Phase transitions of singular values for products of random matrices

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Refs: [1] Forrester-L.: Raney distributions and random matrix theory, J. Stat. Phys., 2015; Singular values for products of complex Ginibre matrices with a source: hard edge limit and phase transition, Comm. Math. Phys., 2016
[2] L.: Singular values for products of two coupled random matrices: hard edge phase transition, arXiv:1602.00634v2.

Outline

Product of Gaussian random matrices

- Product of independent Ginibre matrices $X = G_r \cdots G_1(G_0 + A)$ where G_j is a random $(N + \nu_j) \times (N + \nu_{j-1})$ matrix with i.i.d. standard complex Gaussian entries and A is a non-random matrix; $\nu_r, \ldots, \nu_0 \geq 0$ and $\nu_{-1} = 0$.
- Goal: Hard edge limits of singular values squared for the product X, i.e. eigenvalues $x_1, \ldots, x_N$ of X^*X. Here let $a_1, \ldots, a_N$ be eigenvalues of A^*A.
- Singular values for products of independent random matrices, Akemann-Kieburg-Wei /Akemann-Ipsen-Kieburg '13 (eigenvalue PDF), Kuijlaars-Zhang '14 (Meijer G-kernels, NEW!), L.-Wang-Zhang '14 (Sine-, Airy- kernels), Alexeev, Bai, Burda-Jarosz-Livan- Nowak-Swiech, Claeys, Forrester, Götze, Kosters, Neuschel, Penson, Stivigny, Strahov, Tikhomirov, Zyczkowski, ...

Noncentral complex Wishart matrices

- $X = G + A$, where G is a random $N \times n$ matrix with iid complex normal entries and A is a deterministic matrix.
- Eigenvalue PDF of XX^* ($\alpha = n - N$)

$$P_N(x) = Z^{-1} \det[x_i^{j-1}]_{i,j=1}^N \det[x_i^\alpha e^{-x_i} {}_0F_1(\alpha+1; a_i x_j]_{i,j=1}^N,$$

where $a_1, \ldots, a_N$ are eigenvalues of AA^* and the Bessel function

$$ {}_0F_1(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1+k)} z^k.$$

- ♡ Closely related to non-intersecting squared Bessel paths starting at zero and ending at N points, see e.g. [Kuijlaars, Martínez-Finkelshtein, Wielonsky, '09,'11]

Correlation kernel

[Desrosiers-Forrester 2008] A double integral for correlation kernel

$$K_N(x,y) = y^{\alpha} \frac{1}{2\pi i} \frac{1}{\Gamma^2(\alpha+1)} \int_0^{\infty} du \int_{\mathcal{C}} dv \, u^{\nu_0} e^{-u+v}$$
$$\times \, {}_0F_1(\alpha+1; -ux) \, {}_0F_1(\alpha+1; -vy) \frac{1}{u-v} \prod_{l=1}^{N} \frac{u+a_l}{v+a_l}$$

where $\mathcal{C}$ encircles $-a_1, \ldots, -a_N$ but not u. Suitable for asymptotic analysis at least at the origin!!

Limiting density (Kuijlaars et al, '09, '11)

For fixed $m \geq 0$, suppose $a_{m+1} = \cdots = a_N = bN$. Limiting eigenvalue density

$$\lim_{N\to\infty} K_N(Nt, Nt) = \rho(t; b)$$

with support $[L_1, L_2]$.

- Subcritical $0 \leq b < 1$: $L_1 = 0$ and the hard edge singularity

$$\rho(t; b) \sim ct^{-\frac{1}{2}}, \quad t \to 0^+$$

- Critical $b = 1$: $L_1 = 0$ and the hard edge singularity

$$\rho(t; b) \sim \frac{\sqrt{3}}{2\pi} t^{-\frac{1}{3}}, \quad t \to 0^+$$

- Supcritical $b > 1$: $L_1 > 0$.

Hard edge phase transition

Theorem A

♡ Subcritical $0 < b < 1$ [Kuijlaars et al, '09, '11, $m = 0$; Forrester-L., '16, $m \geq 0$]: with $a_1/N, \cdots, a_m/N$ fixed

$$\lim_{N\to\infty} \frac{1}{(1-b)N} K_N\Big(\frac{\xi}{(1-b)N}, \frac{\eta}{(1-b)N}\Big) = K_{\text{Bessel}}(\xi, \eta)$$

♠ Critical $b = (1 - \tau/\sqrt{N})^{-1}$ [Kuijlaars et al, '09, '11, $m = 0$; Forrester-L., '16, $m \geq 0$]: with $a_j = \sqrt{N}\sigma_j, j = 1, \ldots, m$

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} K_N\Big(\frac{\xi}{\sqrt{N}}, \frac{\eta}{\sqrt{N}}\Big) = \eta^{\alpha} \frac{1}{2\pi i} \frac{1}{\Gamma^2(\alpha+1)} \int_0^{\infty} du \int_{i\mathbb{R}} dv$$

$$u^{\alpha} {}_0F_1(; -u\xi)\, {}_0F_1(; -v\eta) \frac{e^{-\tau u - \frac{1}{2}u^2 + \tau v + \frac{1}{2}v^2}}{u - v} \prod_{j=1}^{m} \frac{u + \sigma_j}{v + \sigma_j}$$

Hard edge phase transition

Theorem A (continued)

♡ Supercritical $b > 1$ [Forrester-L., '16]: with $a_j = \sigma_j b/(b-1)$, $j \le m$ and $m \ge 1$,

$$\lim_{N\to\infty} e^{(1-\frac{1}{b})(\eta-\xi)}(1-\frac{1}{b})K_N((1-\frac{1}{b})\xi,(1-\frac{1}{b})\eta) = \frac{1}{2\pi i}\frac{1}{\Gamma^2(\alpha+1)}$$

$$\eta^{\alpha}\int_0^{\infty} du \int_{\gamma} dv\, u^{\alpha}\,{}_0F_1(;-u\xi)\,{}_0F_1(;-v\eta)\frac{e^{-u+v}}{u-v}\prod_{j=1}^{m}\frac{u+\sigma_j}{v+\sigma_j}.$$

A hard-edge outlier phenomenon is observed at zero and the limiting kernel happens to be the shifted mean $m \times m$ LUE kernel . This type of hard-edge phase transition exists at the hard edge, just like the BBP (Baik-Ben Arous-Péché '05) phase transition at the soft edge!

Correlation kernels for the product

Object: Singular values for product of independent Ginibre matrices $X = G_r \cdots G_1(G_0 + A)$.

- **Proposition**[Forrester-L. '15] With positive eigenvalues $a_1, \ldots, a_N$ for AA^*, the joint density reads off

$$\mathcal{P}_N(x_1, \ldots, x_N) = \frac{1}{N!} \det[K_N(x_i, x_j)]_{i,j=1}^N$$

with correlation kernel $K_N(x, y)$ given by

$$\frac{1}{2\pi i} \int_0^\infty du \int_{\mathcal{C}} dv\, u^{\nu_0} e^{-u+v} \Psi(u; x) \Phi(v; y) \frac{1}{u - v} \prod_{l=1}^N \frac{u + a_l}{v + a_l},$$

where $\mathcal{C}$ encircles $-a_1, \ldots, -a_N$ but not u.

Two auxiliary functions

- $$\Psi(u;x) = \frac{1}{(2\pi i)^r} \frac{1}{\Gamma(\nu_0+1)} \int_{\gamma_1} dw_1 \cdots \int_{\gamma_r} dw_r \prod_{l=1}^{r} w_l^{-\nu_l-1} e^{w_l} \times e^{x/(w_1\cdots w_r)} \, {}_0F_1\left(\nu_0+1; -ux/(w_1\cdots w_r)\right)$$

- $$\Phi(v;y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, y^{-s} \phi(v;s) \prod_{l=1}^{r} \Gamma(\nu_l+s),$$

  where

  $$\phi(v;s) = \frac{\Gamma(\nu_0+s)}{\Gamma(\nu_0+1)} \, {}_1F_1(\nu_0+s;\nu_0+1;-v).$$

Limiting density: $b = 0$

For fixed $m \geq 0$, suppose $a_{m+1} = \cdots = a_N = bN$. Let $\rho(\lambda; b)$ be limiting eigenvalue density.

- Limiting eigenvalue density: Fuss-Catalan law

$$\lim_{N\to\infty} N^r K_N(N^{r+1}\lambda, N^{r+1}\lambda) = \rho(\lambda; 0),$$

  where its k-th moment is given by the Fuss-Catalan number $\frac{1}{(r+2)k+1}\binom{(r+2)k+1}{k}$

- Hard edge singularity with exponent $-1 + \frac{1}{r+2} = -\frac{r+1}{r+2}$

$$\frac{1}{\pi} \sin \frac{\pi}{r+2}\, \lambda^{-1+\frac{1}{r+2}} \qquad \text{as} \quad \lambda \to 0^+,$$

  see e.g. [Banica-Belinschi-Capitaine-Collins '11]

Limiting density: $b = 1$

- [Forrester-L. '15]: The limiting eigenvalue density $\rho(\lambda; 1)$ has k-th moments (Raney numbers)

$$\frac{2}{(2r+3)k+2}\binom{(2r+3)k+2}{k}.$$

- Hard edge singularity with exponent $-1 + \frac{1}{r+3/2} = -\frac{2r+2}{2r+3}$

$$\frac{1}{\pi}\sin\frac{2\pi}{2r+3}\,\lambda^{-1+\frac{1}{r+3/2}} \qquad \text{as} \quad \lambda \to 0^+,$$

see also e.g. [Penson-Zyczkowski '11]

- Compared to the case of $b = 0$ with singularity exponent $-1 + \frac{1}{r+2}$, both are unlikely to be identical for any integer $r \geq 0$. Different exponents exhibit different scaling limits! This implies new kernels different from the Meijer G-kernel.

Meijer G-kernel

[Kuijlaars-Zhang '14] [Bertola-Gekhtman-Szmigielski '14, $r = 1$]
Meijer G-kernel =:

$$\int_0^1 G^{1,0}_{0,r+2}\left({\overline{} \atop 0, -\nu_0, \ldots, -\nu_r} \middle| ux \right) G^{r+1,0}_{0,r+2}\left({\overline{} \atop \nu_0, \ldots, \nu_r, 0} \middle| uy \right) du,$$

where the Meijer G-function $G^{m,n}_{p,q}\left({a_1,\ldots,a_p \atop b_1,\ldots,b_q} \middle| z \right) =$

$$\frac{1}{2\pi i} \int_\gamma \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(a_j + s)} z^{-s} ds.$$

A hard-edge phase transition

Theorem B [Forrester-L. '16]

♡ Subcritical regime: $0 < b < 1$ and with $a_1/N, \cdots, a_m/N$ fixed

$$\lim_{N\to\infty} \frac{1}{(1-b)N} K_N\Big(\frac{\xi}{(1-b)N}, \frac{\eta}{(1-b)N}\Big) = \text{Meijer G-kernel.}$$

♠ Critical regime: $b = 1$. $a_j = \sqrt{N}\sigma_j$, $j = 1, \ldots, m$ and $a_k = N(1-\tau/\sqrt{N})^{-1}$, $k = m+1, \ldots, N$

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} K_N\Big(\frac{\xi}{\sqrt{N}}, \frac{\eta}{\sqrt{N}}\Big) = K^{\text{crit}}(\xi, \eta).$$

A hard-edge phase transition (cotinued)

Deformed critical kernel:

$$K^{\mathrm{crit}}(\xi,\eta) = \frac{1}{2\pi i}\int_0^{\infty} du \int_{-c-i\infty}^{-c+i\infty} dv \left(\frac{u}{v}\right)^{\nu_0} \frac{e^{-\tau u - \frac{1}{2}u^2 + \tau v + \frac{1}{2}v^2}}{u-v} \prod_{j=1}^{m} \frac{u+\sigma_j}{v+\sigma_j}$$

$$G^{1,0}_{0,r+2}\left({- \atop 0, -\nu_0, \ldots, -\nu_r}\,\middle|\, u\xi\right) G^{r+1,0}_{0,r+2}\left({- \atop \nu_0, \ldots, \nu_r, 0}\,\middle|\, v\eta\right),$$

where $0 < c < \min\{\sigma_1, \ldots, \sigma_m\}$.

♡ Supercritical regime: $b > 1$. When $r > 0$, scaling limits?

Two correlated matrices

Cf. both Strahov's talk or Akemann's talk: The product of two coupled matrices and a new interpolating hard-edge scaling limit

- ♡ **Object**: Given two matrices X_1 of size $L \times M$ and X_2 of size $M \times N$ with $L, M \geq N$, the joint PDF (chiral two-matrix model, unsolvable!) defined as

$$\exp\{-\alpha \mathrm{Tr}(X_1 X_1^* + X_2^* X_2) + \mathrm{Tr}(\Omega X_1 X_2 + (\Omega X_1 X_2)^*)\},$$

  and the squared singular values of $Y_2 = X_1 X_2$, where $\alpha > 0$ and Ω is a non-random $N \times L$ coupling matrix such that $\Omega\Omega^* < \alpha^2$.

- ♠ **Goal**: Singular value PDF as a bi-orthogonal ensemble, a double contour integral for correlation kernel, and a phase transition at the origin in four different regimes.

Motivation: complex eigenvalues

- ♠ A, B: independent $N \times M$ matrices with iid standard complex Gaussian entries, Osborn 2004 investigated an analogue of the Dirac operator in the context of QCD with chemical potential $\mu \in [0, 1]$

$$D = \begin{pmatrix} 0 & iA + \mu B \\ iA^* + \mu B^* & 0 \end{pmatrix}.$$

  Complex eigenvalues of D reduce to those of the product $Y = (iA + \mu B)(iA^* + \mu B^*)$.

- ♡ An interpolation model: $\mu = 0$, Wishart distribution; $\mu = 1$, the product of two independent Gaussian matrices.

Motivation: singular values

- Set $X_1 = (A - i\sqrt{\mu}B)/\sqrt{2}$, $\quad X_2 = (A^* - i\sqrt{\mu}B^*)/\sqrt{2}$, then X_1 and X_2 have a joint PDF as defined above

$$\exp\{-\alpha \mathrm{Tr}(X_1X_1^* + X_2^*X_2) + \mathrm{Tr}(\Omega X_1X_2 + (\Omega X_1X_2)^*)\},$$

  but with $L = N$, $\alpha = (1+\mu)/(2\mu)$ and $\Omega = (1-\mu)/(2\mu)I_N$; [Akemann-Strahov '15a] turned to singular values of X_1X_2.

- **Our Object**: Given two matrices X_1 of size $L \times M$ and X_2 of size $M \times N$ with $L, M \geq N$, the joint PDF is proportional to

$$\exp\{-\alpha \mathrm{Tr}(X_1X_1^* + X_2^*X_2) + \mathrm{Tr}(\Omega X_1X_2 + (\Omega X_1X_2)^*)\}$$

  where $\alpha > 0$ and Ω is a non-random $N \times L$ coupling matrix such that $\Omega\Omega^* < \alpha^2$.

- An interpolating ensemble between squared Wishart matrix ($\mu = 0$) and the product of two independent matrices ($\mu = 1$).

Coupled multiplication with a Ginibre matrix

G: $(\kappa+N)\times(\nu+N)$, X: $(\nu+N)\times N$, Ω: $N\times(\kappa+N)$, the joint PDF of G and X proportional to

$$\exp\left\{-\alpha \mathrm{Tr}(GG^*)+\mathrm{Tr}(\Omega GX+(\Omega GX)^*)\right\}h(X)dGdX,$$

where $h(UXV)=h(X)$ for any $U\in U(\nu+N)$ and $V\in U(N)$.

Theorem (Singular value PDF, L. '16)

Let $\delta_1,\ldots,\delta_N$, $\sqrt{t_1},\ldots,\sqrt{t_N}$ be singular values of Ω and X, suppose $h(X)=\frac{1}{\Delta(t)}\det[f_k(t_j)]_{j,k=1}^N$, then the singular values of GX

$$\mathcal{P}_N(x_1,\ldots,x_N)=Z_N^{-1}\det[\xi_i(x_j))]_{i,j=1}^N\det[\eta_i(x_j))]_{i,j=1}^N,$$

where $\xi_i(z)={}_0F_1(\kappa+1;\delta_i^2 z)$ and $\eta_i(z)=\int_0^\infty t^\kappa e^{-\alpha t}\left(\frac{z}{t}\right)^\nu f_i\left(\frac{z}{t}\right)\frac{dt}{t}$.

Transformations of bi-orthogonal ensembles

Cf. both Kuijlaars's and Wang's talks

- ♠ Multiplication with a Ginibre matrix transforms a polynomial ensemble to a polynomial ensemble [Kuijlaars-Stivigny '14, Kuijlaars'15]:

$$Z_N^{-1} \det[x_j^{i-1}]_{i,j=1}^N \det[\eta_i(x_j))]_{i,j=1}^N$$

- ♠ Multiplication with a truncated unitary matrix also preserves the structure of a polynomial ensemble [Kieburg-Kuijlaars-Stivigny 15], so does addition of a GUE matrix [Claeys-Kuijlaars-Wang 15].
- ♡ Coupled multiplication with a Ginibre matrix transforms a bi-invariant polynomial ensemble to a bi-orthogonal ensemble in Borodin's sense.

Correlation kernel

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+1+k)} \left(\frac{z}{2}\right)^{2k+\nu}$$

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu} \int_0^\infty t^{-\nu-1} e^{-t-\frac{z^2}{4t}} dt, \quad |\arg(z)| < \frac{\pi}{2},$$

Theorem (Double integral for correlation kernel, L. '16)

$$K_N(x,y) = \frac{2\alpha^2}{(2\pi i)^2} \int_{\mathcal{C}_{\mathrm{out}}} du \int_{\mathcal{C}_{\mathrm{in}}} dv\, K_{-\kappa}(2\alpha\sqrt{(1-u)x}) \times$$

$$I_\kappa(2\alpha\sqrt{(1-v)y}) \frac{1}{u-v} \left(\frac{1-u}{1-v}\right)^{\kappa/2} \left(\frac{u}{v}\right)^{-\nu-N} \prod_{l=1}^{N} \frac{u-(1-\delta_l^2/\alpha^2)}{v-(1-\delta_l^2/\alpha^2)},$$

where $\mathcal{C}_{\mathrm{in}}$ encircles $1-\delta_1^2/\alpha^2, \ldots, 1-\delta_N^2/\alpha^2$ on the LHS of $\mathcal{C}_{\mathrm{out}}$.

Remarks

- ♠ When $\kappa = 0$ ($L = N$) and all δ_j are equal, singular value PDF has been obtained by [Akemann-Strahov '15a] for two coupled Gaussian matrices.
- ♡ In this case a "double contour integral" for correlation kernel has also been given by [Akemann-Strahov '15a], different from ours.

Limiting kernel (i)

For a given nonnegative integer m, suppose that

$$\delta_{m+1} = \cdots = \delta_N = \delta \text{ and } \alpha = (1+\mu)/(2\mu), \delta = (1-\mu)/(2\mu), 0 < \mu \leq 1.$$

As μN changes from 0 to ∞ at different scales, by tuning the scale of $1 - \delta_1^2/\alpha^2, \ldots, 1 - \delta_m^2/\alpha^2$, the following hard edge phase transition holds true in four different regimes.

Theorem C [Hard edge phase transition, L. '16]

(i) [Meijer G-kernel] When $\mu N \to \infty$,

$$\frac{\mu}{N} K_N\left(\frac{\mu N}{N^2}\xi, \frac{\mu}{N}\eta\right) \longrightarrow K_{\mathrm{I}}(\xi, \eta) :=$$

$$= \left(\frac{\xi}{\eta}\right)^{\kappa/2} \int_0^1 dw\, G^{1,0}_{0,3}\left({- \atop 0, -\nu, -\kappa} \middle| \eta w\right) G^{2,0}_{0,3}\left({- \atop \nu, \kappa, 0} \middle| \xi w\right)$$

Limiting kernels (ii) & (iii)

(ii) [Critical kernel] When $\mu N \to \tau/4$ with $\tau > 0$ and $1 - \delta_l^2/\alpha^2 \to \pi_l \in (0,1)$ for $l = 1, \ldots, m$ (noting $\alpha \sim N$) $\alpha^{-2} K_N(\alpha^{-2}\xi, \alpha^{-2}\eta) \to K_{\mathrm{II}}(\xi,\eta) :=$

$$\frac{2}{(2\pi i)^2} \int_{\mathcal{C}_{\rm out}} du \int_{\mathcal{C}_{\rm in}} dv\, K_{-\kappa}(2\sqrt{(1-u)\xi})\, I_{\kappa}(2\sqrt{(1-v)\eta}) \times e^{-\frac{\tau}{u}+\frac{\tau}{v}} \frac{1}{u-v} \left(\frac{1-u}{1-v}\right)^{\kappa/2} \left(\frac{u}{v}\right)^{-\nu-m} \prod_{l=1}^{m} \frac{u-\pi_l}{v-\pi_l}.$$

(iii) [Perturbed Bessel, cf. Desrosiers-Forrester '06] When $\mu N \to 0$ and $1 - \delta_l^2/\alpha^2 = 4\mu N \pi_l$,

$$e^{\frac{\alpha}{N}(\sqrt{\xi}-\sqrt{\eta})} \frac{1}{4N^2} K_N\left(\frac{1}{4N^2}\xi, \frac{1}{4N^2}\eta\right) \to K_{\mathrm{III}}(\xi,\eta) := \frac{2}{(2\pi i)^2} \times \frac{1}{4(\xi\eta)^{\frac{1}{4}}} \int_{\mathcal{C}_{\rm out}} du \int_{\mathcal{C}_{\rm in}} dv\, e^{\sqrt{\xi}u - \sqrt{\eta}v - \frac{1}{u} + \frac{1}{v}} \frac{1}{u-v} \left(\frac{u}{v}\right)^{-\nu-m} \prod_{l=1}^{m} \frac{u-\pi_l}{v-\pi_l}.$$

Limiting kernel (iv)

(iv) [Finite coupled product kernel] When $\mu N \to 0$ and $1 - \delta_l^2/\alpha^2 \to \pi_l \in (0,1)$ for $l = 1, \ldots, m \geq 1$ (noting $\mu = o(1/N)$),

$$4\mu^2 K_N(4\mu^2\xi, 4\mu^2\eta) \to K_{\mathrm{IV}}(\xi, \eta) :=$$

$$\frac{2}{(2\pi i)^2} \int_{\mathcal{C}_{\mathrm{out}}} du \int_{\mathcal{C}_{\mathrm{in}}} dv \, K_{-\kappa}(2\sqrt{(1-u)\xi}) \, I_\kappa(2\sqrt{(1-v)\eta}) \times \frac{1}{u-v} \left(\frac{1-u}{1-v}\right)^{\kappa/2} \left(\frac{u}{v}\right)^{-\nu-m} \prod_{l=1}^{m} \frac{u-\pi_l}{v-\pi_l}.$$

This is just the same kernel for the coupled product but with $N = m$, $\alpha = 1$ and $1 - \delta_l^2/\alpha^2 = \pi_l$ ($l = 1, \ldots, m$).

A transition

Theorem (L. 2016)

(i)

$$\lim_{\tau \to \infty} \frac{1}{\tau} K_{\mathrm{II}}\left(\frac{x}{\tau}, \frac{y}{\tau}\right) = K_{\mathrm{I}}(x, y).$$

(ii) *Given $q \leq m$, suppose that $\pi_l = \tau \hat{\pi}_l$ for $l = 1, \ldots, q$ and $\pi_{q+1}, \ldots, \pi_m$ are fixed, then*

$$\lim_{\tau \to 0} e^{\frac{2}{\tau}(\sqrt{x} - \sqrt{y})} \frac{1}{\tau^2} K_{\mathrm{II}}\left(\frac{x}{\tau^2}, \frac{y}{\tau^2}\right) = K_{\mathrm{III}}(x, y)\big|_{m \mapsto q, \nu \mapsto \nu + m - q, \pi \mapsto \hat{\pi}}.$$

(iii)

$$\lim_{\tau \to 0} K_{\mathrm{II}}(x, y) = K_{\mathrm{IV}}(x, y).$$

Remarks

When $\kappa = 0$ ($L = N$) and all δ_j are equal, compare relevant results of Akemann-Strahov as follows.
(1) For fixed μ, Part (i) was previously obtained by [Akemann-Strahov 2015a].
(2) Let $\mu = gN^{-\chi}$. In [Akemann-Strahov 2015b], Akemann and Strahov have proved the case $\chi = 0$ which is a special case of Part (i), the critical case $\chi = 1$ corresponding to Part (ii) and the case $\chi \geq 2$ which is part of Part (iii). Although their three double integrals are different from ours, these (critical kernels) are believed to be the same.

Critical kernel K_{II}

Define

$$K_{\mathrm{II}}^{(0)}(\xi,\eta)=\frac{2}{(2\pi i)^2}\int_{\mathcal{C}_{\mathrm{out}}}du\int_{\mathcal{C}_{\mathrm{in}}}dv\,K_{-\kappa}(2\sqrt{(1-u)\xi})\,I_{\kappa}(2\sqrt{(1-v)\eta})\times e^{-\frac{\tau}{u}+\frac{\tau}{v}}\frac{1}{u-v}\left(\frac{1-u}{1-v}\right)^{\kappa/2}\left(\frac{u}{v}\right)^{-\nu-m}, \tag{1}$$

$$K_{\mathrm{II}}(\xi,\eta)=K_{\mathrm{II}}^{(0)}(\xi,\eta)+\sum_{k=1}^{m}\tilde{\Lambda}_{\mathrm{II}}^{(k)}(\xi)\Lambda_{\mathrm{II}}^{(k)}(\eta). \tag{2}$$

$$\tilde{\Lambda}_{\mathrm{II}}^{(k)}(x)=\frac{1}{\pi i}\int_{\mathcal{C}_0}du\;K_{-\kappa}(2\sqrt{(1-u)\xi})e^{-\frac{\tau}{u}}(1-u)^{\kappa/2}u^{-\nu-m}\prod_{l=1}^{k-1}(u-\pi_l), \tag{3}$$

$$\Lambda_{\mathrm{II}}^{(k)}(x)=\frac{1}{2\pi i}\int_{\mathcal{C}_\pi}dv\,I_{\kappa}(2\sqrt{(1-v)\xi})e^{\frac{\tau}{v}}(1-v)^{-\kappa/2}v^{\nu+m}\prod_{l=1}^{k}\frac{1}{v-\pi_l} \tag{4}$$

Integrable form

$$f(x) = \int_0^\infty dt\, t^{-(\frac{\alpha}{2}-\kappa)-3} e^{-xt-\frac{1}{t}} J_\alpha\left(\sqrt{4\tau/t}\right) \tag{5}$$

$$g(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_0} ds\, s^{(\frac{\alpha}{2}-\kappa)-3} e^{xs+\frac{1}{s}} J_\alpha\left(\sqrt{4\tau/s}\right). \tag{6}$$

$$\begin{aligned} K_{\mathrm{II}}^{(0)}(\xi,\eta) \doteq \frac{1}{\eta-\xi}\Big(&\xi\eta f'''(\xi)g'''(\eta) + f''(\xi)\left(g(\eta) - (\alpha - 2\kappa - \tau - 1)g'(\eta\right. \\ &+ g''(\eta)\left(f(\xi) + (\alpha - 2\kappa - \tau + 1)f'(\xi)\right) \\ &- (\xi + \eta + \alpha\kappa - \kappa^2)f''(\xi)g''(\eta) - f'(\xi)g'(\eta)\Big), \end{aligned}$$

and $f(x)$, $g(x)$ are solutions of the fourth order ODEs

$$x^2 f^{(4)} - (\alpha - 2\kappa - 1)xf''' - (2x + \alpha\kappa - \kappa^2)f'' + (\alpha - 2\kappa - \tau + 1)f' + f = 0,$$

$$x^2 g^{(4)} + (\alpha - 2\kappa + 1)xg''' - (2x + \alpha\kappa - \kappa^2)g'' - (\alpha - 2\kappa - \tau - 1)g' + g = 0.$$

Thank you!