Renormalization freedom in curved spacetime

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Outline of the talk

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    - The tadpole in Minkowski spacetime: interaction profile
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Motivations

- Out-of-equilibrium phenomena played an important role during the evolution of the universe
  - Dark matter freeze-out
  - At the inflationary epoch
- In these phenomena one is typically interested in the time evolution of expectation values of observables rather than calculating scattering processes using the S-matrix
- The appropriate framework is the CTP formalism, first developed by Schwinger and Keldysh that allows to choose an arbitrary initial state and to follow its causal evolution including quantum effects

Motivations...

- By considering QFTs with time dependent backgrounds (e.g. quasi de Sitter), the time translation symmetry of the Lagrangian is broken

▶ Time-dependent freedom in the renormalization
  - In the Energy-Momentum Tensor [Baacke, Covi et al, '10]
  - In composite operators [Dresti, Riotto, '13]
- The arbitrariness of the counterterms can have physical implications in out-of equilibrium phenomena
- Better understanding of the formalism
  - Renormalization freedom in the context of distributions
  - Non covariance in the CTP formalism

Renormalization in spacetime: extension of distributions

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ or $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, $u_\lambda(x \mapsto \phi(x)) := u\left(x \mapsto \lambda^{-n}\phi\left(\frac{x}{\lambda}\right)\right)$

Definitions [Steinmann, '71]

- Scaling Degree: $\mathrm{sd}(u) := \inf_\omega \{\omega\colon \lim_{\lambda\to 0} \lambda^\omega u_\lambda = 0\}$
- Singular Order/Divergence Degree: $\mathrm{div}(u) := \mathrm{sd}(u) - n$

Theorem [Epstein, Glaser, '73]

- If $\mathrm{div}(u) < 0$ there exists a unique extension $\tilde{u}$ such that $\mathrm{sd}(\tilde{u}) = \mathrm{sd}(u)$
- If $0 \leq \mathrm{div}(u) < \infty$ there exist extensions $\tilde{u}$ of u with $\mathrm{sd}(\tilde{u}) = \mathrm{sd}(u)$ and they differ by a term $\sum_{|\alpha|\leq \mathrm{div}(u)} c_\alpha \delta^{(\alpha)}$

Curved spacetime: extension of distributions on manifolds

Renormalization in spacetime: 1st example

The singular structure of Δ_F

- The Green's function $\Delta_F = \frac{1}{4\pi^2}\frac{1}{x^2}$ is singular at $x = 0$, i.e

$$\{0\} \in \operatorname{sing\,supp}(\Delta_F) = \{x \in \mathbb{R}^4 \colon \nexists U_x \,/\, \Delta_F|_{U_x} \in \mathcal{C}^\infty(U_x)\}$$

- $\mathrm{sd}(\Delta_F) = 2 \Rightarrow \mathrm{div}(\Delta_F) = -2$

The product of two Δ_F distributions

- The product Δ_F^2 is ill-defined for functions that have support at 0
- $\Delta_F^2 \in \mathcal{D}'(\mathbb{R}^4 \setminus \{0\})$
- $\mathrm{sd}(\Delta_F^2) = 4 \Rightarrow \mathrm{div}(\Delta_F^2) = 0$

Then we can conclude that

- The extension of Δ_F^2 to $\mathcal{D}(\mathbb{R}^4)$ exists but is not unique
- Let $\tilde{\Delta}_F^2$ and $\bar{\Delta}_F^2$ be two extensions, then there exists a constant c such that $\tilde{\Delta}_F^2 = \bar{\Delta}_F^2 + c\delta$

Renormalization in spacetime: 2nd example

Let $\Delta_F(x-y) \propto \frac{1}{|x-y|^2}$. The tadpole can be seen as a product of two singular distributions

$$\propto \int dz^4 \Delta_F(x-z)\Delta_F(z-z)\Delta_F(z-y)$$

$$\propto \int dz_1^4\, dz_2^4\, \Delta_F(x-z_1)\left[\Delta_F(z_1-z_2)\delta_4(z_1-z_2)\right]\Delta_F(z_2-y)$$

The scaling degree is 6 and the degree of divergence is 2. This means that the extension is not unique and that after a choice of the scheme the freedom is

$$c_0\delta_4 + c_1\delta_4' + c_2\delta_4'',$$

where the free constants c_i should depend on m^2 [Hollands, Wald, '01].

Renormalization in momentum space

$$\propto \int_{t_{in}}^{t} dt_1 \left[\left(\frac{-i\lambda}{2}\right)\int \frac{dp^3}{(2\pi)^3} F(p,t_1,t_1)\right] \times$$
$$\times \left[\left(-iG^R(k,t,t_1)\right) F(k,t_1,t) + F(k,t,t_1)\left(-iG^A(k,t_1,t)\right)\right]$$

The divergent part is $\int \frac{dp^3}{(2\pi)^3} F(p,t_1,t_1)$.

Renormalization

- Regularization: $\int^{\Lambda} \frac{dp^3}{(2\pi)^3} F(p,t_1,t_1)$
- Identify the divergence: $\left(\frac{-i\lambda}{16\pi^2}\right)\Lambda^2 \; - C$
- Renormalization: $\int^{\Lambda} \frac{dp^3}{(2\pi)^3} F(p,t_1,t_1) - [\;\;]_{\mathrm{div}} = \frac{C\lambda}{128\pi^2 w_k^3}$

The tadpole in Minkowski spacetime: CTP formalism

$\Big($  $\Big)_{\mathrm{CTP}} = \Big($  $\Big)_{\mathrm{STD}} + \dfrac{\lambda m^2 C}{128\pi^2 w_k^3}\cos\left(2w_k(t - t_{in})\right)$

Is the extra term physical?

Properties

- It can be canceled with a proper choice of the scheme (i.e. $C = 0$)
- It depends on the elapsed time
- It is not covariant

Since the spacetime is time-translational invariant we expect to recover the Poincaré symmetry for $t - t_{in} \gg 0$ The extra contribution is an artifact of the fact that the interaction profile contains a Heaviside distribution, i.e. $\lambda(t) = \lambda\,\theta(t - t_{in})$

Constructing the coupling constant profile...

We are looking for a construction of the profile that is 0 for $t < t_{in}$ and 1 for $t > t_{in} + \Delta t$. In the intermediate region $t_{in} < t < t_{in} + \Delta t$ the coupling is switched-on continuously.

Let f be a function of regularity $\mathcal{C}^n(\mathbb{R})$ which is 0 for $t < t_{in}$, then the extension is given by

$$g(t) = \frac{f(t - t_{in})}{f(t - t_{in}) + f(\Delta t - t + t_{in})}$$

Example, $f : x \mapsto x^3$

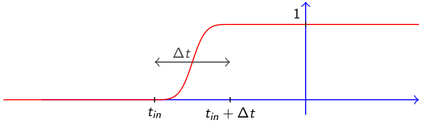


Figure: $\mathcal{C}^2(\mathbb{R})$ extension

The tadpole in Minkowski spacetime: interaction profile

Time-dependence of the non covariant term in the renormalization of the tadpole (with Hadamard propagators):

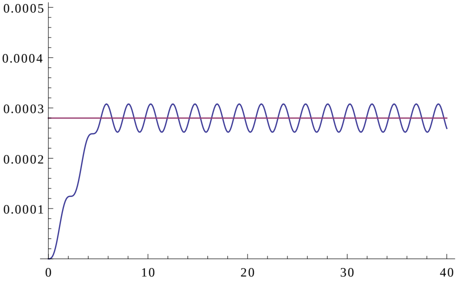


Figure: linear and quadratic transient extension for $\Delta t = 5$. The amplitude tends to a constant value (for large Δt) which is profile-independent

The tadpole with a linear profile

With a linear profile:

$$\overset{t>t_{in}+\Delta t}{=} \frac{C\lambda}{256\pi^2}\left[\frac{2}{w_k^3} + \frac{\sin\left(2w_k(t-t_{in}-\Delta t)\right) - \sin\left(2w_k(t-t_{in})\right)}{\Delta t\, w_k^4}\right]$$

Properties

- In the limit $\Delta t \to \infty$ the tadpole tends to the constant value $\frac{C\lambda}{128\pi^2 w_k^3}$
- In the limit $\Delta t \to 0$ we recover the usual result with the θ-profile

Considerations

- The constant value seems to be profile-independent (different profiles tested)
- The t_{in} dependence is an artifact of the switching-on profile

Curved spacetime: how can we recognize a true time-dependence when we break the time-translational symmetry?

The tadpole in de Sitter spacetime

Tadpole in de Sitter scale for external legs taken at lowest order in k

$$= \frac{\lambda H^2}{48\pi^2 k^3}\frac{1}{2}\left[\left(\frac{1}{\epsilon}-\frac{C}{H^2}+\log\left(\frac{\mu^2}{H^2}\right)\right)\left(\log\left(\frac{\tau}{\tau_{in}}\right)+\frac{1}{3}-\frac{\tau^3}{3\tau_{in}^3}\right)\right]$$

General result

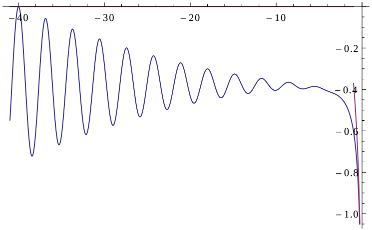


Figure: Time-dependence of the tadpole assuming $\tau_{in} = -40$ and $k = 1$

- Using different profiles it is possible to get rid to the rational dependence in τ_{in}
- The result has in any case a logarithmic dependence in time $\log(\tau)$

Conclusions

- Time-dependent contributions appear in the CTP formalism in Minkowski spacetime because the non-covariant interaction profile $\lambda(t) = \lambda\,\theta(t - t_{in})$
- The non-covariant time dependence can be removed by choosing a better profile
- In de Sitter spacetime one has in addition that the background depends on time (through the scale factor)
- In the analyzed case the time-dependent logarithm is the only term that survives after a proper choice of the interaction profile
- The algebraic approach predicts that the it should be possible to write the counterterms in a covariant form (i.e. as an analytic function of the curvature and the metric).

Backup

Scaling of a function/distribution

Let f be a function in $\mathbb{R}^n$.

Definitions

- Scaling Degree of f: $\mathrm{sd}(f) := \inf_\omega \{\omega\colon \lim_{\lambda\to 0} \lambda^\omega f(\lambda x) = 0\}$
- Distribution $u_f : g \mapsto \langle f, g\rangle$

Because duality, the definition of the scaling at the level of the test functions is

$$\begin{aligned}\langle x \mapsto \lambda^\omega f(\lambda x), g\rangle &= \int d^n x\, \lambda^\omega f(\lambda x) g(x)\\ &= \int d^n x\, f(x)\lambda^{\omega-n} g\left(\frac{x}{\lambda}\right) = \left\langle f, x\mapsto \lambda^{\omega-n} g\left(\frac{x}{\lambda}\right)\right\rangle\end{aligned}$$

Example: the function (and then the associated distribution) $x \mapsto x^{-2}$ has scaling degree 2, because $\lambda^\omega \frac{1}{(\lambda x)^2} \to 0$ for $\omega > 2$

Extension of a distribution

Let $u \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ and $\mathrm{div}(u) < 0$, the extension of u to $\mathcal{D}(\mathbb{R}^n)$ is constructed through a sequence of smooth functions

$$\theta_n = \begin{cases} 0 & , \quad x = 0 \\ 1 & , \quad x \in U^c \end{cases},$$

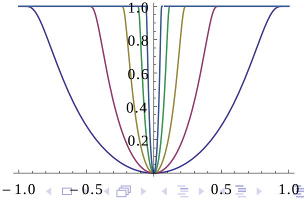
where U is a neighborhood of the origin such that

$$\theta_n \overset{n\to\infty}{\to} \begin{cases} 0 & , \quad x = 0 \\ 1 & , \quad \text{otherwise} \end{cases}$$

The extended distribution is given by $\tilde{u} = \lim_{n\to\infty} \theta_n u$ and is independent of the choice of the sequence $\{\theta_n\}_{n\in\mathbb{N}}$.

Example:

$$\theta_n(x) = 1 - e^{\frac{(2^n x)^2}{(2^n x)^2 - 1}} \chi_{[-2^{-n}, 2^{-n}]}$$



Extension for positive singular order $\omega > 0$

The extension theorem guarantees a unique extension on test function that vanish at the origin up to order ω

Definition (W-operation)

Let $\mathcal{D}^\omega(\mathbb{R}^n) \subset \mathcal{D}(\mathbb{R}^n)$ the subspace of function vanishing up to order ω at 0. The function W is a projection into that subspace

$$W_\omega \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}^\omega(\mathbb{R}^n), \quad \varphi \mapsto W_\omega \varphi$$

where

$$(W_\omega \varphi)(x) = \varphi(x) - w(x) \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{\alpha!} \left(\partial^\alpha \frac{\varphi}{w} \right)(0),$$

with $w \in \mathcal{D}(\mathbb{R}^n), w(0) \neq 0$.

Remark: The following relation holds $W_\omega^w(w\varphi) = w\, W_\omega^{x \mapsto 1}(\varphi)$

Extension for positive singular order - theorem

Theorem [Brunetti, Fredenhagen, '97]

Let $u_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ be a distribution with singular order ω. Given a W_ω-operation and constants $C^\alpha \in \mathbb{C}$, then there is one distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ with singular order ω such that

- $\langle u, \varphi \rangle = \langle u_0, \varphi \rangle\,,\ \forall \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$
- $\langle u, wx^\alpha \rangle = C^\alpha,\ \alpha \leq \omega.$

u is then given by

$$\langle u, \varphi \rangle = \left\langle u_0^{\mathrm{ext}}, W_\omega \varphi \right\rangle + \sum_{|\alpha| \leq \omega} \frac{C^\alpha}{\alpha!} \left(\partial^\alpha \frac{\varphi}{w} \right) (0)$$

Remark: w is a function used to make quantities like $\langle u, wx^\alpha \rangle$ meaningful

Wavefront Set

Definition (Wavefront Set)

For a distribution $u \in \mathcal{D}'(\mathcal{U})$ the wavefront set $\mathrm{WF}(u)$ is the complement in $\mathcal{U} \times \mathbb{R}^n \setminus \{0\}$ of the set of points $(x, \xi) \in \mathcal{U} \times \mathbb{R}^n \setminus \{0\}$ such that there exist

- a function $f \in \mathcal{D}(\mathcal{U})$ with $f(x) = 1$
- an open conic neighborhood Γ of ξ with

$$\sup_{\xi \in \Gamma}(1 + |\xi|)^N |\widehat{f \cdot u(\xi)}| < \infty, \ \forall N \in \mathbb{N}_0$$

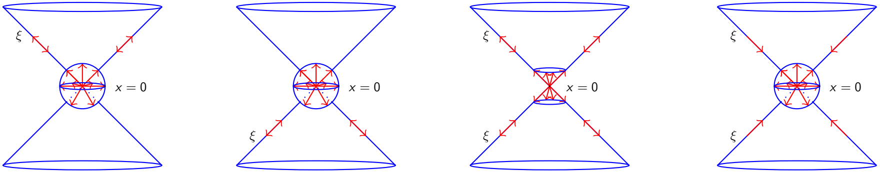
- The Wave Front Set contains the information of the singular structure of a distribution
- The WF is a local concept, it does not depend on the global structure of $\mathbb{R}^n$
- The WF is related to the possibility to define the product of two distributions when in general it is not allowed

Wavefront Set of propagators

Reminder

- $\mathrm{WF}(u)|_x = \mathrm{sing\,supp}(u)$
- $\mathrm{WF}(u)|_\xi = \Sigma(u)$ is the frequency set of u

$\Delta_R = \theta(x_0)\Delta$ $\quad$ $\Delta_A = -\theta(-x_0)\Delta$ $\quad$ $\Delta = i[\phi(f), \phi(g)]$ $\quad$ $\Delta_F = i\langle \mathcal{T}\phi(f), \phi(g)\rangle$



The Wavefront Set of the usual propagators are

- $\mathrm{WF}(\tilde{\Delta}_{R/A}) = (\{0\} \times (\mathbb{R}^n \setminus \{0\})) \cup \{(x,\xi) \colon x \in V, \pm x_0 > 0, \xi = \lambda x, \lambda \in \mathbb{R}, \xi \neq 0\}$
- $\mathrm{WF}(\tilde{\Delta}_F) = (\{0\} \times (\mathbb{R}^n \setminus \{0\})) \cup \{(x,\xi) \colon x \in V, \xi = \lambda x, \lambda < 0, \xi \neq 0\}$
- $\mathrm{WF}(\tilde{\Delta}) = \{(x,\xi) \colon 0 \neq \xi \in V, x = \lambda\xi, \lambda \in \mathbb{R}\}$

Wavefront Set: Product of distributions

Product of functions: The usual multiplication of functions $f_1, f_2 \in \mathcal{C}(\mathcal{U})$ can be understood as the tensor product $f_1 \otimes f_2$ restricted to the diagonal in $\mathcal{U} \times \mathcal{U}$, i.e. $(f_1 \otimes f_2)(x, x) = f_1(x) f_2(x)$

Corollary [Hormander criterion]

The product $u_1 \, u_2$ of two distributions $u_1, u_2 \in \mathcal{D}'(\mathcal{U})$ can be defined as the restriction of $u_2 \otimes u_1$ to the diagonal if the following condition is satisfied

$$(x, \xi) \in \mathrm{WF}(u_1) \Rightarrow (x, -\xi) \notin \mathrm{WF}(u_2)$$

- The corollary says that we can define a meaningful product of two distributions because the growth of u_1 in the ξ direction is compensated by the decay of u_2 in the $-\xi$ direction
- When the *Hormander criterion* is not satisfied it might be possible to define the product of two distributions

Causal Perturbation Theory

Formal S-Matrix power series

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n T_n(x_1, \dots x_n) g(x_1) \cdots g(x_n)$$

- $g \in \mathcal{D}(\mathbb{R}^4)$ play the role of the coupling constant
- T_n are time ordered functions and can be computed recursively
- $T_1 = \mathcal{L}_{int} = -\frac{\lambda}{4!} : \phi^4(x) :$

We can use Wick's theorem and the extension of distributions to estimate the S-Matrix

Causal Perturbation Theory - Wick ordering

Definition (Wick ordering)

A normal ordering prescription : : for the operator ϕ is defined by the following recursion relation

$$:\phi: = \phi$$

$$:\phi(x_1)\cdots\phi(x_{n+1}): = :\phi(x_1)\cdots\phi(x_n):\phi(x_{n+1}) - \\ - \sum_l :\phi(x_1)\cdots\hat{\phi}(x_l)\cdots\phi(x_n): \omega_2(x_l, x_{n+1}),$$

where $\hat{}$ means omitting the corresponding element and ω_2 denotes the two-point distribution of the state ω

Remark: $:\phi(x)\phi(y): = \phi(x)\phi(y) - \omega_2(x,y)$ is well defined since the divergent contribution is canceled in the coinciding-point limit

Causal Perturbation Theory - T_2 in ϕ^4-theory

Example: $T_2(x,y)$

- $T_1(x) = -\frac{\lambda}{4!} :\phi^4(x):$
- $T_2(x,y) = \theta(x-y)\,T_1(x)\,T_1(y) + \theta(y-x)\,T_1(y)\,T_1(x)$

Property (e.g. [K. Hepp, '69])

$$:\phi(x)^r::\phi(y)^s: = \sum_{t=0}^{\min\{r,s\}} t! \binom{r}{t}\binom{s}{t} (i\omega_2(x-y))^t :\phi(x)^{r-t}\phi(y)^{s-t}:$$

$$\begin{aligned} T_2(x,y) = \frac{\lambda^2}{4!^2}\Big[&:\phi^4(x)\phi^4(y): + 16 i\Delta_F(x-y)\ :\phi^3(x)\phi^3(y): \\ &+ 72(i\Delta_F(x-y))^2 :\phi^2(x)\phi^2(y): \\ &+ 96(i\Delta_F(x-y))^3 :\phi(x)\phi(y): \\ &+ 24(i\Delta_F(x-y))^4\Big] \end{aligned}$$

Renormalization of composite operators

Statement

Let A be a bare local composite operator, consider all local operators B such that

- B has the same symmetry property of A
- $\mathrm{dim} B \leq \mathrm{dim} A$

then it is possible to define a divergent coefficient Z_{AB} for each B such that

$$A_R(x) \leq \sum_B Z_{AB} \{B(x) - \langle B \rangle\}$$

is finite[a].

[a] i.e. all the n-point functions $\langle A_R(x)\phi(x_1)\ldots\phi(x_n)\rangle$ are finite

GNS construction

Theorem (GNS construction)

Let ω be a state on a C^*-algebra $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}$, a representation π and a unit vector $\Omega \in \mathcal{H}$ such that $\mathcal{H} = \pi(\mathcal{A})\Omega$ and $\omega(A \in \mathcal{A}) = \langle \Omega, \pi(A)\Omega \rangle$

The representation $(\mathcal{H}, \langle \cdot, \cdot \rangle, \pi)$ is called the Gelfand–Naimark–Segal (GNS) representation

- $\mathcal{A}$ is the algebra of the operators
- A state on a C^*-algebra $\mathcal{A}$ is a linear functional $\tau \colon \mathcal{A} \to \mathbb{C}$
- $\mathcal{H}$ is the Hilbert space, $\mathcal{L}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$
- π is a representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$, i.e. $\pi \colon \mathcal{A} \to \mathcal{L}(\mathcal{H})$

QFT in the sense of distributions

Free scalar field theory ($P = \Box_g + m^2 + \xi R$)

- State: linear functional ω on the algebra of observable with $\omega(A^* A) \geq 1$ and $\omega(\mathbb{I}) = 1$
- A quantum field is a distribution $\varphi \in \mathcal{D}'(M)$: $f \to \varphi(f)$, where $\varphi(f) \in \mathcal{A}$, the algebra of observables
- The field equation in the sense of distributions: $\varphi(Pf) = 0$
- Canonical commutation relation ($G = G_R - G_A$):

$$[\varphi(f), \varphi(h)] = i \langle f, Gh \rangle \mathbb{I} = i \int dx\, dy\, f(x) G(x, y) h(y)\, \mathbb{I}$$

Renormalization

Definitions

- Renormalization: technique used to treat infinities arising in calculated quantities
- Regularization scheme: scheme used to absorb the infinities that arise in the perturbative approach
- Renormalization prescription: set of rules that describe what finite part one has to consider in the counterterms
- Renormalization conditions: conditions used to fix the the freedom of the counterterms.