Interpolating between Wishart and inverse-Wishart distributions

Topological phase transitions in 1D multichannel disordered wires with a chiral symmetry

Christophe Texier

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with **Aurélien Grabsch**, cond-mat arXiv:1506.05322

Wishart distribution (Laguerre ensemble)

$$M : N \times N \text{ Hermitian matrix with } \begin{cases} \text{real } (\beta = 1) \\ \text{complex } (\beta = 2) \\ \text{quaternionic } (\beta = 4) \end{cases} \text{ elements}$$

and *positive* eigenvalues

$$\mathrm{D}M\, P(M) \propto \overbrace{\mathrm{D}M}^{\text{Haar}}\, (\det M)^{\eta - 1 - \frac{\beta}{2}(N-1)} \exp\left[-\frac{1}{2}\,\mathrm{Tr}\left\{M\right\}\right]$$

for $\eta > \frac{\beta}{2}(N-1)$

Inverse-Wishart

$$\mathrm{D}M\, \widetilde{P}(M) \propto \mathrm{D}M\, (\det M)^{-\eta - 1 - \frac{\beta}{2}(N-1)} \exp\left[-\frac{1}{2}\,\mathrm{Tr}\left\{M^{-1}\right\}\right]$$

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Consider $G = \mathbf{1}_N$:

- $k^2 = 0$: Wishart with $\eta = -\mu > \frac{\beta}{2}(N-1)$

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Consider $G = \mathbf{1}_N$:

- $k^2 = 0$: Wishart with $\eta = -\mu > \frac{\beta}{2}(N-1)$
- $k^2 \to \infty$: inverse-Wishart with $\eta = +\mu > \frac{\beta}{2}(N-1)$

Physical context : Multichannel disordered wires with chiral symmetry

1D Dirac Hamiltonian with a random mass :

$$\boxed{\mathcal{H}_D = \mathrm{i}\,\sigma_2 \otimes \mathbf{1}_N\,\partial_x + \sigma_1 \otimes M(x)}$$

acting on a spinor with $2N$ components

chiral symmetry : $\sigma_3 \mathcal{H}_D \sigma_3 = -\mathcal{H}_D$

Disorder :

Gaussian white noise ($N \times N$ matrix) $\rightarrow \begin{cases} \text{real } (\beta = 1) \\ \text{complex } (\beta = 2) \\ \text{quaternion } (\beta = 4) \end{cases}$

$$P\left[M(x)\right] \propto \exp\left[-\frac{1}{2}\int \mathrm{d}x\ \mathrm{Tr}\left\{(M(x)-\mu\,G)^\dagger G^{-1}(M(x)-\mu\,G)\right\}\right]$$

where $\mu \in \mathbb{R}$ and $G = G^\dagger$.

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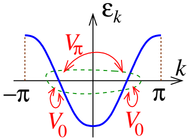
Motivations

The strictly 1D Dirac equation with random mass

$$\Big[\underbrace{\mathrm{i}\,\sigma_2\,\partial_x + \sigma_1\, m(x)}_{=\mathcal{H}_D}\Big]\Psi(x) = \varepsilon\,\Psi(x)$$

In condensed matter and statistical physics :



- 1D disordered metal at half filling

  $\widetilde{\mathcal{H}}_{\mathrm{D}} = -\mathrm{i}\sigma_3\partial_x + V_0(x) + \sigma_1\,V_\pi(x)$

  $\longrightarrow$ Kappus-Wegner (band center) anomaly

- Disordered spin chains and spin ladders

  (Fisher '94, '95 ; Steiner, Fabrizio, Gogolin, Mélin, Steiner, '97 ; ...)

- *Classical* diffusion in a random environment

  $\mathcal{H}_{\mathrm{D}}^2 \to H_\pm = -\partial_x^2 + m^2 \pm m' \to$ FPE $\partial_t P = \partial_x(\partial_x - 2m)P$

  $\to$ Sinai diffusion ($\langle m\rangle = 0$), etc.

  Bouchaud, Comtet, Georges & Le Doussal, 1990 ; ...

- $\cdots$

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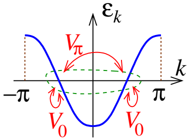

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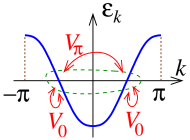

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Multichannel disordered wires

- Standard classes (orthogonal, unitary, symplectic)

  Dorokhov, 1982 ; 1988 ; ... Mello, Pereyra, Kumar, 1988 : « DMPK »

  Beenakker, Rev. Mod. Phys. (1997)

- Additional discrete symmetries : chiral & BdG classes

  → spectral properties, localisation

  Brouwer, Mudry, Simons, Altland, PRL (1998)

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  Ludwig & Schulz-Baldes, J. Stat. Phys. (2013) (localisation)

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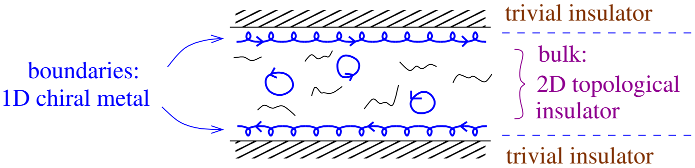
Topological insulators and quantum phase transitions

Example : Integer Quantum Hall Effect (class A)



« bulk-edge correspondence »

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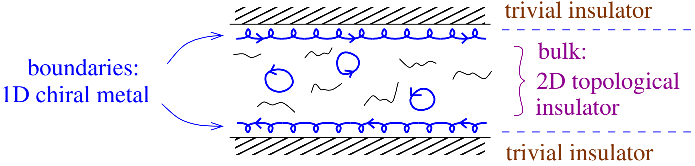

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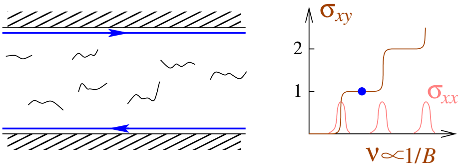
Quantum phase transition $\sigma_{xy} = n \to n+1$



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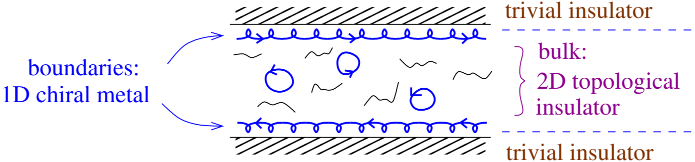

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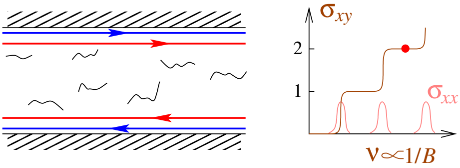
Quantum phase transition without symmetry breaking



« topological phase transition »

What are the possible topological insulators ?

Symmetries of RMT and disordered systems :

(Altland & Zirnbauer, ’96 ; ’97)

Two discrete symmetries in condensed matter :

Chiral (sublattice) & particle-hole (superconductors)

Periodic table of topological insulators :

						topological index		
		TRS	p-h S	ch. S	SRS	0D	1D	2D $\cdots$
Wigner-Dyson	AI (orthogonal)	$+1$	no	no	yes	$\mathbb{Z}$	no	no
	A (unitary)	no	no	no	indiff.	$\mathbb{Z}$	no	$\mathbb{Z}$
	AII (symplectic)	-1	no	no	no	$\mathbb{Z}$	no	$\mathbb{Z}_2$
Chiral	BDI (chiral orth.)	$+1$	$+1$	yes	yes	$\mathbb{Z}_2$	$\mathbb{Z}$	no
	AIII (chiral unit.)	no	no	yes	indiff.	no	$\mathbb{Z}$	no
	CII (chiral sympl.)	-1	-1	yes	no	no	$\mathbb{Z}$	no
BdG	D	no	$+1$	no	no	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	DIII	-1	$+1$	yes	no	no	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	C	no	-1	no	yes	no	no	$\mathbb{Z}$
	CI	$+1$	-1	yes	yes	no	no	no

Kitaev (2009) ; Ryu, Schnyder, Furusaki, Ludwig, New J. Phys. (2010)

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Question

1D multichannel Dirac equation with random mass :

Q : How does disorder drive topological phase transitions ?

$\rightarrow$ Relation with the matrix model

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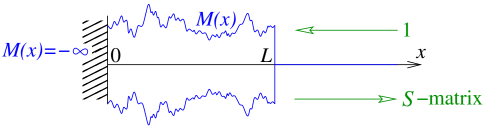
The scattering problem



Weyl (chiral) representation :

$$\widetilde{\mathcal{H}}_D = -\mathrm{i}\,\sigma_3 \otimes \mathbf{1}_N\,\partial_x + \sigma_1 \otimes M(x)$$

Scattering state : $2N \times N$ « spinor » : (for $\varepsilon > 0$)

$$\widetilde{\Psi}_\varepsilon(x) = \begin{pmatrix} 0 \\ \mathbf{1}_N \end{pmatrix} e^{-\mathrm{i}\varepsilon(x-L)} + \begin{pmatrix} \mathcal{S}(\varepsilon) \\ 0 \end{pmatrix} e^{+\mathrm{i}\varepsilon(x-L)} \qquad \text{for } x > L$$

$\mathcal{S}(\varepsilon)$: $N \times N$ scattering matrix

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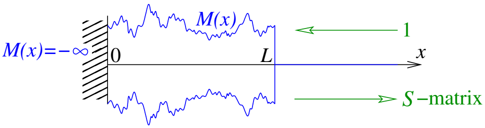

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$2N \times N$ « spinor » :

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \qquad \Rightarrow \qquad Z \overset{\text{def}}{=} -\varepsilon\, \chi\, \varphi^{-1}$$

obeys

$$Z'(x) = -\varepsilon^2 - Z(x)^2 - M(x)\, Z(x) - Z(x)\, M(x)$$

Relation to the $\mathcal{S}$-matrix :

$$S(\varepsilon) = -\mathrm{i} \left[\varepsilon - \mathrm{i}\, Z(L) \right] \left[\varepsilon + \mathrm{i}\, Z(L) \right]^{-1}$$

Riccati matrix Z and scattering matrix $\mathcal{S}$

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Plan

Random matricial process

$M(x)$ a Gaussian white noise ($N \times N$ matrix)

Stochastic matricial differential equation

$$Z'(x) = -\varepsilon^2 - Z(x)^2 - M(x)\,Z(x) - Z(x)\,M(x) \qquad \text{(Stratonovich)}$$

Matricial Fokker-Planck equation

$\varepsilon = \mathrm{i}k \in \mathrm{i}\mathbb{R} \Rightarrow$ if $M(x) = 0$, fixed point $Z(x) = k\,\mathbf{1}_N$

if $M(x) \neq 0$: stationary (equil.) distribution (proven for $\beta = 1$ & 2) :

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Density of states (DoS)

Krein-Friedel relation :

DoS $\leftrightarrow$ Scattering

$$\mathcal{S}(\varepsilon) = -\mathrm{i}\,[\varepsilon - \mathrm{i}Z(L)]\,[\varepsilon + \mathrm{i}Z(L)]^{-1}$$

integrated DoS $\mathcal{N}(\varepsilon)$ $\leftrightarrow$ $\langle Z(L)\rangle = \int \mathrm{D}Z\, f(Z)\, Z$

($+$ analytic continuation, $\varepsilon = \mathrm{i}k \in \mathrm{i}\mathbb{R}$)

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Integrated DoS $\mathcal{N}(\varepsilon)$:

$$G = \mathrm{diag}(g_1, \cdots, g_N)$$

$$\Omega = \langle \mathrm{Tr}\{Z\} \rangle = \sum_{i=1}^{N} g_i^2 \frac{\partial \ln \mathcal{C}_{N,\beta}}{\partial g_i} \quad \xrightarrow{G=g\mathbf{1}_N} \quad \Omega = -g\left(N\mu + k\frac{\partial \ln \mathcal{C}_{N,\beta}}{\partial k}\right)$$

$$\mathcal{N}(\varepsilon) = -\frac{1}{\pi}\,\mathrm{Im}\left[\Omega|_{k=-\mathrm{i}\varepsilon}\right]$$

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Determinantal representations ($G = g\mathbf{1}_N$)

Chiral orthogonal ($\beta = 1$, BDI)

Pfaffian

$$\mathcal{C}_{N,1} \propto \mathrm{pf}\Big[\cdots\Big]$$

Chiral unitary ($\beta = 2$, AIII)

Hankel determinant

$$\mathcal{C}_{N,2} = N!\,2^N\,k^{-N\mu}\,\det\Big[K_{\mu+1+N-i-j}(k/g)\Big]_{1\leqslant n,\,m\leqslant N}$$

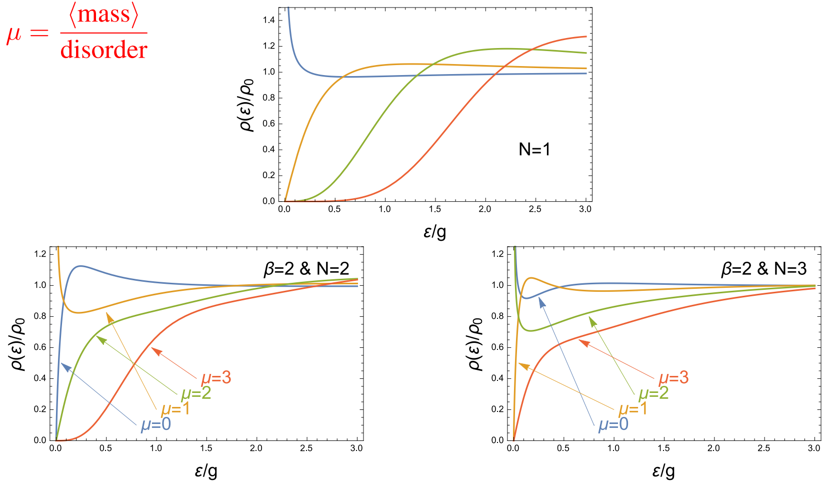
Chiral symplectic ($\beta = 4$, CII)

$$\mathcal{C}_{N,4} = N!\,2^N\,k^{-N\mu}\,\mathrm{pf}\Big[(m-n)\,K_{\mu+1+2N-n-m}(k/g)\Big]_{1\leqslant n,\,m\leqslant 2N}$$

similar result in : Titov, Brouwer, Furusaki & Mudry, PRB (2001)

Density of states $\rho(\varepsilon) = \mathcal{N}'(\varepsilon)$

$$\mu = \frac{\langle \text{mass} \rangle}{\text{disorder}}$$

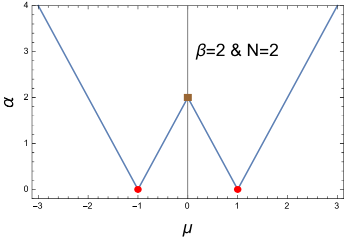


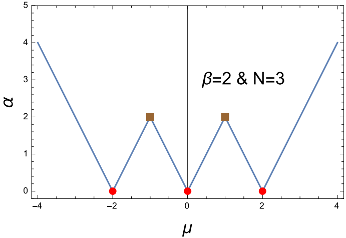
$$\mathcal{N}(\varepsilon) \sim \varepsilon^{\alpha} \quad \text{as} \quad \varepsilon \to 0$$

$\mu = 0$: parity effect (Brouwer, Furusaki, Mudry, PRL '00 ; Physica E '01)

Exponent of the DoS $\rho(\varepsilon) = \mathcal{N}'(\varepsilon) \sim \varepsilon^{\alpha-1}$

$$\mu = \frac{\langle \text{mass} \rangle}{\text{disorder}}$$





■ : $\mathcal{N}(\varepsilon) \sim \varepsilon^{\beta} |\ln \varepsilon|$

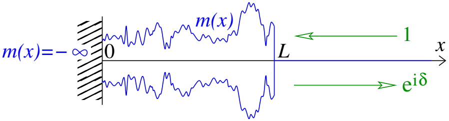
● : $\mathcal{N}(\varepsilon) \sim 1/\ln^2 \varepsilon$ (Dyson singularity $\forall\, \beta$) $\rightarrow$ superuniversality

(Gruzberg, Read, Vishveshwara, PRB 2005)

$\alpha = 0 \Leftrightarrow$ topological phase transition

Plan

Topological quantum number ($N = 1$ channel case)



set $\langle m(x) \rangle = \pm \mu g = m_0$

$\rightarrow$ Phase-shift $\delta_\pm(\varepsilon)$

$\rightarrow$ IDoS $\mathcal{N}_L^\pm(\varepsilon) = \frac{\delta_\pm(\varepsilon)}{2\pi}$

Bulk-edge correspondence

$\rightarrow$ case with no disorder – subgap state :

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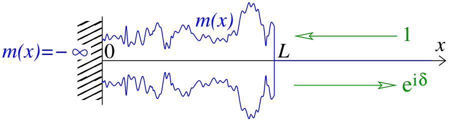

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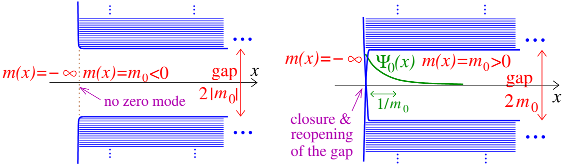

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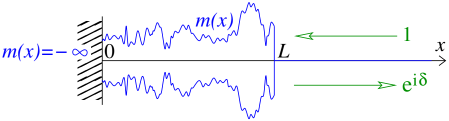

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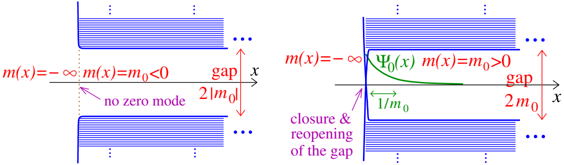
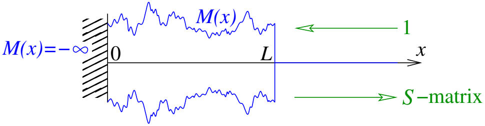


$\Rightarrow$ $\boxed{\Delta(\infty) = \frac{1}{2}}$

Witten index ($\frac{1}{2}$ # of zero modes) :

$$\mathcal{N}_L^+(0) - \mathcal{N}_L^-(0) = \frac{\delta_+(0) - \delta_-(0)}{2\pi} = \Delta(\infty) \quad \underset{\tilde{\beta}\to\infty}{\longleftarrow} \quad \Delta(\tilde{\beta}) = \mathrm{Tr}\left\{\sigma_3\, e^{-\tilde{\beta}\mathcal{H}_{\mathrm{D}}^2}\right\}$$

Witten index and Riccati matrix



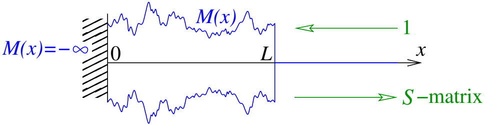
$$S(\varepsilon) = -\mathrm{i}\big[\varepsilon - \mathrm{i}\,Z(L)\big]\big[\varepsilon + \mathrm{i}\,Z(L)\big]^{-1}$$

Eigenvalues : $z_n = \varepsilon\, \tan\delta_n$ (phase shift)

Witten index ($\leftrightarrow \frac{1}{2}$ # of zero modes)

$$\delta_\pm(\varepsilon) = -\mathrm{i}\ln\det S(\varepsilon) \text{ for } \langle M(x)\rangle = \pm\mu g\,\mathbf{1}_N$$

$$\Delta(\infty) \underset{\varepsilon\to0}{\longleftarrow} \frac{\langle\delta_+(\varepsilon)-\delta_-(\varepsilon)\rangle}{2\pi} = \begin{cases} \frac{1}{2}\sum_n\langle\mathrm{sign}(z_n)\rangle & \text{for } \varepsilon\in\mathbb{R} \\ \frac{1}{2}\sum_n\langle\mathrm{sign}(z_n-k)\rangle & \text{for } \varepsilon=\mathrm{i}k\in\mathrm{i}\mathbb{R}\end{cases}$$



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$k = -\mathrm{i}\varepsilon \to 0$ limit of the distribution $f(Z)$

$$P(z_1, \cdots, z_N) = \mathcal{C}_{N,\beta}^{-1} \prod_{i<j} |z_i - z_j|^\beta \prod_l z_l^{\mu-\beta(N-1)/2-1} \, e^{-\frac{1}{2g}(z_l + k^2/z_l)}$$

$\mu - \beta(N-1)/2 > 0$:

$$P(z_1, \cdots, z_N) \underset{k\to 0}{\longrightarrow} \mathrm{Wishart}$$

$\mu - \beta(N-1)/2 < 0$

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Condensation of n charges to $z = 0$

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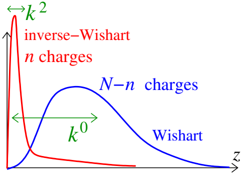

Condensation of n charges to $z = 0$

$k \to 0$: two *uncorrelated* sets of "charges"

$$\Delta(\infty) = \lim_{k\to 0} \frac{1}{2} \sum_n \langle \mathrm{sign}(z_n - k) \rangle$$

- Case $\theta = \mu - \beta(N-1)/2 > 0$

- Case $\theta = \mu - \beta(N-1)/2 \in]-n\beta, -(n-1)\beta[$

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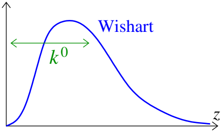

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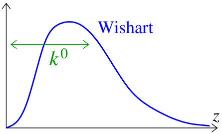

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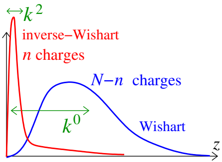

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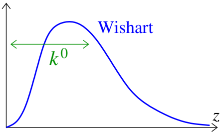

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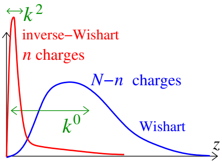

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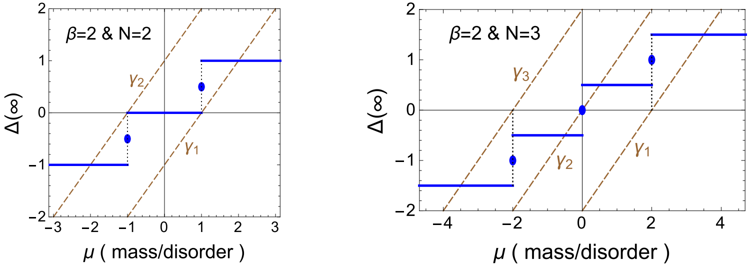
Witten index and phase diagram

Witten index



Phase diagram : plane (m_0, g)

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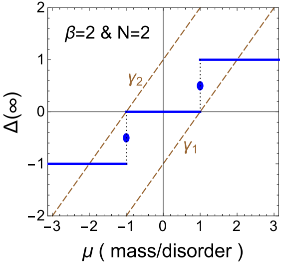

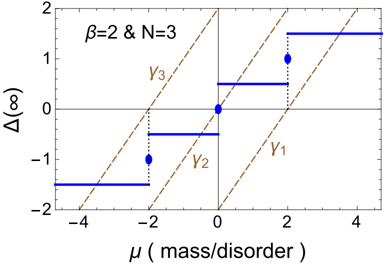

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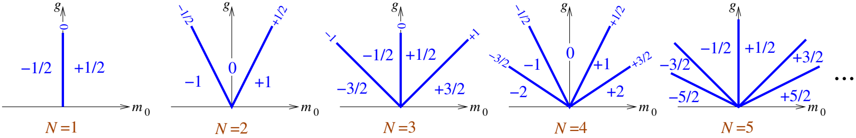

Plan

Conclusions

$$\mathcal{H}_D = \mathrm{i}\,\sigma_2 \otimes \mathbf{1}_N\,\partial_x + \sigma_1 \otimes M(x)$$

- Relation to Random Matrix Products $\Pi_N = M_N \cdots M_1$

$$M_n = \begin{pmatrix} \mathbf{1}_N \cos\theta & -\mathbf{1}_N \sin\theta \\ \mathbf{1}_N \sin\theta & \mathbf{1}_N \cos\theta \end{pmatrix} \begin{pmatrix} e^W & 0\,\mathbf{1}_N \\ 0\,\mathbf{1}_N & e^{-W} \end{pmatrix} \quad \in \ \mathrm{Sp}(2N)$$

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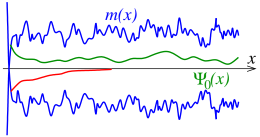
Lyapunov spectrum ($\varepsilon = 0$) : $\gamma_n = \left[\mu - \frac{\beta}{2}(N - 2n + 1)\right] g$



with $n \in \{1, \cdots, N\}$

- 1D : two symmetry classes (D, DIII) $\to \mathbb{Z}_2$-insulators

  Morimoto, Furusaki, Mudry, PRB (june 2015)

  $\to$ application of our method ?

- No isotropy among channels ($G \neq \mathbf{1}_N$).

  Most general : $P[M(x)] \propto \exp\left[-\int \mathrm{d}x \, \mathrm{Tr}\left\{A^{-1}M(x)B^{-1}M(x)\right\}\right]$

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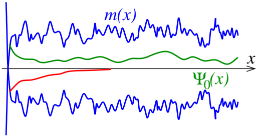

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Thank you.

A. Grabsch & C. Texier, preprint cond-mat arXiv:1506.05322

Appendices

Appendix A : Matricial Fokker-Planck equation

$$(*) \qquad \boxed{Z' = -Z^2 - E - \mu G Z - \mu Z G - Z\widetilde{M} - \widetilde{M} Z} \qquad \text{with } E = \varepsilon^2$$

Case $\beta = 1$ (class BDI)

$$\widetilde{M}_{ab}(x) = \sqrt{\sigma_{ab}}\ \underbrace{\zeta_{ab}(x)}_{\text{normalised}} \qquad \text{with } \sigma_{ab} \stackrel{\text{def}}{=} \begin{cases} g_a & \text{if } a = b, \\ \frac{g_a g_b}{g_a + g_b} & \text{if } a \neq b. \end{cases}$$

$$(*) \ \Rightarrow \qquad Z'_{mn} = \left[-Z^2 - E - \mu G Z - \mu Z G\right]_{mn} + \sum_{k \leqslant l} B_{mn,kl}(Z)\, \zeta_{kl},$$

where

$$B_{mn,kl}(Z) \stackrel{\text{def}}{=} -\frac{2 - \delta_{kl}}{2} \sqrt{\sigma_{kl}}\, (Z_{mk}\delta_{nl} + Z_{ml}\delta_{nk} + Z_{ln}\delta_{km} + Z_{kn}\delta_{lm})$$

Forward generator $\mathscr{G}^\dagger$

Define

$$\left(\frac{\partial}{\partial Z}\right)_{mn} \stackrel{\text{def}}{=} \frac{1+\delta_{mn}}{2}\frac{\partial}{\partial Z_{mn}},$$

and

$$\tilde{\sigma}_{ab} \stackrel{\text{def}}{=} \frac{2-\delta_{ab}}{2}\sigma_{ab}.$$

$$\mathscr{G}^\dagger = 2\,\mathrm{Tr}\left\{\frac{\partial}{\partial Z}Z\left[\tilde{\sigma}\circ\left[\frac{\partial}{\partial Z}Z+\left(\frac{\partial}{\partial Z}Z\right)^T\right]\right]\right\} + \mathrm{Tr}\left\{\frac{\partial}{\partial Z}\left(Z^2+E+\mu GZ+\mu ZG\right)\right\},$$

$[A\circ B]_{mn} = A_{mn}B_{mn}$: Hadamard product

Stationary solution

$$\mathscr{G}^\dagger f(Z) = 0$$