \log \rho$
- take logarithmic transform
- do same recipe, transform back in the end

Knowledge: PDF of initial densities $\Psi(\tau) = \dfrac{\tau^2}{2\sigma_\tau^2}$

LDP @ Large Scale Structure

Our desire: PDF of final densities

Inverse Laplace transform simple

- path integration in complex plane
- saddle-point approximation good

$$P(\rho)$$

$$P(\rho) \simeq \sqrt{\frac{\Psi''(\rho) + \Psi'(\rho)/\rho}{2\pi}} \exp[-\Psi(\rho)]$$

Legendre transform straightforward

- rate function everywhere convex
- valid everywhere $\lambda = \Psi'$

$$\mu = \log \rho \qquad \Psi(\mu)$$

Spherical collapse piece of cake

$$\rho(\tau) = (1 - \tau/\nu)^{-\nu}, \; r = R\rho^{1/3}$$

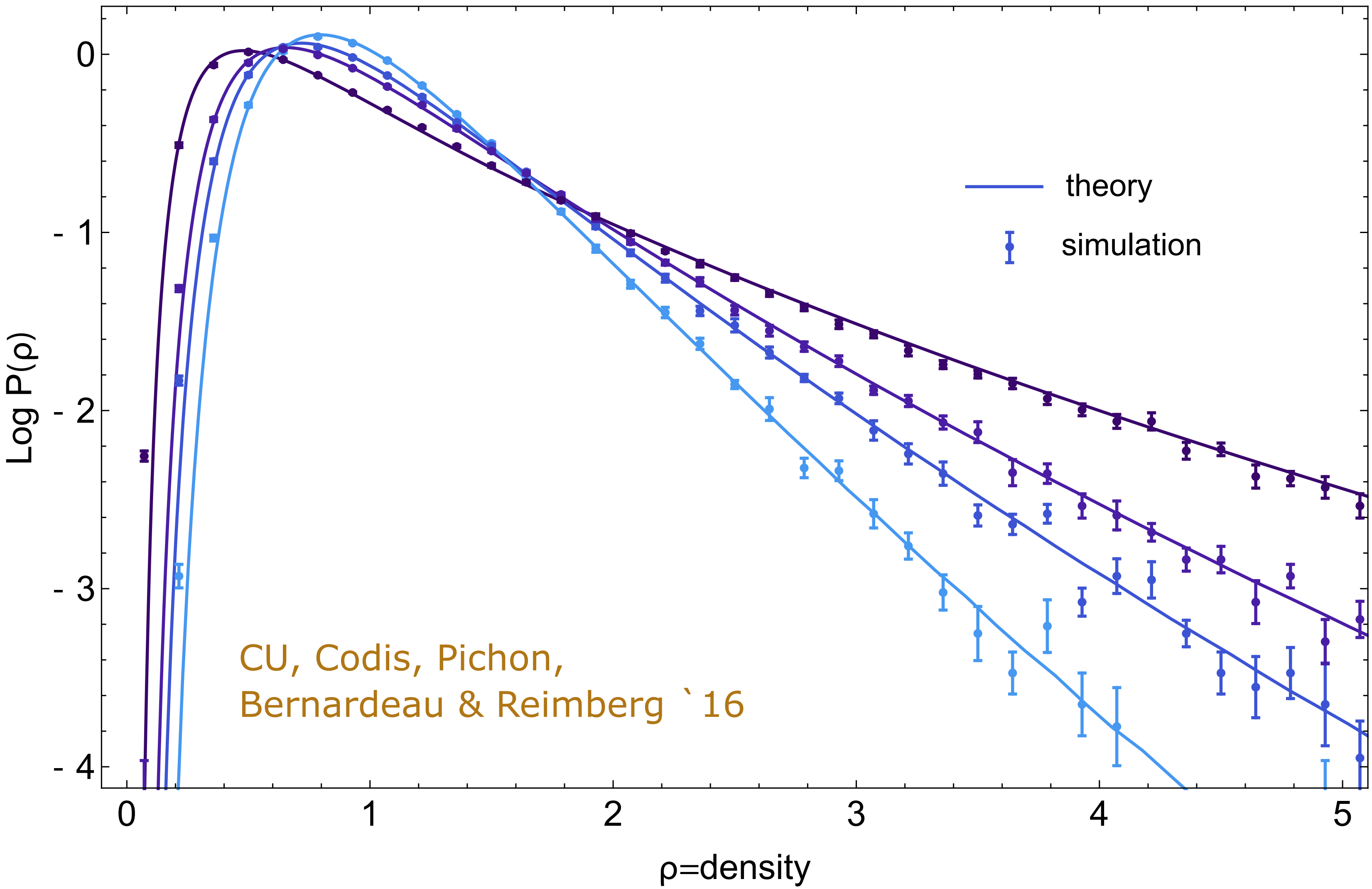
The log is a game-changer!

Uhlemann, Codis, Pichon, Bernardeau & Reimberg `16

Knowledge: PDF of initial densities

$$\Psi(\tau) = \frac{\tau^2}{2\sigma_\tau^2}$$

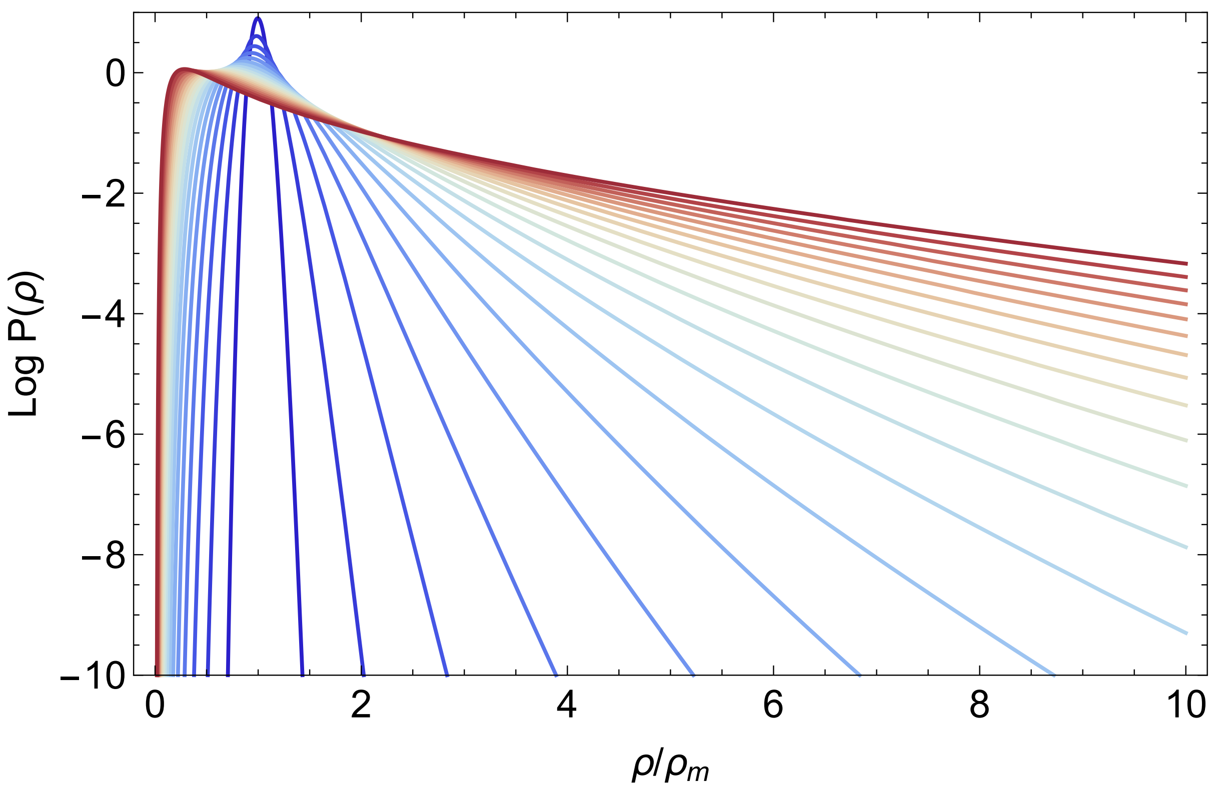
Saddle point approximation for log density: 1 cell



Saddle point approximation for log density: 1 cell

- incredibly simple, for example for power law initial spectrum

$$\mathcal{P}_R(\rho|\sigma) = \exp\left[-\frac{\nu^2\left(\rho^{\frac{1}{\nu}}-1\right)^2\rho^{-\frac{2}{\nu}+\frac{n}{3}+1}}{2\sigma^2}\right]\sqrt{\frac{\nu^2(n+3)^2\left(\rho^{\frac{1}{\nu}}-1\right)^2+12\nu(n+3)\left(\rho^{\frac{1}{\nu}}-1\right)-18\left(\rho^{\frac{1}{\nu}}-2\right)}{36\pi\sigma^2\rho^{\frac{2}{\nu}+1-\frac{n}{3}}}}.$$



- generalisation for arbitrary initial power spectra

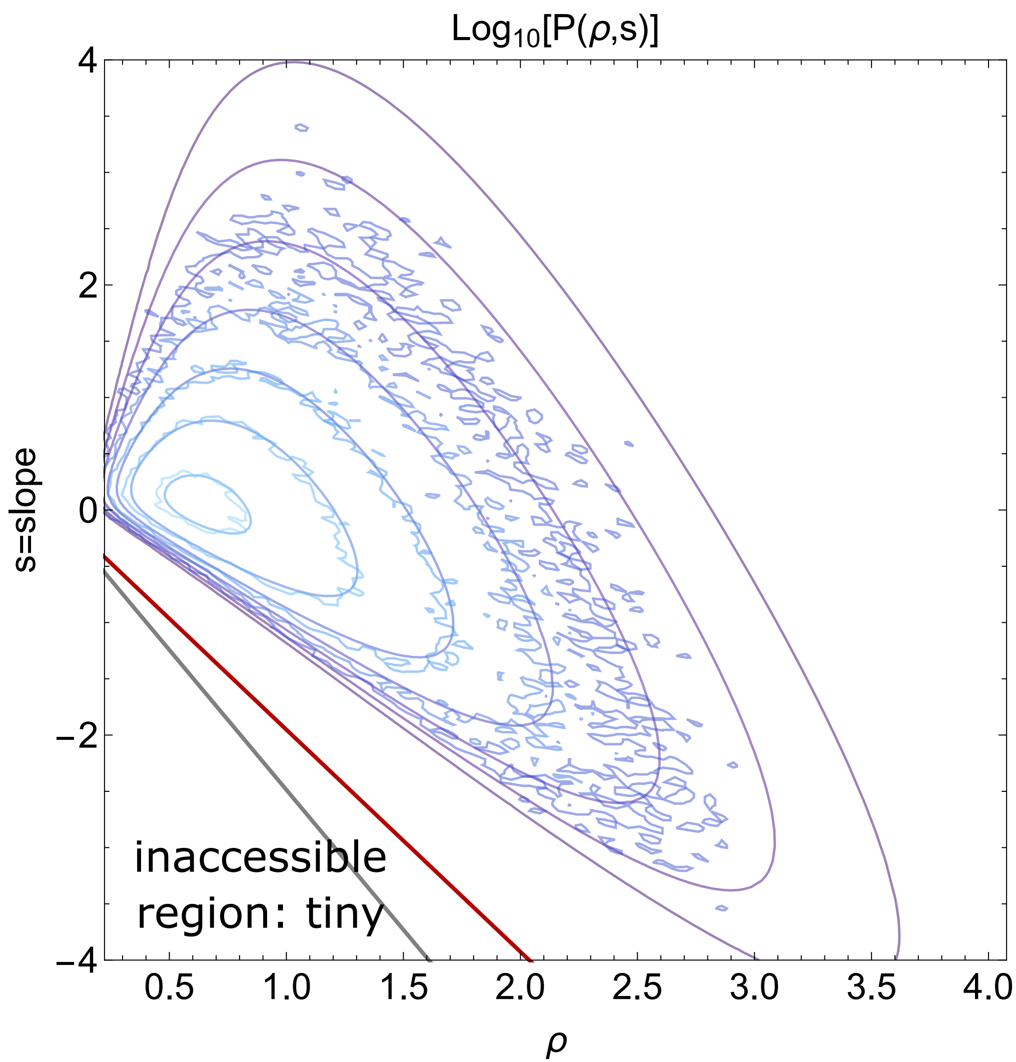
LSSFast

Mathematica code

http://cita.utoronto.ca/~codis/LSSFast.html

try your favourite initial power spectrum

Saddle point approximation for log density: 2 cell



- generalize formalism to 2-cell
- allows for conditionals
  - over-/underdense regions
- slope carries information about density profile

$$s = \frac{\rho_2 - \rho_1}{r - 1}$$

similar mapping

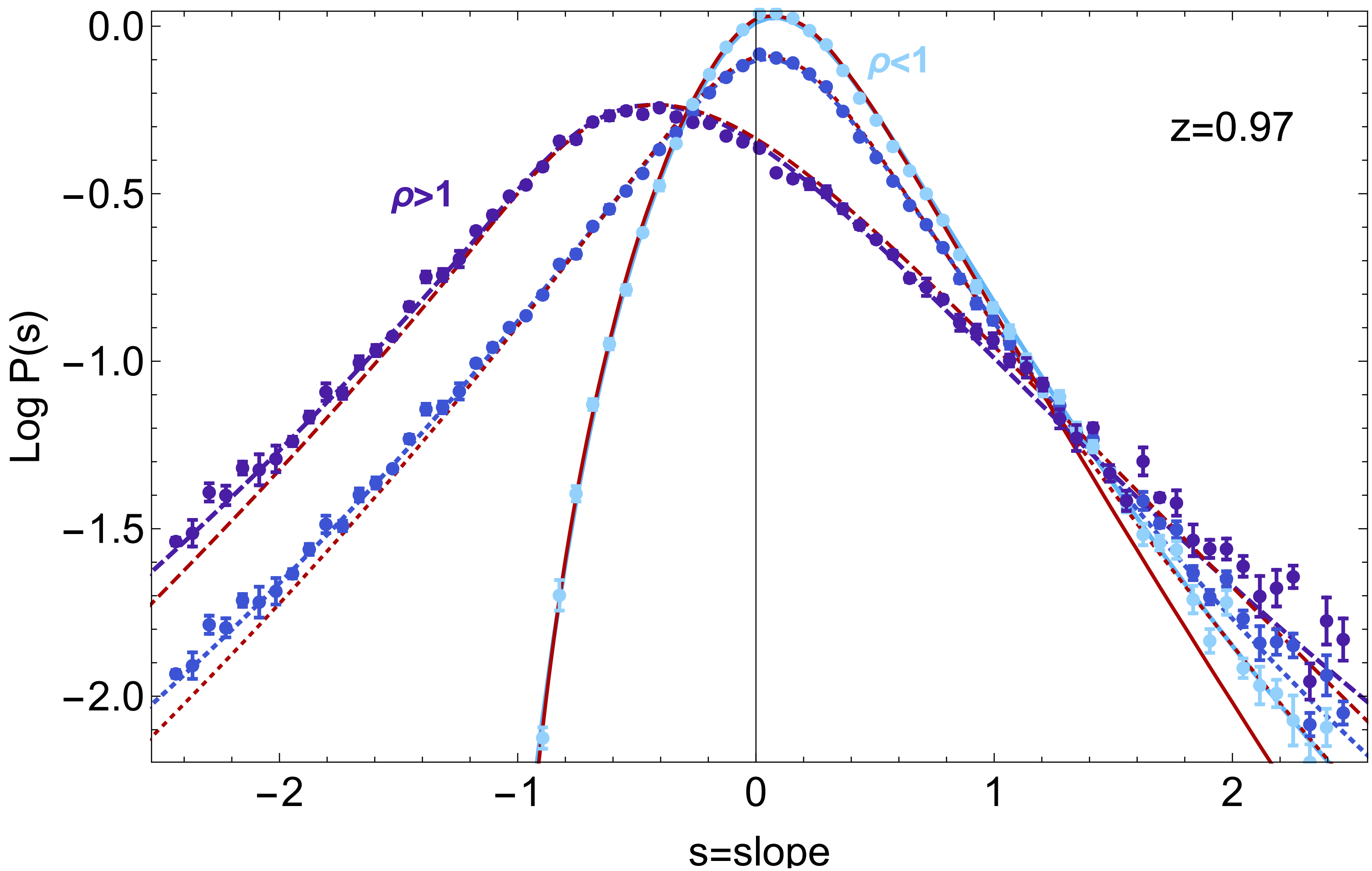
- sum and difference of mass

$$\mu_1 = \log(r^3\rho_2 + \rho_1)$$

$$\mu_2 = \log(r^3\rho_2 - \rho_1)$$

$$r = R_2/R_1$$

Saddle point approximation for log density: 2 cell marginal



- differences between overdensities (peaks) & underdensities (voids)

Outline

- From the CMB to the Cosmic Web
- Dark matter dynamics
- Large deviation principle
- Probability distribution of density in cells
- Interesting (future) applications: clustering & constraining cosmology

Testing Cosmology with PDFs

Dark energy dependence

- density variance: linear growth $\sigma(R,z) \propto D(z|w_0,w_a)$
- linear growth: depends on Hubble function

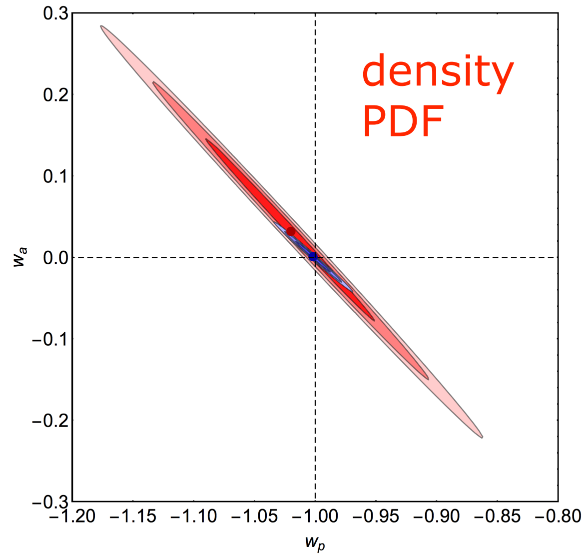
$$H^2(z) = H_0^2\left[\Omega_m(1+z)^3 + \Omega_\Lambda \exp\left(\int_0^z \frac{1+w(z')}{1+z'}dz'\right)\right]$$

- Hubble function: dark energy equation of state

$$w(z) = w_0 + \frac{w_a z}{1+z}$$

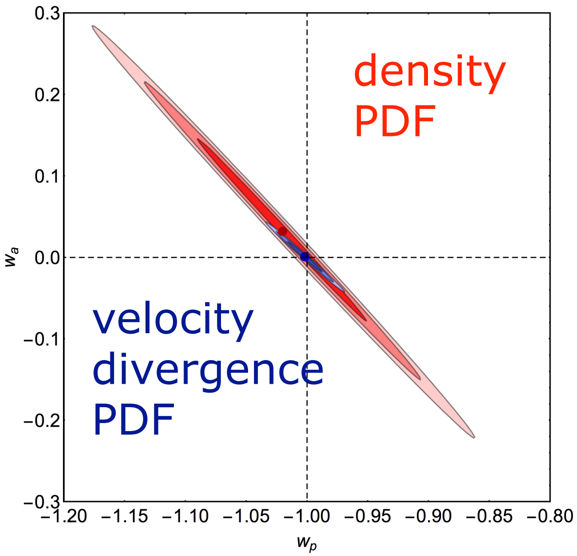
Codis, Pichon, Bernardeau, CU & Prunet `16

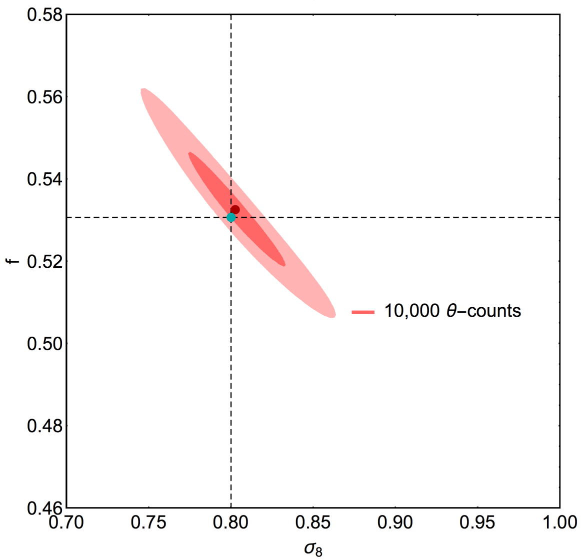
Maximum likelihood estimator



Testing Cosmology with PDFs

Maximum likelihood estimator





Dark energy dependence

- density variance: linear growth $\sigma(R,z) \propto D(z|w_0,w_a)$
- linear growth: depends on Hubble function

$$H^2(z) = H_0^2\left[\Omega_m(1+z)^3 + \Omega_\Lambda \exp\left(\int_0^z \frac{1+\boxed{w(z')}}{1+z'}dz'\right)\right]$$

- Hubble function: dark energy equation of state

$$w(z) = \boxed{w_0} + \frac{\boxed{w_a}z}{1+z}$$

Codis, Pichon, Bernardeau, CU & Prunet `16

Potential improvement: velocity divergence

- more sensitive to growth
- less nonlinear, robust to bias
- also spherical collapse

$$\theta(\rho) = f(\Omega)\nu(1-\rho^{1/\nu})$$

work in progress with Codis, Bernardeau, Pichon & Hahn

Testing Cosmology with PDFs: Cell bias

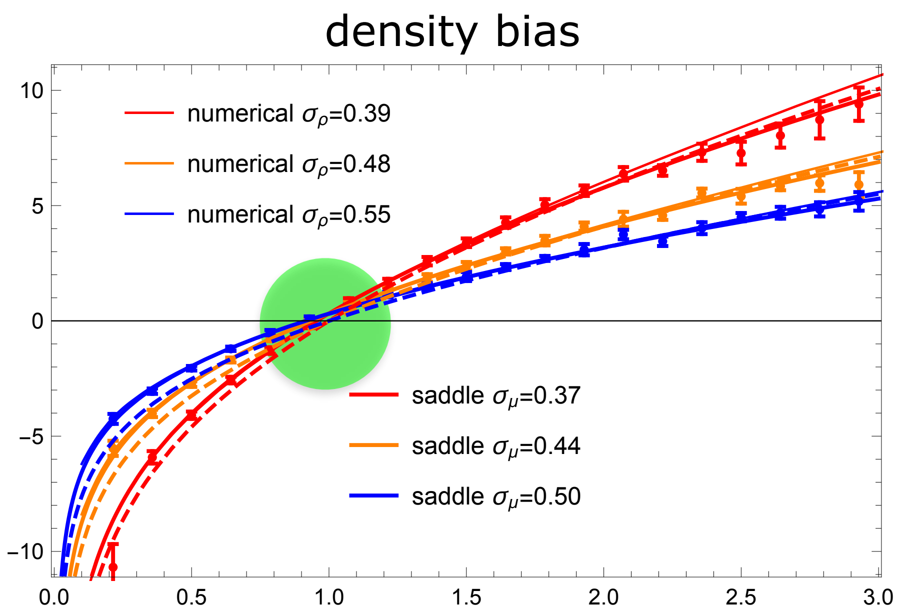
(In)dependence of spheres for finite separation

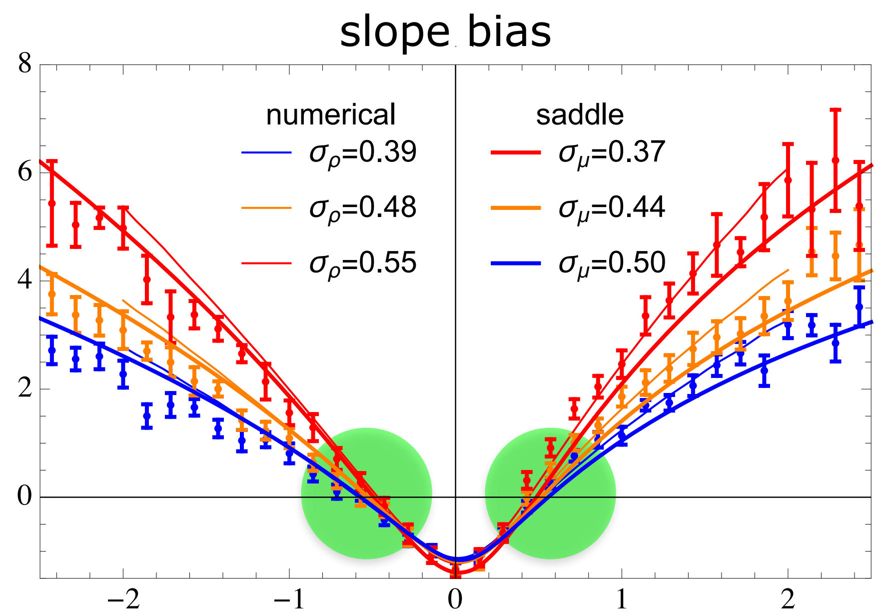
- correlations of density in spheres
- large separation regime predictable
- as for PDF: numerical integration or saddle

$$\frac{P(\rho(x),\rho'(x+r))}{P(\rho)P(\rho')} = 1 + \xi(r) b(\rho) b(\rho')$$

DM correlation

cell bias





numerical integration: Codis, Bernardeau & Pichon `16
saddle: work in progress with Codis, Bernardeau & Pichon

Correlation functions from PDFs: BAO signature

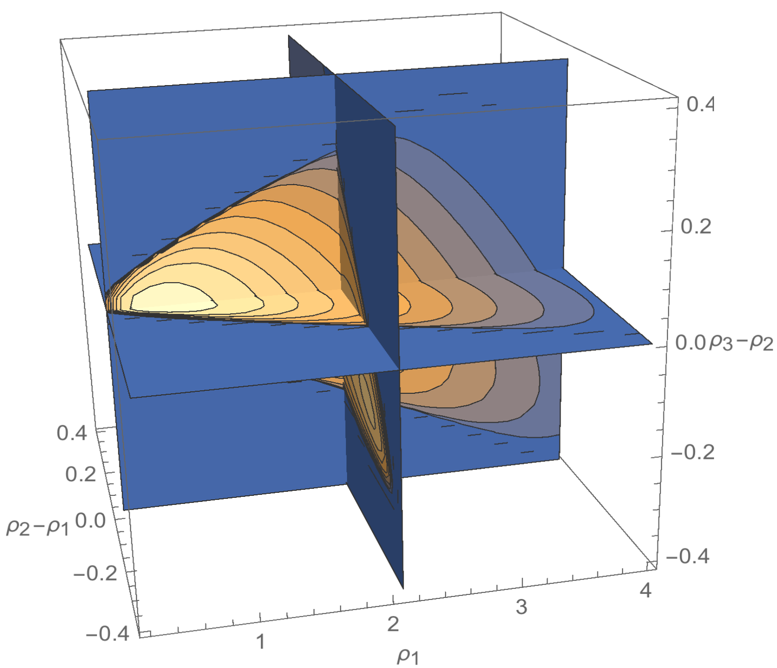
work in progress
with Codis, Bernardeau, Pichon

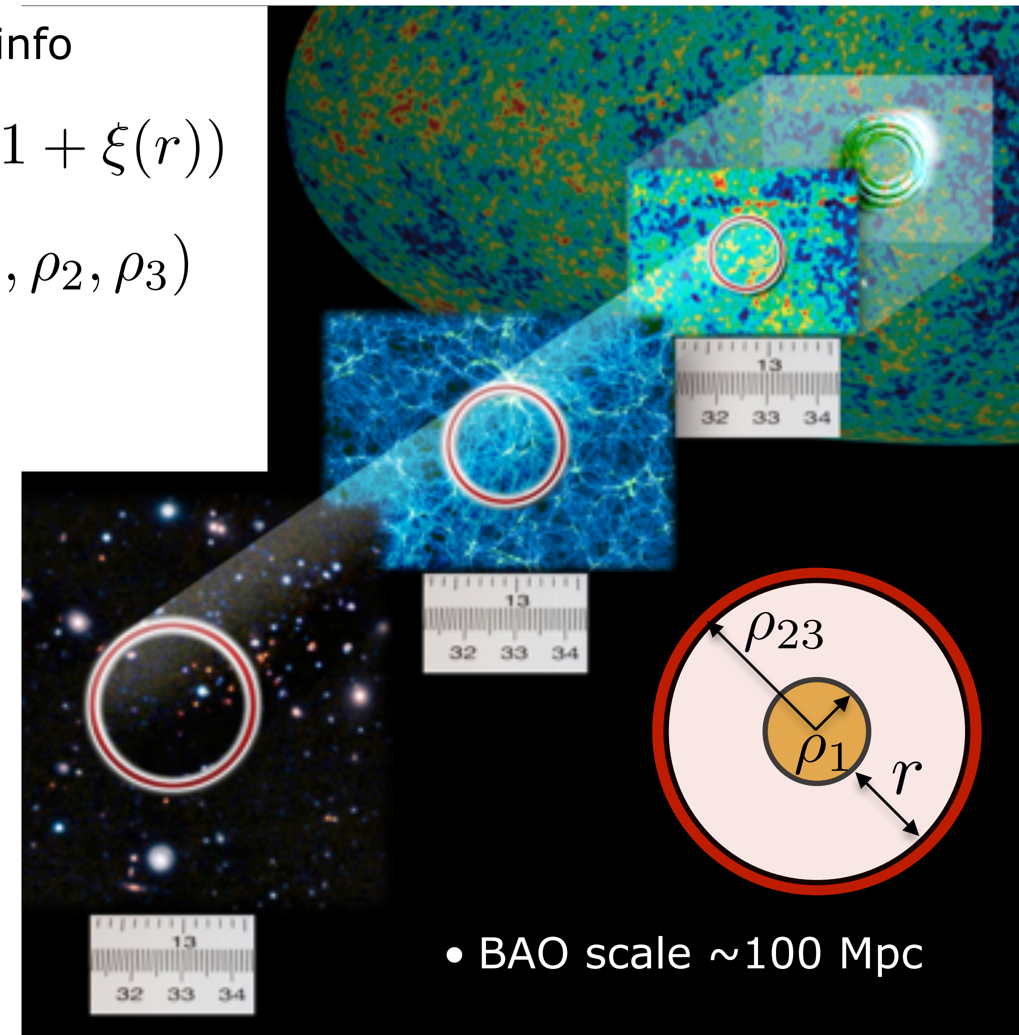
Perspective: Clustering from 3 cell PDF

- from 3-cell PDF: correlation function info

$$P(\rho_1, \rho_{23}|r) = P(\rho_1)P(\rho_{23}) \times (1 + \xi(r))$$

- demands computing 3-cell PDF $P(\rho_1, \rho_2, \rho_3)$
  - logarithmic transform trick again





- BAO scale ~100 Mpc

Take home messages

From the CMB to the Cosmic Web

- CMB provides Gaussian initial conditions for structure formation
- gravitational evolution is **nonlinear** and leads to **non-Gaussianity**

Dark matter dynamics

- in general: perturbation theory based on fluid or simulations
- for highly symmetrical situation: **spherical collapse**

Large deviation principle

- rate function to identify **`least unlikely`** path in the evolution
- **contraction principle**: rate <-> cumulant generating fct., mappings

Probability distribution of density in spheres

- densities in spheres allows to use spherical collapse
- **log** is clever variable, fully analytical **saddle point approximation**

Applications: Clustering, Constraining, …

- joint PDFs of densities as alternative to **correlation functions**
- **constraints** on cosmology (dark energy) and gravity