Time-delay matrix for chaotic cavities: new results and a conjecture

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Time-delay matrix

$$\boxed{Q = -i\hbar S^\dagger(E)\frac{\mathrm{d}S(E)}{\mathrm{d}E}}$$

$S(E)$: $N \times N$ scattering matrix; N: number of channels;
Conservation of probability: $S(E)$ unitary; Note that $Q = Q^\dagger$.

Invented by Eisenbud, Wigner and Smith in the 50ies in order to characterise temporal aspects of a quantum scattering process.

Proper delay times

Eigenvalues of the Wigner-Smith matrix Q

$$\tau_1, \tau_2, \ldots, \tau_N$$

Wigner time delay

Time ‘spent’ by the particle in the scattering region

$$\mathrm{tr}Q = \tau_1 + \tau_2 + \cdots + \tau_N$$

Chaotic cavities

Time-delay matrix in chaotic regime

Within the random-matrix ansatz, the proper delay-times have joint law [Brouwer, Prahm and Beenakker (1997)]:

$$P(\tau_1, \cdots, \tau_N) \propto \prod_k f(\tau_k) \cdot \underbrace{\prod_{i<j} |\tau_i - \tau_j|^\beta}_{\text{repulsion}}$$

$f(\tau) = \theta(\tau)\tau^{\beta(1-\frac{3}{2}N)-2} \exp(-\beta\tau_H/2\tau)$; β: Dyson index; τ_H: Heisenberg time.

Remark: The inverse $L = \frac{1}{N} Q^{-1}$ belongs to the Laguerre ensemble. (a 'classical distribution' in RMT).

Let $\lambda_i = \frac{1}{N\tau_i} \geq 0$. The joint law of the inverse delay times is ($\tau_H = 1$)

$$\boxed{P(\lambda_1, \cdots, \lambda_N) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_k \lambda_k^{\beta N/2} e^{-\beta N \lambda_k/2}}$$

Review of previous results

Despite the joint distribution for τ_i was explicitly known, the statistical properties of their sum (Wigner time delay) remained unknown for a while. The same for higher moments.

$$\boxed{\mathcal{T}_\kappa = N^{\kappa-1}\mathrm{tr}Q^\kappa}$$

- Distribution of $\mathcal{T}_1$ for $N = 1$, Gopar, Mello and Büttiker (1996);
- Distribution of $\mathcal{T}_1$ for $N = 2$, Savin, Fyodorov and Sommers (2001);
- Averages $\langle \mathcal{T}_k \rangle$ for $N \gg 1$, Berkolaiko and Kuipers (2010);
- First large-N corrections to $\langle \mathcal{T}_k \rangle$ for $\beta = 1$ and 2, Berkolaiko and Kuipers (2010), Mezzadri and Simm (2012), Kuipers, Savin and Sieber (2014);
- Recursion for cumulants of $\mathcal{T}_1$ for all N, Mezzadri and Simm (2013);
- Averages and covariances of $\mathcal{T}_1$, $\mathcal{T}_2$ for $N \gg 1$, Grabsch and Texier (2014);
- Representation for cumulants of $\mathcal{T}_k$ for $\beta = 2$ for all N, Novaes (2015);

How? RMT & semiclassical methods. For $\beta = 2$ orthogonal polynomials, representation theory of the symmetric group and connection with Painlevé III.

Large number of scattering channels N

$$\boxed{\mathcal{T}_\kappa = N^{\kappa-1}\mathrm{tr}Q^\kappa}$$

Cumulants (connected averages)

$$\boxed{\langle \mathcal{T}_{\kappa_1}\cdots\mathcal{T}_{\kappa_\nu}\rangle_c \sim \left(\frac{1}{\beta N^2}\right)^{\nu-1}\alpha[\kappa_1,\ldots,\kappa_\nu]}$$

$\nu = 1$: averages $\langle \mathcal{T}_\kappa\rangle_c \sim \alpha[\kappa]$;

$\nu = 2$: covariances $\langle \mathcal{T}_{\kappa_2}\mathcal{T}_{\kappa_2}\rangle_c \sim \dfrac{1}{\beta N^2}\alpha[\kappa_1,\kappa_2]$

Main result: Generating functions

$$\boxed{F_\nu(z_1,\ldots,z_\nu) = \frac{1}{\beta^{\nu-1}}\sum_{\kappa,\ldots,\kappa_\nu\geq 0}\alpha[\kappa_1,\ldots,\kappa_\nu]z_1^{\kappa_1}\cdots z_\nu^{\kappa_\nu}}$$

[FDC, PRE **91***, 060102(R) (2015), FDC, F. Mezzadri, N. Simm & P. Vivo, JPA* **49***, 18LT01 (2016)]*

Large number of scattering channels N

Theorem 1. The generating function of cumulants $\langle \mathcal{T}_{\kappa_1} \cdots \mathcal{T}_{\kappa_\nu} \rangle_c$ at leading order in N is

$$\boxed{F_\nu(z_1, \ldots, z_\nu) = (-1)^\nu z_1 \cdots z_\nu \, g_\nu(z_1, \ldots, z_\nu) + \delta_{1,\nu}(2 - z)]}$$

where the large-N ν-point connected resolvents

$$g_\nu(z_1, \ldots, z_\nu) = \lim_{N\to\infty} N^{2(\nu-1)} \left\langle \frac{1}{N}\sum_{i_1=1}^{N} \frac{1}{z_1 - \lambda_{i_1}} \cdots \frac{1}{N}\sum_{i_\nu=1}^{N} \frac{1}{z_\nu - \lambda_{i_\nu}} \right\rangle_c$$

can be computed recursively (in ν).

[FDC, Phys. Rev. E **91**, *060102(R) (2015)]*

[FDC, F. Mezzadri, N. Simm & P. Vivo, J. Phys. A: Math. Theor. **49**, *18LT01 (2016)]*

Large number of scattering channels N

$$F_v(z_1,\dots,z_v) = \frac{1}{\beta^{v-1}} \sum_{\kappa_,\dots,\kappa_v \geq 0} \alpha[\kappa_1,\dots,\kappa_v] z_1^{\kappa_1}\cdots z_v^{\kappa_v}$$

$$F_1(z_1) = \frac{1}{2}(3 - z_1 - \sqrt{z_1^2 - 6z_1 + 1}),$$

$$F_2(z_1, z_2) = \frac{1}{\beta}\frac{z_1 z_2}{(z_1 - z_2)^2}\left[\frac{z_1 z_2 - 3(z_1 + z_2) + 1}{\sqrt{(z_1^2 - 6z_1 + 1)(z_2^2 - 6z_2 + 1)}} - 1\right],$$

$$F_{3,0}(z_1, z_2, z_3) = \frac{16}{\beta^2} z_1 z_2 z_3 \frac{[z_1 z_2 z_3 - (z_1 + z_2 + z_3) + 6]}{\prod_{i=1}^{3}(z_i^2 - 6z_i + 1)^{3/2}}$$

$$F_{v\geq 3}(z_1,\dots,z_v) = \widehat{D}_v(z_v) F_{v-1}(z_1,\dots,z_{v-1}) = \frac{e_v}{\beta^{v-1}} \frac{\sum C_{p_1,\dots,p_v} e_1^{p_1}\cdots e_v^{p_v}}{\prod_{i=1}^{v}(z_i^2 - 6z_i + 1)^{v-3/2}}.$$

Sketch of the proof I

Working with the Laguerre ensemble $\lambda_i = \frac{1}{N\tau_i}$

$$\mathcal{T}_\kappa = N^{\kappa-1} \sum_i \tau_i^\kappa = \frac{1}{N} \sum_i \frac{1}{\lambda_i^\kappa}$$

Key identity

$$\boxed{F_v(z_1, \ldots, z_v) = (-1)^v z_1 \cdots z_v \, g_v(z_1, \ldots, z_v), \qquad v \geq 2}$$

from Cauchy's integral formula:

$$\langle \mathcal{T}_{\kappa_1} \cdots \mathcal{T}_{\kappa_v} \rangle_c = \oint_{\mathcal{C}} \frac{\mathrm{d}z_1}{2\pi i} \cdots \oint_{\mathcal{C}} \frac{\mathrm{d}z_v}{2\pi i} \frac{g_v(z_1, \ldots, z_v)}{z_1^{\kappa_1} \cdots z_v^{\kappa_v}} \quad (\mathcal{C} \text{ does not enclose } z_i = 0)$$

$$= (-1)^v \oint_{|z_1|=\epsilon} \frac{\mathrm{d}z_1}{2\pi i} \cdots \oint_{|z_v|=\epsilon} \frac{\mathrm{d}z_v}{2\pi i} \frac{g_v(z_1, \ldots, z_v)}{z_1^{\kappa_1} \cdots z_v^{\kappa_v}} \quad (\text{for } \epsilon \text{ sufficiently small})$$

$$= (-1)^v \times \text{coefficient of } z_1^{\kappa_1 - 1} \cdots z_v^{\kappa_v - 1} \text{ in the power series of } g_v \text{ at } z_i = 0$$

Sketch of the proof II

Main ingredients for the recursion

1. The inverse delay times λ_i's form a one-cut β-ensemble.
2. The v-point resolvents have a honest $1/N$-expansion for all $\beta > 0$ [G. Borot and A. Guionnet (2013)].
3. At leading order in N a 'functional derivative method':

$$g_v(z_1, \ldots, z_v) = -\frac{1}{\beta}\frac{\delta}{\delta V(z_v)} g_{v-1}(z_1, \ldots, z_{v-1}).$$

4. Total Derivative Formula

$$\frac{\delta}{\delta V(z)} = \frac{\partial}{\partial V(z)} + \frac{\delta A}{\delta V(z)}\frac{\partial}{\partial A} + \frac{\delta B}{\delta V(z)}\frac{\partial}{\partial B} + \sum_{\ell \geq 1}\left\{\frac{\delta M_\ell}{\delta V(z)}\frac{\partial}{\partial M_\ell} + \frac{\delta \mathcal{J}_\ell}{\delta V(z)}\frac{\partial}{\partial \mathcal{J}_\ell}\right\}.$$

$\frac{\partial}{\partial V(z)}$: 'loop insertion operator';
A, B : edges of the cut;
$M_\ell, \mathcal{J}_\ell$: suitable coordinates [J. Ambjørn et al. (1993)].

Averages and covariances

κ	0	1	2	3	4	5	6	7	8	9	$\cdots$
Averages $\langle \mathcal{T}_\kappa \rangle_c \sim$	1	1	2	6	22	90	394	1806	8558	41586	$\cdots$

Known in combinatorics as the 'Large Schröder numbers'.

Covariances $$\langle \mathcal{T}_{\kappa_1} \mathcal{T}_{\kappa_2} \rangle_c \sim \frac{1}{\beta N^2} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 4 & 24 & 132 & 720 & \cdots \\ 24 & 160 & 936 & 5312 & \\ 132 & 936 & 5700 & 33264 & \\ 720 & 5312 & 33264 & 198144 & \\ \vdots & \vdots & & & \ddots \end{bmatrix}$$

[*FDC, Phys. Rev. E* **91**, *060102(R) (2015)*]

Higher order cumulants

Table: A few values of cumulants $\langle \mathcal{T}_{\kappa_1}, \cdots \mathcal{T}_{\kappa_v} \rangle \sim (\beta N^2)^{1-v} \alpha[\kappa_1, \ldots, \kappa_v]$.

(κ_1)	$\alpha[\kappa_1]$	(κ_1, κ_2)	$\alpha[\kappa_1, \kappa_2]$	$(\kappa_1, \kappa_2, \kappa_3)$	$\alpha[\kappa_1, \kappa_2, \kappa_3]$
(1)	1	$(1,1)$	4	$(1,1,1)$	96
(2)	2	$(1,2)$	24	$(1,1,2)$	848
		$(2,2)$	160	$(1,2,2)$	7488
				$(2,2,2)$	66112

$(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$	$\alpha[\kappa_1, \kappa_2, \kappa_3, \kappa_4]$	$(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5)$	$\alpha[\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5]$
$(1,1,1,1)$	5088	$(1,1,1,1,1)$	437760
$(1,1,1,2)$	54720	$(1,1,1,1,2)$	5303808
$(1,1,2,2)$	569600	$(1,1,1,2,2)$	61526016
$(1,2,2,2)$	5792256	$(1,1,2,2,2)$	690596352
$(2,2,2,2)$	57876480	$(1,2,2,2,2)$	7553912832
		$(2,2,2,2,2)$	80925462528

[FDC, F. Mezzadri, N. Simm & P. Vivo, J. Phys. A: Math. Theor. **49**, *18LT01 (2016)]*

Further results and a conjecture

Q: Are these cumulants integer-valued? If yes, what do they count?

Affirmative in a few cases.

Theorem 2. Let $\langle \mathcal{T}_{\kappa_1}, \cdots \mathcal{T}_{\kappa_\nu} \rangle \sim (\beta N^2)^{1-\nu} \alpha[\kappa_1, \ldots, \kappa_\nu]$.

(i) $\alpha[\kappa] \in \mathbb{N}$ (known in combinatorics as ‘large Schröder numbers’);

(ii) $\alpha[\kappa_1, \kappa_2] \in \mathbb{N}$;

(iii) $\alpha[\kappa_1, \kappa_2, \kappa_3] \in \mathbb{N}$;

(iv) $\alpha[\underbrace{1, \ldots, 1}_{\nu \text{ times}}] \in \mathbb{N}$ for all $\nu \geq 1$.

Proof by brute force.

(i)+(ii)+(iii)+(iv) + numerical evidences suggest the following

Conjecture. $\alpha[\kappa_1, \ldots, \kappa_\nu] \in \mathbb{N}$.

[FDC, F. Mezzadri, N. Simm & P. Vivo, J. Phys. A: Math. Theor. **49**, *18LT01 (2016)]*

'Genus' expansion

Q: next-to-leading orders?
For generic β and N, let $\tau_k(N) = \langle \mathcal{T}_k \rangle$.

$$\tau_k(N) = \sum_{g=0}^{\infty} \tau_{k,g} N^{-g}.$$

The leading order is $\tau_{k,0} = \alpha[k]$ (independent of β).

Finite-N formulae for $\tau_k(N)$ at $\beta = 2$:

$$\tau_k(N) = \frac{1}{k} \sum_{j=0}^{N-1} \binom{k+j-1}{k-1} \binom{k+j}{k-1} \frac{\Gamma(2N-k-j)}{\Gamma(N-j)} \frac{\Gamma(N+1)}{\Gamma(2N)} \quad \text{[Mezzadri-Simm'12]}$$

$$= \frac{1}{k!} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{\Gamma(N-j+k)}{\Gamma(N-j)} \frac{\Gamma(N+j+1-k)}{\Gamma(N+j+1)} \quad \text{[Novaes'15].}$$

Both hard to extract large-N asymptotics from.
[First three terms $\tau_{k,0}, \tau_{k,2}, \tau_{k,4}$ of the asymptotics in Mezzadri-Simm '12, Berkolaiko-Kuipers '10, Kuipers-Savin-Sieber '14].

‘Genus’ expansion

$$\tau_k(N) = \sum_{g=0}^{\infty} \tau_{k,g} N^{-g}.$$

Theorem 3. For $\beta = 2$, we have

$$(N^2 - k^2)(k+1)\tau_{k+1} - 3N^2(2k-1)\tau_k + N^2(k-2)\tau_{k-1} = 0$$

(an analog of Harer-Zagier recursion of GUE).

The large-N coefficients $\tau_{k,g}$ satisfy the homogeneous linear recurrence

$$(k+1)\tau_{k+1,2(g+1)} - 3(2k-1)\tau_{k,2(g+1)} + (k-2)\tau_{k-1,2(g+1)} - k^2(k+1)\tau_{k+1,2g} = 0,$$

with initial conditions $\tau_{k,0} = \alpha[k]$, $\tau_{0,2g} = \delta_{0,2g}$, $\tau_{1,2g} = \delta_{0,2g}$.
All coefficients $\tau_{k,g}$ with odd g vanish identically.

Generating functions

$$f_g(z) = \sum_{k=0}^{\infty} \tau_{k,g} z^k$$

Def: $y(z) = z^2 - 6z + 1$.
Corollary. If g is odd, $f_g(z) = 0$. For g even

$$\begin{cases} f_{g+2}(z) = \sqrt{y(z)} \int_0^z \frac{\mathrm{d}x}{y(x)^{3/2}} \left\{ x^2 f_g'''(x) + x f_g''(x) \right\}, \\ f_0(z) = \frac{3 - z - \sqrt{y(z)}}{2}. \end{cases}$$

Remark. The recursion is systematic; $f_g(z)$, for $g = 0, \ldots, 4$ have been computed recently (Kuipers-Savin-Sieber '14).

Similar results for $\beta = 1$ (inhomogeneous recursions).

Methods similar to those of Haagerup-Thorbjoernsen for $\beta = 2$ and Ledoux for $\beta = 1$.

‘Genus’ expansion

$$\tau_k(N) = \sum_{g=0}^{\infty} \tau_{k,g} N^{-g}.$$

$(\beta = 2)$				g			
k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	0	2	0	2	0	2
3	6	0	30	0	126	0	510
4	22	0	310	0	3262	0	31270
5	90	0	2730	0	57330	0	1048410
6	394	0	21980	0	805854	0	24848560

$(\beta = 1)$				g			
k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	0	0	0	0	0	0
2	2	2	6	10	22	42	86
3	6	18	102	378	1638	6426	26214
4	22	128	1142	7048	47454	291696	1821094
5	90	840	10650	96000	904530	7786680	66945450
6	394	5306	89576	1092460	13529862	152881422	1704027412

‘Genus’ expansion and a conjecture

Numerical evidences suggest the following

Conjecture. $\tau_{k,g} \in \mathbb{N}$ for $\beta = 1$ and 2.

We proved that infinitely many $\tau_{k,g}$’s are integers. More precisely:

Theorem 4. For $\beta = 2$: $\tau_{k,g} \in \mathbb{N}$ for $k \leq 10000$ or $g \leq 40$.

Indication of proof. We show that

$$\mathcal{J}_k(\zeta) = \sum_{g=0}^{\infty} \tau_{k,g}\zeta^g = \frac{P_k(\zeta^2)}{\prod_{j=0}^{k-1}(1 - j^2\zeta^2)}$$

where $P_k(\zeta)$ are polynomials satisfying a three term recursion. The denominator is a product of geometric series.

i) Compute $P_k(\zeta)$ using the recursion;

ii) Verify that $P_k(\zeta) \in \mathbb{N}[\zeta]$.

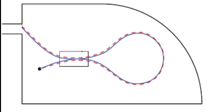
Use a symbolic algebra system. Same reasoning in the g ‘direction’.

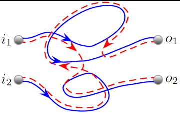
The conjectures

$$F_v(z_1,\ldots,z_v) = \frac{1}{\beta^{v-1}} \sum_{\kappa,\ldots,\kappa_v \geq 0} \underbrace{\alpha[\kappa_1,\ldots,\kappa_v]}_{\text{integers ?}} z_1^{\kappa_1}\cdots z_v^{\kappa_v}$$

Why should we care about this?

- For $v = 1$: $\alpha[\kappa] =$ Large Schröder numbers ;
  What is the underlying counting problem for $v > 1$ and for $g > 0$?
- Semiclassics: sum rules of wave amplitudes over classical trajectories
  (Sum over ‘diagrams’ with ‘encounters’ [J. Kuipers et al., New. J. Phys. (2014)]).





$$c(\mathcal{D}) = \frac{N^{\#I(\mathcal{D})}}{N^{\#L(\mathcal{D})}} \prod_{e\in E(\mathcal{D})} (-N)^{1-\#\text{end points in } e}$$

If semiclassics $\equiv$ RMT: heuristic argument supporting the conjecture.

Take-home messages

- New results on the Wigner-Smith matrix of ballistic quantum dots: $\langle \mathrm{tr} Q^{\kappa_1} \cdots \mathrm{tr} Q^{\kappa_v} \rangle$ for large N in terms of generating functions;
- Evidences that the statistical behavior of dynamics in chaotic cavities is associated to a counting problem: partial results and a conjecture. Another connection between RMT and semiclassical approximation.

For more details:

J. Ambjørn, **L. Chekhov**, **C. F. Kristjansen** and **Yu. M. Makeenko**, *Nucl. Phys. B* **404**, 127 (1993).

Y. V. Fyodorov, **D. V. Savin**, **H.-J. Sommers**, *Phys. Rev .E* **55**, R4857 (1997).

B. Eynard, *Nucl. Phys. B* **506**, 633 (1997).

P. W. Brouwer, **K. M Prahm** and **C. W. J. Beenakker**, Phys. Rev. Lett. **78**, 4737 (1997).

C. W. J. Beenakker, *Rev. Mod. Phys.* **69**(3), 731 (1997).

T. Guhr, **A. Müller-Groeling**, and **H. A. Weidenmüller**, *Phys. Rep.* **299** (4/6), 189 (1998).

U. Haagerup and S. Thorbjoernsen, *Expo. Math.* **21**, 293 (2003).

M. Ledoux, *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques* **45**, 754 (2009).

G. Berkolaiko and **J. Kuipers**, *New J. Phys* **13**, 063020 (2011).

F. Mezzadri and **N. J. Simm**, *Commun. Math. Phys.* **324**, 465 (2013).

J. Kuipers, **D. V. Savin** and **M. Sieber**, *New J. Phys.* **16**, 123018 (2014).

M. Novaes, *J. Math. Phys.* **56**, 062110 (2015).

F. D. Cunden, *Phys. Rev. E* **91**, 060102(R) (2015).

F. D. Cunden, **F. Mezzadri**, **N. Simm** and **P. Vivo**, J. Phys. A: Math. Theor. **49**, 18LT01 (2016).