Double contour integral formulas for the sum of GUE and one matrix model

Based on arXiv:1608.05870 with Tom Claeys, Arno Kuijlaars, and Karl Liechty

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Gaussian Unitary Ensemble, everyone knows it

Let $\mathcal{M}$ be an $n \times n$ Hermitian matrix, with diagonal entries in i.i.d. normal distribution $N(1,0)$, and upper-triangular entries in i.i.d. complex normal distribution, with real and imaginary parts in $N(0, \frac{1}{2})$. Then the random matrix $\mathcal{M}$ is in the **Gaussian Unitary Ensemble**.

Its eigenvalues are a **determinantal process**, described by the correlation functions, which are

$$R_k(x_1, \dots, x_k) = \det(K_n(x_i, x_j))_{i,j=1}^k,$$

where K_n is the **correlation kernel**

$$K_n(x,y) = \sum_{k=0}^{n-1} \frac{1}{\sqrt{2\pi} k!} H_k(x) e^{-x^2/4} H_k(y) e^{-y^2/4}.$$

It is well known (in [Abramowitz–Stegun])

$$H_k(x) = \frac{k!}{2\pi i} \oint_0 e^{xt - t^2/2} t^{-k-1} dt = \frac{1}{\sqrt{2\pi} i} \int_{-i\infty}^{i\infty} e^{(s-x)^2/2} s^k ds.$$

Double contour integral formula for GUE

Taking the sum with the help of the "telescoping trick", we have

$$K_n(x,y) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint_0 dt \frac{e^{(s-x)^2/2}}{e^{(t-y)^2/2}} \left(\frac{s}{t}\right)^n \frac{1}{s-t}.$$

The local statistics of the GUE, namely the limiting Airy distribution and limiting Sine distribution, can be derived by the saddle point analysis of the double contour integral formula.

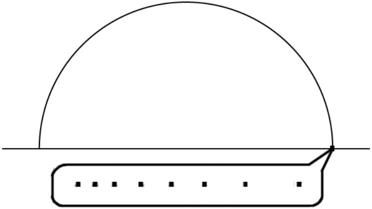


Figure: The space between consecutive eigenvalues is $O(n^{-2/3})$. Airy distribution.

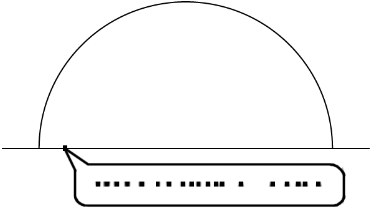


Figure: The space between consecutive eigenvalues is $O(n^{-1})$. Sine distribution.

Deformation of contours: Airy

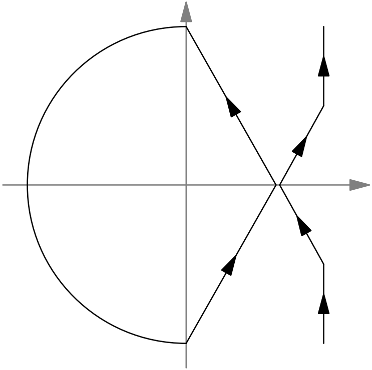


Figure: The deformed contours for Airy limit.

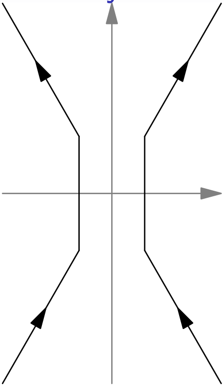


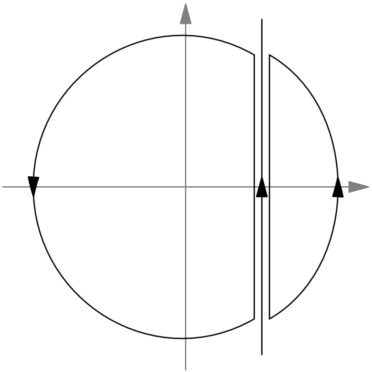
Figure: The limiting local shape of the contours.

The integral converges to

$$K_{\text{Airy}}(x, y) = \frac{1}{(2\pi i)^2} \int ds \int dt \frac{e^{s^3/3 - xs}}{e^{t^3/3 - yt}} \frac{1}{s - t}.$$

Deformation of contours: Sine

We need to deform the contour to a greater degree, and allow the vertical contour to cut through the circular one.



plus

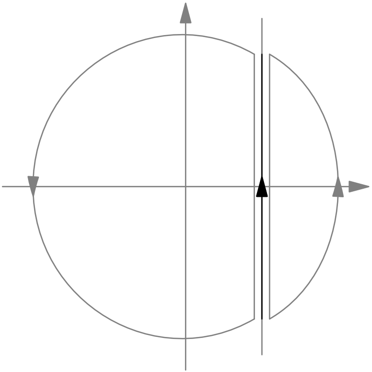


Figure: The deformed contours for Sine limit.

Figure: The contour for the residue integral (wrt s).

The two intersecting points are the saddle points. It turns out that the integral on the right is bigger, and gives the Sine kernel.

Other models solved by double contour integrals

1. GUE/Wishart with external source (GUE + $\mathrm{diag}(a_1, \dots, a_n)$) [Brezin–Hikami], [Zinn-Justin], [Tracy–Widom], [Baki–Peche–Ben Arous], [El Karoui], [Bleher–Kuijlaars]
2. GUE/Wishart minor process (all upper-left corners together) [Johansson–Nordenstam], [Dieker–Warren]
3. Upper-triangular ensemble (XX^*, upper-triangular entries of X are i.i.d. complex normal, diagonal ones are in independent gamma distribution) and the related Muttalib–Borodin model [Adler–van Moerbeke–W], [Cheliotis], [Forrester–W], [Zhang]
4. Determinantal particle systems (TASEP, polynuclear growth model, Schur process, etc) Johansson, Spohn, Borodin, Farrari, and too many others

Pros and cons

Pro When a double contour integral formula is obtained, the computation of asymptotics is a straightforward application of saddle point analysis.

Con All models are related to special functions which have their own contour integral representations. If the model is defined by functions that are not special, or not special enough, then there is little hope to find a double contour formula.

Pro Thanks to the recent development of models related to the product random matrices, we have abundant of models associated to special functions, namely Meijer-G functions. [all participants here, and many more]

Proof of Airy and Sine universality for the product of Ginibre matrices and the Muttalib–Borodin model

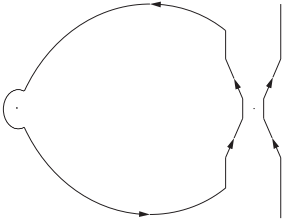


Figure: The deformed contours for Airy limit.

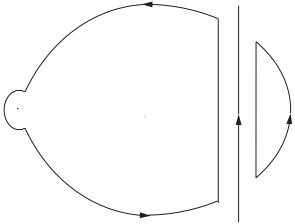


Figure: For Sine limit. (The residue integral is omitted).

The schematic figures applies for both the two models [Liu–W–Zhang], [Forrester–W]. The computation of the "hard edge" limit, which is more interesting, can be done in a technically easier way by double contour integral formula. [Kuijlaars–Zhang]

Review of one matrix model

Consider the n-dimensional random Hermitian matrix M with pdf

$$\frac{1}{C}\exp(-n \operatorname{Tr} V(M)), \quad V \text{ is a potential.}$$

The distribution of eigenvalues is a determinantal process. To express the kernels, we consider **orthogonal polynomials** with weight $e^{-nV(x)}$:

$$\int p_j(x)p_k(x)e^{-nV(x)}dx = \delta_{jk}h_k, \quad p_k(x) = x^k + \cdots .$$

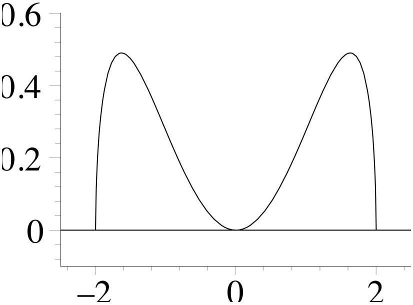
and then we have two equivalent kernels:

$$K_n^{\text{alg}}(x,y) = \frac{1}{h_{n-1}}\frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}e^{-nV(y)},$$

$$K_n(x,y) = \frac{1}{h_{n-1}}\frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x-y}e^{-\frac{n}{2}V(x)}e^{-\frac{n}{2}V(y)}.$$

Cases to be considered

We are interested in a particular potential function $V = x^4/4 - px^2$, especially if $p = 1$ or very close to 1, or more precisely, $p - 1 = \mathcal{O}(n^{-2/3})$. The density of eigenvalues is shown in the figure (from [Claeys–Kuijlaars]).



Note that at $x = 0$, the density vanishes like a square function, in contrast with the vanishing of density two edges that hase the square root behaviour. We say that $x = 0$ is an (interior) singular point of the potential $V(x) = x^4/4 - x^2$.

Solution to 1MM

We need to compute the limit of $K_n(x,y)$, which is reduced to the asymptotics of $p_n(x)$ and $p_{n-1}(x)$. They trick is that they satisfy the following Riemann–Hilbert problem for

$$Y(z) = \begin{pmatrix} \frac{1}{h_n} p_n(z) & \frac{1}{h_n} Cp_n(z) \\ -2\pi i h_{n-1} p_{n-1}(z) & -2\pi i h_{n-1} Cp_{n-1}(z) \end{pmatrix}$$

where

$$Cp_n(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{p_n(s)e^{-nV(s)}}{s-z} ds,$$

$$Cp_{n-1}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{p_n(s)e^{-nV(s)}}{s-z} ds,$$

such that

1. $Y_+(x) = Y_-(x)\begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$.
2. $Y(z) = (1 + \mathcal{O}(z^{-1}))\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.

- The asymptotics of Y is obtained by the Deift–Zhou nonlinear steepest-descent method [Bleher–Its], [DKMVZ].
- Around the singular point 0,

$$K_n(n^{-1/3}x, n^{-1/3}y) \sim K_{\mathrm{PII}}(x, y) = \frac{\Phi_1(x;\sigma)\Phi_2(y;\sigma) - \Phi_2(x;\sigma)\Phi(y;\sigma)}{\pi(x-y)},$$

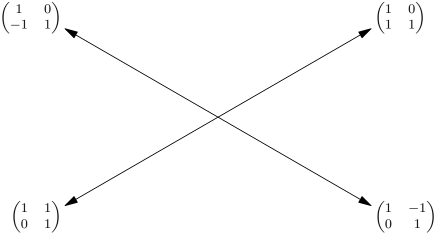
where σ is propotional to $n^{2/3}(p-1)$, and ($\psi^{(1)}$ and $\psi^{(2)}$ are defined in next slide)

$$\Phi_1(x;\sigma) = \psi_1^{(1)}(x;\sigma) + \psi_1^{(2)}(x;\sigma),$$
$$\Phi_2(x;\sigma) = \psi_2^{(1)}(x;\sigma) + \psi_2^{(2)}(x;\sigma).$$

2x2 Riemann–Hilbert problem associated to the Hastings–McLeod solution to Painlevé II

Let Ψ be a 2x2 matrix valued function, such that

1. Ψ is analytic on $\mathbb{C}\backslash$ the four rays and continuous up to the boundary.
2. $\Psi_+ = \Psi_- A_j$ on each ray, where the jump matrix A_j is given in the figure.
3. $\Psi(\zeta) = \Psi(\zeta;\sigma) = (I + \mathcal{O}(\zeta^{-1})e^{-i(\frac{4}{3}\zeta^3+\sigma\zeta)\sigma_3}$ as $\zeta \to \infty$, where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.



Now denote the $\Psi(\zeta;\sigma)$ in the left and right sectors by $(\psi^{(1)}(\zeta;\sigma), \psi^{(2)}(\zeta;\sigma))$, where $\psi^{(1)}$ and $\psi^{(2)}$ are 2-vectors. (Yes, Ψ in these two sectors are identical.)

Review of two matrix model

- In the general form, the two matrix model has two $n \times n$ random Hermitian matrices M_1, M_2 with pdf

$$\frac{1}{C} \exp[-n \operatorname{Tr}(V(M_1) + W(M_2) - \tau M_1 M_2)],$$

where V, W are potentials and τ is the interaction factor.

- We are interested in the case $V(x) = x^2/2$ and $W(y) = y^4/2 + (\alpha/2)y^2$, and are interested in the distribution of the eigenvalues of M_1. The distribution of eigenvalues of M_2 is much easier, and we are going to explain it below.
- Then the eigenvalues of M_1 are a determinantal process, with the correlation kernel

$$K_n(x, y) = \frac{(0, w_{0,n}(y), w_{1,n}(y), w_{2,n}(y)) Y_+^{-1}(y) Y_+(x)(1, 0, 0, 0)^T}{2\pi i(x - y)},$$

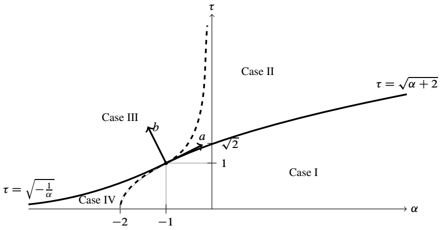
where Y is defined by a RHP in next slide.

4×4 Riemann–Hilbert problem

Consider the following Riemann–Hilbert problem

1. $Y_+(x) = Y_-(x) \times \begin{pmatrix} 1 & w_{0,n}(x) & w_{1,n}(x) & w_{2,n}(x) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, for $x \in \mathbb{R}$, where the exact formulas of $w_{i,n}(x)$ are omitted.
2. As $z \to \infty$, $Y(z) = (1 + \mathcal{O}(z^{-1})) \times \mathrm{diag}(z^n, z^{-n/3}, z^{-n/3}, z^{-n/3})$.

The RHP seems not too bad, while how hard it is depends on the value of τ and α in the following phase diagram (Duits, Geudens, Kuijlaars, Mo, Delvaux, Zhang ..., figure from [Duits]):



2MM with quadratic potential

Consider the 2MM with pdf $C^{-1}\exp[-n\,\mathrm{Tr}(M_1^2/2 + W(M_2) - \tau M_1 M_2)]$, that is, the potential V is quadratic. Then let

$$\tilde{W}(x) = W(x) - \frac{\tau^2}{2}x^2, \quad \text{and} \quad \tilde{M}_1 = M_1 - \tau M_2.$$

Then the joint pdf for $\tilde{M}_1$ and M_2 is

$$\frac{1}{C}\exp\left[-n\,\mathrm{Tr}(\tilde{M}_1^2/2 + \tilde{W}(M_2))\right],$$

and then $\tilde{M}_1$ and M_2 are independent. We can think M_1 as $\tilde{M}_1 + \tau M_2$. So the two matrix model with one quadratic potential is equivalent to the sum of a GUE matrix (i.e. a random matrix in 1MM with quadratic potential) and a random matrix in a 1MM [Duits].

Correlation functions for GUE + (fixed) external source

Let H be a GUE and A a fixed Hermitian matrix with eigenvalues $a_1, \ldots, a_n$, then the correlation kernel of the eigenvalues of $A + H$ is

$$\frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint dt \frac{e^{\frac{n}{2}(s-x)^2}}{e^{\frac{n}{2}(t-y)^2}} \prod_{k=1}^{n} \left(\frac{s - a_k}{t - a_k} \right) \frac{1}{s - t},$$

where Γ encloses $a_1, \ldots, a_n$.

Here we can allow $a_1, \ldots, a_n$ to be random, and need to integrate over the distribution of $a_1, \ldots, a_n$. How if they are eigenvalues of a matrix model too?

Correlation function for GUE + 1MM

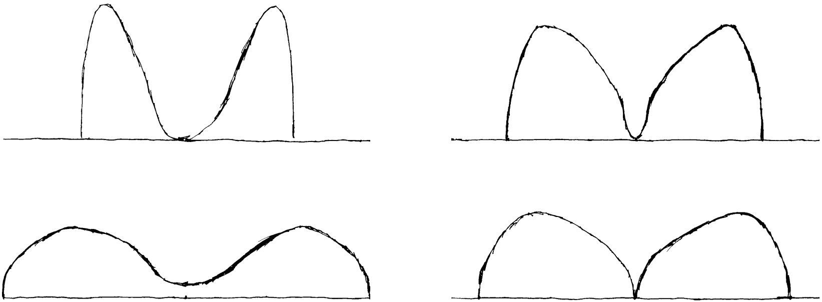
Theorem

[Claeys–Kuijlaars–W] Let M be a random matrix in 1MM, with random eigenvalues $a_1, \ldots, a_n$, then the correlation kernel of the eigenvalues of $M + H$ is

$$
\begin{aligned}
&K_{n,2\mathrm{MM}}(x,y) \\
&= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} ds \oint dt \frac{e^{\frac{n}{2}(s-x)^2}}{e^{\frac{n}{2}(t-y)^2}} \mathbb{E}\left[\prod_{k=1}^{n} \left(\frac{s-a_k}{t-a_k}\right)\right] \frac{1}{s-t} \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \int_{\mathbb{R}} dt \frac{e^{\frac{n}{2}(s-x)^2}}{e^{\frac{n}{2}(t-y)^2}} \underbrace{\frac{e^{-nV(t)}}{h_{n-1}} (p_n(s)p_{n-1}(t) - p_{n-1}(s)p_n(t)) \frac{1}{s-t}}_{K_n^{\mathrm{alg}}(s,t)} \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \int_{\mathbb{R}} dt \frac{e^{\frac{n}{2}[(s-x)^2+V(s)]}}{e^{\frac{n}{2}[(t-y)^2+V(t)]}} K_n(s,t).
\end{aligned}
$$

Result in figure

Suppose the distribution of eigenvalues in for M is given in the upper-left Figure, then as τ becomes larger, the distribution of eigenvalues of $M + \tau H$ evolves, shown in figures clockwise, into subcritical, critical), and then supercritical phases.



Below we consider $M + \sqrt{r} H$, whose kernel is given by

$$K^r_{n,2\mathrm{MM}}(x,y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \int_{\mathbb{R}} dt \frac{e^{\frac{n}{2}[(s-x)^2/r + V(s)]}}{e^{\frac{n}{2}[(t-y)^2/r + V(t)]}} K_n(s,t).$$

Derivation: subcritical

Suppose the critical value for r is $r_{\mathrm{cr}} \in (0, \infty)$. Then for $r \in (0, t_{\mathrm{cr}})$ there are c_r, c'_r depending on r in the way that as r runs from 0 to r_{cr}, then $c_r, c'_r \to 0$. such that for any $\xi, \eta \in \mathbb{R}$, if $x = c_t n^{-1/3}\xi$, $y = c_t n^{-1/3}\eta$, we have that the the functions $(s-x)^2/r + V(s)$ and $(t-y)^2/r + V(t)$ have the saddle point approximation

$$(s-x)^2/r+V(s) \sim c'_r n^{-2/3}(u-\xi)^2, \quad (t-y)^2/r+V(t) \sim c'_r n^{-2/3}(v-\eta)^2,$$

where $u = n^{1/3}s$, $v = n^{1/3}t$. Thus we have

$$\begin{aligned} K^r_{n,\mathrm{2MM}}(x,y) &\sim \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du \int_{\mathbb{R}} dv \frac{e^{\frac{n^{1/3}c'_r}{2}[(u-\xi)^2]}}{e^{\frac{n^{1/3}c'_r}{2}[(v-\eta)^2]}} K_n(n^{-1/3}u, n^{-1/3}v) \\ &\sim \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du \int_{\mathbb{R}} dv \frac{e^{\frac{n^{1/3}c'_r}{2}[(u-\xi)^2]}}{e^{\frac{n^{1/3}c'_r}{2}[(v-\eta)^2]}} K_{\mathrm{PII}}(u,v) \\ &\sim K_{\mathrm{PII}}(\xi,\eta). \end{aligned}$$

Derivation: critical

When $r = r_{\mathrm{cr}}$, both c_r and c'_r become 0, and the argument in last slide breaks down. However, we can still assume that as $r = r_{\mathrm{cr}} + n^{-1/3}\tau$, $x = n^{-2/3}\xi$, $y = n^{-2/3}\eta$, and have that

$$(s-x)^2/r+V(s) \sim n^{-1}(bu^2+\xi u), \quad (t-y)^2/r+V(t) \sim n^{-1}(bv^2-\eta v)^2,$$

where $u = n^{1/3}s$, $v = n^{1/3}t$ and b depends on τ. So we have

$$K^r_{n,2\mathrm{MM}}(x,y) \sim \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du \int_{\mathbb{R}} dv \frac{e^{bu^2-\xi u}}{e^{bv^2-\eta v}} K_{\mathrm{PII}}(u,v).$$

The problem is that the integral may not be well defined, even if we consider it formally. The reason is that the sign of b depends on the sign of τ, and can be either positive or negative, while as $v \to \pm\infty$, $K_{\mathrm{PII}}(u,v)$ does not vanish. (The correct form can be written down with the help of longer formulas, and we omit them.)

Result in formula

- Here we note that the PII singularity is quite robust. If M has the PII singularity, then $M + \sqrt{r}H$ has too, if $r < r_{\text{cr}}$.
- The critical kernel, the most interesting one, has the kernel [Claeys–Kuijlaars–Liechty–W] (formally)

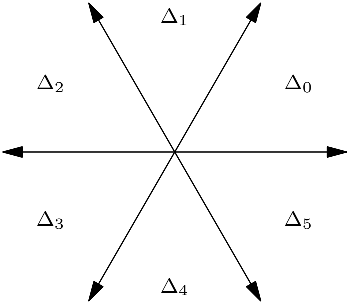
$$K^r_{n,2\text{MM}}(x,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} du \int_{\mathbb{R}} dv \, \frac{e^{au^3+bu^2+\xi u}}{e^{av^3+bv^2+\eta v}} K_{\text{PII}}(u,v).$$

- If the potential is symmetric, then parameter a vanishes, as we discussed in previous slide. But our method allows us to consider asymmetric potentials, and generally there is a cubic term in the exponents.
- We can also deal with higher singularities, or singularities at the edge.
- The equivalence to the previous result by 4×4 RHP is obtained [Liechty-W].

Tacnode Riemann–Hilbert problem

Let M be a 4×4 matrix-valued function, and suppose it satisfies the following Riemann–Hilbert problem:

$$J_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix} \qquad J_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$J_3 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad J_0 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$J_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \qquad J_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. M is analytic in each of the sectors Δ_j, continuous up to the boundaries, and $M(z) = \mathcal{O}(1)$ as $z \to 0$.
2. On the boundaries of the sectors Δ_j, $M = M^{(j)}$ satisfies the jump conditions

$$M^{(j)}(z) = M^{(j-1)}(z) J_j, \quad \text{for } j = 0, \dots, 5, \quad M^{(-1)} \equiv M^{(5)},$$

for the jump matrices $J_0, \dots, J_5$ with constant entries specified in the figure in last page.
3. As $z \to \infty$, $M(z)$ satisfies the asymptotics

$$M(z) = \left(1 + \mathcal{O}(z^{-1})\right) \left(v_1(z), v_2(z), v_3(z), v_4(z)\right),$$

where v_1, v_2, v_3, and v_4 are defined as

$$\begin{aligned} v_1(z) &= \frac{1}{\sqrt{2}} e^{-\theta_1(z) + \tau z} \left((-z)^{-\frac{1}{4}}, 0, -i(-z)^{\frac{1}{4}}, 0\right)^T, \\ v_2(z) &= \frac{1}{\sqrt{2}} e^{-\theta_2(z) - \tau z} \left(0, z^{-\frac{1}{4}}, 0, i z^{\frac{1}{4}}\right)^T, \\ v_3(z) &= \frac{1}{\sqrt{2}} e^{\theta_1(z) + \tau z} \left(-i(-z)^{-\frac{1}{4}}, 0, (-z)^{\frac{1}{4}}, 0\right)^T, \\ v_4(z) &= \frac{1}{\sqrt{2}} e^{\theta_2(z) - \tau z} \left(0, i z^{-\frac{1}{4}}, 0, z^{\frac{1}{4}}\right)^T, \end{aligned}$$

where

$$\begin{aligned} \theta_1(z) &= \frac{2}{3} r_1 (-z)^{\frac{3}{2}} + 2 s_1 (-z)^{\frac{1}{2}}, \\ & z \in \mathbb{C} \setminus [0, \infty), \\ \theta_2(z) &= \frac{2}{3} r_2 z^{\frac{3}{2}} + 2 s_2 z^{\frac{1}{2}}, \\ & z \in \mathbb{C} \setminus (-\infty, 0], \end{aligned}$$

Integral representation

Then define the 4-vectors

$$n^{(k)}(z) = n^{(k)}(z; r_1, r_2, s_1, s_2, \tau) = \mathcal{Q}_{\Gamma^{(k)}}(f^{(k)}, g^{(k)}), \quad k = 0, \dots, 5,$$

where

$$\mathcal{Q}_\Gamma(f,g)(z) :=$$

$$\mathcal{M}\begin{pmatrix} \int_{\Gamma_1} e^{\frac{2iz\zeta}{C}} f_1(\zeta)G_1(\zeta)d\zeta + \int_{\Gamma_2} e^{\frac{2iz\zeta}{C}} g_1(\zeta)G_1(\zeta)d\zeta + \int_{\Gamma_3} e^{\frac{2iz\zeta}{C}} (f_1(\zeta)+g_1(\zeta))G_1(\zeta)d\zeta \\ \int_{\Gamma_1} e^{\frac{2iz\zeta}{C}} f_2(\zeta)G_2(\zeta)d\zeta + \int_{\Gamma_2} e^{\frac{2iz\zeta}{C}} g_2(\zeta)G_2(\zeta)d\zeta + \int_{\Gamma_3} e^{\frac{2iz\zeta}{C}} (f_2(\zeta)+g_2(\zeta))G_2(\zeta)d\zeta \\ \int_{\Gamma_1} e^{\frac{2iz\zeta}{C}} f_1(\zeta)G_3(\zeta)d\zeta + \int_{\Gamma_2} e^{\frac{2iz\zeta}{C}} g_1(\zeta)G_3(\zeta)d\zeta + \int_{\Gamma_3} e^{\frac{2iz\zeta}{C}} (f_1(\zeta)+g_1(\zeta))G_3(\zeta)d\zeta \\ \int_{\Gamma_1} e^{\frac{2iz\zeta}{C}} f_2(\zeta)G_4(\zeta)d\zeta + \int_{\Gamma_2} e^{\frac{2iz\zeta}{C}} g_2(\zeta)G_4(\zeta)d\zeta + \int_{\Gamma_3} e^{\frac{2iz\zeta}{C}} (f_2(\zeta)+g_2(\zeta))G_4(\zeta)d\zeta \end{pmatrix},$$

and

$$\mathcal{M} = e^{-\tau z\left(\frac{r_1^2-r_2^2}{r_1^2+r_2^2}\right)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{i}{r_1}\left(\tau\frac{r_1^2-r_2^2}{r_1^2+r_2^2} + \tau - s_1^2 + \frac{u}{C}\right) & \frac{i}{r_1}\frac{\sqrt{r_2}q}{\gamma\sqrt{r_1}C} & \frac{-i}{r_1} & 0 \\ \frac{-i}{r_2}\frac{\gamma\sqrt{r_1}q}{\sqrt{r_2}C} & \frac{i}{r_2}\left(\tau\frac{r_1^2-r_2^2}{r_1^2+r_2^2} - \tau + s_2^2 - \frac{u}{C}\right) & 0 & \frac{-i}{r_2} \end{pmatrix},$$

such that (C is defined below)

$$a = \frac{4}{3}\left(\frac{r_1^2 - r_2^2}{r_1^2 + r_2^2}\right), \quad b = \frac{8\tau}{C^2(r_1^2 + r_2^2)}, \quad c = \frac{1}{C}\left[\frac{4\tau^2(r_1^2 - r_2^2)}{(r_1^2 + r_2^2)^2} - 2\left(\frac{s_1}{r_1} - \frac{s_2}{r_2}\right)\right],$$

$$\gamma_1 = \exp\left(-\frac{8r_1^4\tau^3}{3(r_1^2 + r_2^2)^3} + \frac{4r_1 s_1 \tau}{r_1^2 + r_2^2}\right), \quad \gamma_2 = \exp\left(-\frac{8r_2^4\tau^3}{3(r_1^2 + r_2^2)^3} + \frac{4r_2 s_2 \tau}{r_1^2 + r_2^2}\right),$$

and then the function

$$G(\zeta) = \exp\left(ia\zeta^3 + b\zeta^2 + ic\zeta\right),$$

and the related functions

$$G_1(\zeta) = \sqrt{\frac{2}{\pi}}\frac{\gamma_1}{C\sqrt{r_1}}G(\zeta), \quad G_2(\zeta) = \sqrt{\frac{2}{\pi}}\frac{\gamma_2}{C\sqrt{r_2}}G(\zeta), \quad G_3(\zeta) = \frac{2i}{C}\zeta G_1(\zeta), \quad G_4(\zeta) = \frac{2i}{C}\zeta G_2(\zeta).$$

The entries of $\mathcal{M}$ are expressed in

$$C = (r_1^{-2} + r_2^{-2})^{1/3}, \qquad \gamma = \exp\left(\frac{8}{3}\frac{r_1^2 - r_2^2}{(r_1^2 + r_2^2)^2}\tau^3 - 4\frac{r_1 s_1 - r_2 s_2}{r_1^2 + r_2^2}\tau\right),$$

and q and u are functions of

$$\sigma := \frac{2}{C}\left(\frac{s_1}{r_1} + \frac{s_2}{r_2} - \frac{2\tau^2}{r_1^2 + r_2^2}\right).$$

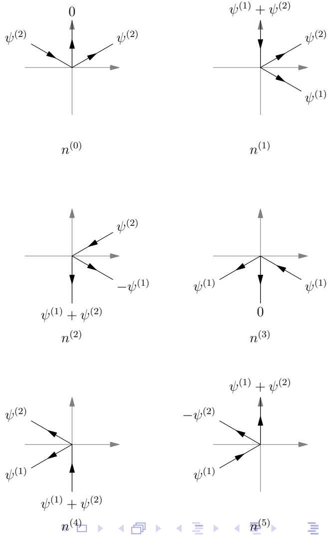
Furthermore, $q = q(\sigma)$ satisfies the Painlevé II equation with Hastings–McLeod initial condition (Ai is the Airy function)

$$q''(\sigma) = \sigma q + 2q^3, \quad q(\sigma) \sim \mathrm{Ai}(\sigma) \quad \text{as } \sigma \to +\infty,$$

$q' = q'(\sigma)$ is the derivative with respect to σ, and u is the PII Hamiltonian

$$u(\sigma) := q'(\sigma)^2 - q(\sigma)^2 - q(\sigma)^4.$$

At last, we can specify the contours $\Gamma_j^{(k)}$ and functions $f^{(k)}$ and $g^{(k)}$ in the integrands as in the figure.



Theorem

[Liechty-W] The 4x4 RHP M can be expressed by $n^{(k)}$, the integrals involving entries of Ψ.

$$M^{(0)} = \left(n^{(5)} - n^{(0)}, n^{(0)}, n^{(1)}, -n^{(2)}\right),$$

$$M^{(1)} = \left(-n^{(3)}, n^{(0)}, n^{(1)}, -n^{(2)}\right),$$

$$M^{(2)} = \left(-n^{(3)}, -n^{(4)}, n^{(1)} + n^{(2)}, -n^{(2)}\right),$$

$$M^{(3)} = \left(-n^{(3)}, -n^{(2)} - n^{(3)}, -n^{(5)}, n^{(4)}\right),$$

$$M^{(4)} = \left(-n^{(3)}, n^{(0)}, -n^{(5)}, n^{(4)}\right),$$

$$M^{(5)} = \left(n^{(1)}, n^{(0)}, -n^{(5)}, n^{(4)} + n^{(5)}\right).$$