Smallest eigenvalue distribution for product of Ginibre matrices

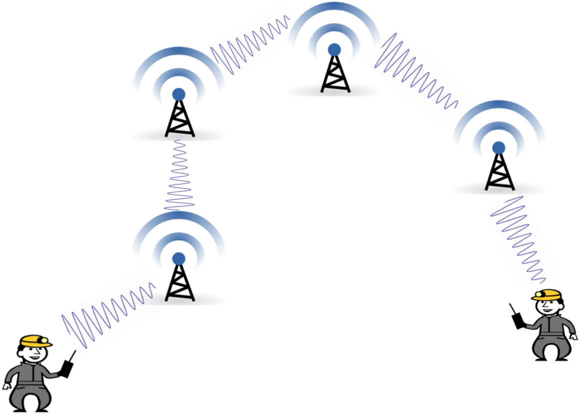
Dries Stivigny
KU Leuven (Belgium)

Joint work with Tom Claeys and Manuela Girotti (UC Louvain)

Random Product Matrices,
New Developments & Applications,
22-26 August 2016

Motivation - Wireless telecommunication

Consider a MIMO (Multiple-Input Multiple-Output) communication channel from one source to one destination via $r-1$ clusters of scatterers.



Let us assume that the source and destination are equipped with $n + \nu_0$ transmitting and $n + \nu_r$ receiving antennas, respectively, and each cluster of scatterers is assumed to have $n + \nu_j$ $(1 \leq j \leq r - 1)$ scattering objects.

Let us assume that the source and destination are equipped with $n+\nu_0$ transmitting and $n+\nu_r$ receiving antennas, respectively, and each cluster of scatterers is assumed to have $n+\nu_j$ $(1 \leq j \leq r-1)$ scattering objects.

The transmitted signal propagates from the transmitter array to the first cluster of scatterers, from the first to the second cluster, and so on, until it is received from the $(r-1)$st cluster by the receiver array.

Let us assume that the source and destination are equipped with $n + \nu_0$ transmitting and $n + \nu_r$ receiving antennas, respectively, and each cluster of scatterers is assumed to have $n + \nu_j$ $(1 \leq j \leq r - 1)$ scattering objects.

The transmitted signal propagates from the transmitter array to the first cluster of scatterers, from the first to the second cluster, and so on, until it is received from the $(r-1)$st cluster by the receiver array.

Such a communication link is canonically described by a "channel matrix"

$$Y_r = G_r G_{r-1} \dots G_1$$

and the singular values of Y_r "describe" the information that is transmitted (Müller, 2002).

Product of Ginibre random matrices

Let $\{G_j\}_{j=1,\ldots,r}$ be rectangular complex Ginibre matrices (i.i.d. complex Gaussian entries) of size $(n+\nu_j) \times (n+\nu_{j-1})$, with $\nu_0 = 0$ and $\nu_j \in \mathbb{N}$, and consider their product

$$Y_r := G_r \ldots G_1.$$

Product of Ginibre random matrices

Let $\{G_j\}_{j=1,\ldots,r}$ be rectangular complex Ginibre matrices (i.i.d. complex Gaussian entries) of size $(n+\nu_j)\times(n+\nu_{j-1})$, with $\nu_0=0$ and $\nu_j\in\mathbb{N}$, and consider their product

$$Y_r := G_r \ldots G_1.$$

Combining the work of Akemann, Kieburg, Wei and Akemann, Ipsen, Kieburg one obtains that the ensemble of squared singular values, i.e.

$$\text{the set of eigenvalues } \{x_k\}_{k=1,\ldots,n} \text{ of } Y_r^* Y_r,$$

is a (determinantal) point process on $\mathbb{R}^+$ with joint p.d.f.

$$\frac{1}{Z_n}\prod_{j<k}(x_j-x_k)\det\left[w_k(x_j)\right]_{j,k=1,\ldots,n}$$

and symbol $w_k(x) = G^{r,0}_{0,r}\left(\begin{array}{c} - \\ \nu_r,\ldots,\nu_2,\nu_1+k-1 \end{array}\middle| x\right)$ a Meijer-G function.

Product of Ginibre random matrices

$$\frac{1}{Z_n} \prod_{j<k} (x_j - x_k) \det \left[w_k(x_j) \right]_{j,k=1,\ldots,n}$$

A nice property of such distributions is that they form a determinantal point process.

Product of Ginibre random matrices

$$\frac{1}{Z_n}\prod_{j<k}(x_j - x_k)\det\left[w_k(x_j)\right]_{j,k=1,\dots,n}$$

A nice property of such distributions is that they form a determinantal point process. This means there exists a correlation kernel $K_n(x,y)$ such that all k-point correlations can be written as

$$\rho^{(k)}(x_1,\dots,x_k) = \det\left[K_n(x_i,x_j)\right]_{i,j=1}^{k}.$$

The kernel can be expressed in terms of biorthogonal functions.

Product of Ginibre random matrices

$$\frac{1}{Z_n} \prod_{j<k} (x_j - x_k) \det \left[w_k(x_j) \right]_{j,k=1,\ldots,n}$$

A nice property of such distributions is that they form a determinantal point process. This means there exists a correlation kernel $K_n(x, y)$ such that all k-point correlations can be written as

$$\rho^{(k)}(x_1, \ldots, x_k) = \det \left[K_n(x_i, x_j) \right]_{i,j=1}^{k}.$$

The kernel can be expressed in terms of biorthogonal functions.

Product of Ginibre matrices: double contour integral representation for K_n (Kuijlaars, Zhang).

1. Macroscopic limit: Fuss-Catalan distribution (Penson, Życzkowski and Neuschel)

1. Macroscopic limit: Fuss-Catalan distribution (Penson, Życzkowski and Neuschel)

2. Microscopic limit: different regimes
   - Bulk: sine kernel (Liu, Wang, Zhang)
   - Soft edge: Airy kernel (Liu, Wang, Zhang)

1. Macroscopic limit: Fuss-Catalan distribution (Penson, Życzkowski and Neuschel)

2. Microscopic limit: different regimes
   - Bulk: sine kernel (Liu, Wang, Zhang)
   - Soft edge: Airy kernel (Liu, Wang, Zhang)
   - Hard edge: Meijer G-kernel (Kuijlaars, Zhang)

Scaling limit at hard edge

Starting from the correlation kernel K_n, Kuijlaars and Zhang performed a scaling limit at the hard edge $x = 0$, while keeping the parameters $\nu_1, \ldots, \nu_r$ fixed:

Scaling limit at hard edge

Starting from the correlation kernel K_n, Kuijlaars and Zhang performed a scaling limit at the hard edge $x = 0$, while keeping the parameters $\nu_1, \ldots, \nu_r$ fixed:

Theorem (Kuijlaars and Zhang, 2014)

$$\lim_{n \to +\infty} \frac{1}{n} K_n\left(\frac{x}{n}, \frac{y}{n}\right) = \mathbb{K}(x, y) \qquad x, y \in \mathbb{R}^+$$

Scaling limit at hard edge

Starting from the correlation kernel K_n, Kuijlaars and Zhang performed a scaling limit at the hard edge $x = 0$, while keeping the parameters $\nu_1, \dots, \nu_r$ fixed:

Theorem (Kuijlaars and Zhang, 2014)

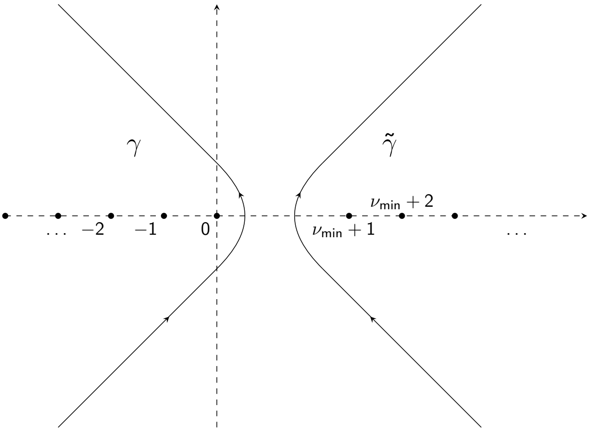
$$\lim_{n \to +\infty} \frac{1}{n} K_n\left(\frac{x}{n}, \frac{y}{n}\right) = \mathbb{K}(x,y) \qquad x, y \in \mathbb{R}^+$$

Integral representation (Kuijlaars and Zhang, 2014)

$$\mathbb{K}(x,y) = \int_{\tilde{\gamma}} \frac{\mathrm{d}u}{2\pi i} \int_{\gamma} \frac{\mathrm{d}v}{2\pi i} \prod_{i=1}^{r} \frac{\Gamma(-u+1+\nu_i)}{\Gamma(-v+1+\nu_i)} \frac{\Gamma(v)}{\Gamma(u)} \frac{x^{-v} y^{u-1}}{v-u}$$

Hard edge kernel: integral representation

$$\mathbb{K}(x,y) = \int_{\tilde{\gamma}} \frac{\mathrm{d}u}{2\pi i} \int_{\gamma} \frac{\mathrm{d}v}{2\pi i} \prod_{i=1}^{r} \frac{\Gamma(-u+1+\nu_i)}{\Gamma(-v+1+\nu_i)} \frac{\Gamma(v)}{\Gamma(u)} \frac{x^{-v} y^{u-1}}{v-u}$$



Meijer G-kernel

This kernel is a generalization of the Bessel kernel (which we obtain if $r = 1$) and has appeared in other models, including

- Product of truncated unitary matrices (Kieburg, Kuijlaars, Stivigny)
- Borodin biorthogonal ensembles (Borodin)
- Cauchy two-matrix model ($r = 2$) (Bertola, Gekhtman, Szmigielski)

Meijer G-kernel

This kernel is a generalization of the Bessel kernel (which we obtain if $r = 1$) and has appeared in other models, including

- Product of truncated unitary matrices (Kieburg, Kuijlaars, Stivigny)
- Borodin biorthogonal ensembles (Borodin)
- Cauchy two-matrix model ($r = 2$) (Bertola, Gekhtman, Szmigielski)

Kuijlaars and Zhang showed it is integrable in the sense of IIKS. This leads naturally to the study of the Fredholm determinant

- System of differential equations (Strahov and Zhang)
- Small gap asymptotics (Zhang)

Meijer G-kernel

This kernel is a generalization of the Bessel kernel (which we obtain if $r = 1$) and has appeared in other models, including

- Product of truncated unitary matrices (Kieburg, Kuijlaars, Stivigny)
- Borodin biorthogonal ensembles (Borodin)
- Cauchy two-matrix model ($r = 2$) (Bertola, Gekhtman, Szmigielski)

Kuijlaars and Zhang showed it is integrable in the sense of IIKS. This leads naturally to the study of the Fredholm determinant

- System of differential equations (Strahov and Zhang)
- Small gap asymptotics (Zhang)
- Large gap asymptotics

Gap probability and smallest particle distribution

We are interested in the so-called "gap probability", i.e. the probability of finding no particles in a given domain. In a determinantal point process with kernel K, it is well known that the smallest particle x^* has a distribution

$$\begin{aligned}\mathrm{Prob}(x^* > s) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[0,s]^k} \det \left[K(x_i, x_j)\right]_{i,j=1,\dots,k} \mathrm{d}x_1 \dots \mathrm{d}x_k \\ &=: \det\left(1 - K|_{[0,s]}\right)\end{aligned}$$

Gap probability and smallest particle distribution

We are interested in the so-called "gap probability", i.e. the probability of finding no particles in a given domain. In a determinantal point process with kernel K, it is well known that the smallest particle x^* has a distribution

$$\begin{aligned}\mathrm{Prob}(x^* > s) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left[K(x_i, x_j)\right]_{i,j=1,\ldots,k} \mathrm{d}x_1 \ldots \mathrm{d}x_k \\ &=: \det\left(1 - K|_{[0,s]}\right)\end{aligned}$$

Now one can show that in our case

$$\det\left(1 - K_n|_{[0,s/n]}\right) \to \det\left(1 - \mathbb{K}|_{[0,s]}\right) \qquad \text{as } n \to \infty$$

Large gap asymptotics

Theorem (Claeys, Girotti, Stivigny, '16)

As $s \to +\infty$ we have

$$\det\left(1 - \mathbb{K}|_{[0,s]}\right) = c e^{-K_1 s^{\frac{2}{r+1}} + K_2 s^{\frac{1}{r+1}} + K_3 \ln(s)} (1 + o(1))$$

for some unknown constant $c > 0$ and with K_1, K_2, K_3 (known) constants depending on the parameters of the model.

Large gap asymptotics

Theorem (Claeys, Girotti, Stivigny, '16)

As $s \to +\infty$ we have

$$\det\left(1 - \mathbb{K}|_{[0,s]}\right) = c e^{-K_1 s^{\frac{2}{r+1}} + K_2 s^{\frac{1}{r+1}} + K_3 \ln(s)} (1 + o(1))$$

for some unknown constant $c > 0$ and with K_1, K_2, K_3 (known) constants depending on the parameters of the model.

$$K_1 = \frac{r^{\frac{1-r}{1+r}} (r+1)^2}{4}$$

$$K_2 = (r+1) r^{-\frac{r}{(1+r)}} \left[\left(4\sqrt{r} - r\right) \left(\sum_{j=1}^{r} \nu_j - \frac{r}{2} \nu_{\min} \right) - \frac{\nu_{\min}}{2} \right]$$

Large gap asymptotics: remarks

1. One can check that for $r = 1$

$$K_1 = 1, \qquad K_2 = 2\nu_1$$

and for $r = 2$

$$K_1 = \frac{9}{2^{\frac{7}{3}}}, \qquad K_2 = \frac{3}{2^{\frac{2}{3}}}\left(\left(4\sqrt{2}-2\right)(\nu_1+\nu_2-\nu_{\min}) - \frac{\nu_{\min}}{2}\right)$$

which agrees with the results from Deift, Krasovsky, Vasilevska ($r = 1$) and Witte, Forrester ($r = 2$)

Large gap asymptotics: remarks

1. One can check that for $r = 1$

$$K_1 = 1, \qquad K_2 = 2\nu_1$$

and for $r = 2$

$$K_1 = \frac{9}{2^{\frac{7}{3}}}, \qquad K_2 = \frac{3}{2^{\frac{2}{3}}}\left(\left(4\sqrt{2} - 2\right)(\nu_1 + \nu_2 - \nu_{\min}) - \frac{\nu_{\min}}{2}\right)$$

which agrees with the results from Deift, Krasovsky, Vasilevska ($r = 1$) and Witte, Forrester ($r = 2$)

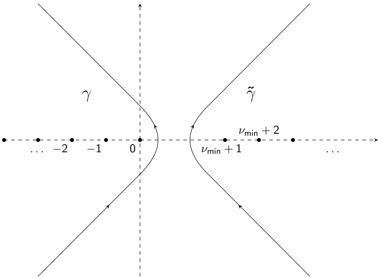
2. We obtain similar formulas for products of truncated unitary matrices and for Muttalib-Borodin Laguerre biorthogonal ensembles.

Gap probabilities of our kernel

Given a bounded closed interval $I = [0, s]$ and the kernel

$$\mathbb{K}(x,y) = \int_{\tilde{\gamma}} \frac{\mathrm{d}u}{2\pi i} \int_{\gamma} \frac{\mathrm{d}v}{2\pi i} \frac{F(v)}{F(u)} \frac{x^{-v} y^{u-1}}{v-u}$$

$$F(\lambda) := \frac{\Gamma(\lambda)}{\prod_{j=1}^{r} \Gamma(1+\nu_j - \lambda)},$$



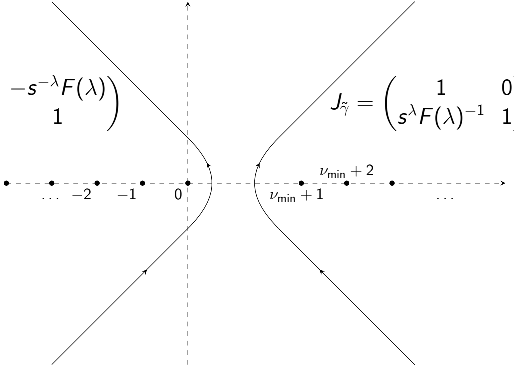
We are interested in the Fredholm determinant of $\mathbb{K}$ restricted on $[0, s]$:

$$\det\left(1 - \mathbb{K}\Big|_{[0,s]}\right).$$

A 2×2 RH Problem

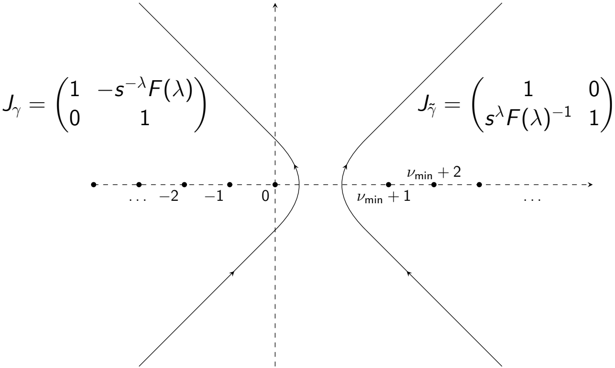
The relevant RH problem Y for us is the following:
$Y : \mathbb{C} \setminus (\gamma \cup \tilde{\gamma}) \to \mathbb{C}^{2\times 2}$ is analytic, normalized at infinity
$Y = I + \mathcal{O}\left(\lambda^{-1}\right)$ as $\lambda \to \infty$ and with jumps of the form

$$J_\gamma = \begin{pmatrix} 1 & -s^{-\lambda}F(\lambda) \\ 0 & 1 \end{pmatrix} \qquad J_{\tilde{\gamma}} = \begin{pmatrix} 1 & 0 \\ s^{\lambda}F(\lambda)^{-1} & 1 \end{pmatrix}$$



A 2×2 RH Problem

The relevant RH problem Y for us is the following:
$Y : \mathbb{C} \setminus (\gamma \cup \tilde{\gamma}) \to \mathbb{C}^{2\times 2}$ is analytic, normalized at infinity
$Y = I + \mathcal{O}\left(\lambda^{-1}\right)$ as $\lambda \to \infty$ and with jumps of the form



Remark: In our RH Problem s is a parameter!

Auxiliary result

Theorem (Claeys, Girotti, Stivigny, '16)

The following differential formula for gap probabilities holds:

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det \left(1 - \mathbb{K}\Big|_{[0,s]} \right) = \frac{1}{s} \left(Y_1(s) \right)_{2,2}$$

where $(Y_1(s))_{2,2}$ *is the* $(2,2)$*-entry of the residue matrix appearing in the asymptotic expansion at infinity of the matrix* Y

$$Y(\lambda; s) = I + \frac{Y_1(s)}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \quad \text{as } \lambda \to \infty$$

and Y *is the solution to the RH problem previously described.*

Auxiliary result

Theorem (Claeys, Girotti, Stivigny, '16)

The following differential formula for gap probabilities holds:

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det \left(1 - \mathbb{K}\Big|_{[0,s]} \right) = \frac{1}{s} \left(Y_1(s)\right)_{2,2}$$

where $\left(Y_1(s)\right)_{2,2}$ *is the* $(2,2)$*-entry of the residue matrix appearing in the asymptotic expansion at infinity of the matrix* Y

$$Y(\lambda; s) = I + \frac{Y_1(s)}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \quad \text{as } \lambda \to \infty$$

and Y *is the solution to the RH problem previously described.*

This RH representation is especially usefull to obtain large s asymptotics. We will obtain these asymptotics for Y using Deift/Zhou steepest descent method.

Deift/Zhou steepest descent analysis

In order to achieve the result, we apply a series of invertible transformations to our original RH Problem Y until we arrive at a RH Problem R for which we can easily get the asymptotics. This method is known as Deift/Zhou steepest descent and consists (in our case) of the following steps:

Deift/Zhou steepest descent analysis

In order to achieve the result, we apply a series of invertible transformations to our original RH Problem Y until we arrive at a RH Problem R for which we can easily get the asymptotics. This method is known as Deift/Zhou steepest descent and consists (in our case) of the following steps:

1. $Y \mapsto T$: "Esthetic" transformation
2. $T \mapsto S$: Introduction of a g-function to normalize the problem

Deift/Zhou steepest descent analysis

In order to achieve the result, we apply a series of invertible transformations to our original RH Problem Y until we arrive at a RH Problem R for which we can easily get the asymptotics. This method is known as Deift/Zhou steepest descent and consists (in our case) of the following steps:

1. $Y \mapsto T$: "Esthetic" transformation
2. $T \mapsto S$: Introduction of a g-function to normalize the problem
3. Construction of a global parametrix P^{∞}
4. Construction of local parametrices near the edge points

Deift/Zhou steepest descent analysis

In order to achieve the result, we apply a series of invertible transformations to our original RH Problem Y until we arrive at a RH Problem R for which we can easily get the asymptotics. This method is known as Deift/Zhou steepest descent and consists (in our case) of the following steps:

1. $Y \mapsto T$: "Esthetic" transformation
2. $T \mapsto S$: Introduction of a g-function to normalize the problem
3. Construction of a global parametrix P^{∞}
4. Construction of local parametrices near the edge points
5. $S \mapsto R$: Final transformation leading to a RH Problem normalized at infinity and with jump matrices all close to identity

Back to gap probabilities

Following backwards all the transformations $Y \mapsto T \mapsto S \mapsto R$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det \left(1 - \mathbb{K}\Big|_{[0,s]} \right) = i\, g_1\, s^{\frac{1-r}{1+r}} - i(P_1^\infty(s))_{2,2} s^{-\frac{r}{r+1}} + \mathcal{O}\left(s^{-\frac{r}{r+1}} \right)$$

with g_1, resp. P_1^∞ the residue of g, resp. P^∞ at infinity.

Back to gap probabilities

Following backwards all the transformations $Y \mapsto T \mapsto S \mapsto R$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det \left(1 - \mathbb{K}\Big|_{[0,s]} \right) = i\, g_1\, s^{\frac{1-r}{1+r}} - i(P_1^\infty(s))_{2,2} s^{-\frac{r}{r+1}} + \mathcal{O}\left(s^{-\frac{r}{r+1}} \right)$$

with g_1, resp. P_1^∞ the residue of g, resp. P^∞ at infinity. One can now check that

$$g_1 = i \frac{(r+1) r^{\frac{1-r}{1+r}}}{2}$$
$$(P_1^\infty(s))_{2,2} = -p_1(s)$$

and after integration the result follows.

Auxiliary result

Theorem (Claeys, Girotti, Stivigny, '16)

The following differential formula for gap probabilities holds:

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det \left(1 - \mathbb{K}\Big|_{[0,s]} \right) = \frac{1}{s} \left(Y_1(s) \right)_{2,2}$$

where $(Y_1(s))_{2,2}$ *is the* $(2,2)$*-entry of the residue matrix appearing in the asymptotic expansion at infinity of the matrix* Y

$$Y(\lambda; s) = I + \frac{Y_1(s)}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \quad \text{as } \lambda \to \infty$$

and Y *is the solution to the RH problem of before.*

Integrable kernels: Izergin-Its-Korepin-Slavnov procedure

Let $\Sigma \subset \mathbb{C}$ be a set of contours and let $\mathbb{M}$ be an integral operator acting on $L^2(\Sigma)$ with kernel

$$\mathbb{M}(\lambda, \mu) = \frac{\mathbf{f}(\lambda)^T \mathbf{g}(\mu)}{\lambda - \mu}$$

with $\mathbf{f}, \mathbf{g}$ p-dimensional column-vectors of functions such that

$$\mathbf{f}(\lambda)^T \mathbf{g}(\lambda) = 0$$

Integrable kernels: Izergin-Its-Korepin-Slavnov procedure

Let $\Sigma \subset \mathbb{C}$ be a set of contours and let $\mathbb{M}$ be an integral operator acting on $L^2(\Sigma)$ with kernel

$$\mathbb{M}(\lambda, \mu) = \frac{\mathbf{f}(\lambda)^T \mathbf{g}(\mu)}{\lambda - \mu}$$

with $\mathbf{f}, \mathbf{g}$ p-dimensional column-vectors of functions such that

$$\mathbf{f}(\lambda)^T \mathbf{g}(\lambda) = 0$$

Proposition

We have the identity

$$\det\left(1 - \mathbb{K}|_{[0,s]}\right) = \det\left(1 - \mathbb{M}_s\right)$$

where $\mathbb{M}_s$ is an integral operator of the type above.

Integrable kernels: IIKS procedure (cont.)

To such operators, one can associate a RH problem, analytic on $\mathbb{C} \setminus \Sigma$, of size $p \times p$ with jump matrix $J(\lambda) = 1 - \mathbf{f}(\lambda)\mathbf{g}(\lambda)^T$ (and normalized at infinity)

Integrable kernels: IIKS procedure (cont.)

To such operators, one can associate a RH problem, analytic on $\mathbb{C} \setminus \Sigma$, of size $p \times p$ with jump matrix $J(\lambda) = 1 - \mathbf{f}(\lambda)\mathbf{g}(\lambda)^T$ (and normalized at infinity)

Proposition (cont.)

In our case, the functions $\mathbf{f}, \mathbf{g}$ are given by

$$\mathbf{f}(\lambda) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \lambda \in \gamma \\ \begin{pmatrix} 0 \\ s^{\lambda} \end{pmatrix} & \text{if } \lambda \in \tilde{\gamma} \end{cases}, \qquad \mathbf{g}(\mu) = \begin{cases} \begin{pmatrix} 0 \\ s^{-\mu} F(\mu) \end{pmatrix} & \text{if } \mu \in \gamma \\ \begin{pmatrix} -F(\mu)^{-1} \\ 0 \end{pmatrix} & \text{if } \mu \in \tilde{\gamma} \end{cases}.$$

Integrable kernels: IIKS procedure (cont.)

To such operators, one can associate a RH problem, analytic on $\mathbb{C} \setminus \Sigma$, of size $p \times p$ with jump matrix $J(\lambda) = 1 - \mathbf{f}(\lambda)\mathbf{g}(\lambda)^T$ (and normalized at infinity)

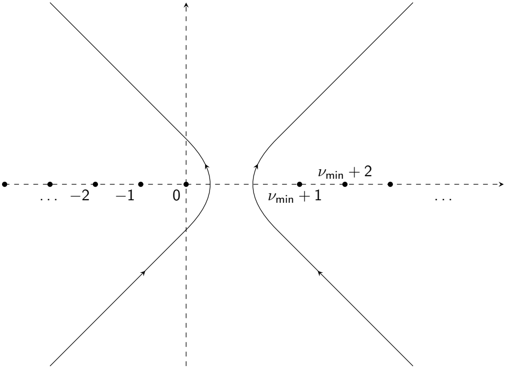
Proposition (cont.)

In our case, the functions $\mathbf{f}, \mathbf{g}$ are given by

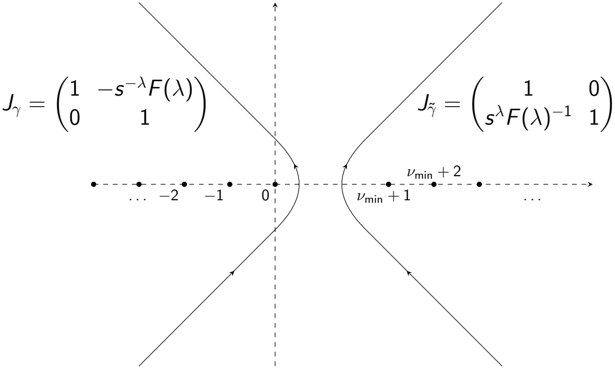
$$\mathbf{f}(\lambda) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } \lambda \in \gamma \\ \begin{pmatrix} 0 \\ s^{\lambda} \end{pmatrix} & \text{if } \lambda \in \tilde{\gamma} \end{cases}, \qquad \mathbf{g}(\mu) = \begin{cases} \begin{pmatrix} 0 \\ s^{-\mu} F(\mu) \end{pmatrix} & \text{if } \mu \in \gamma \\ \begin{pmatrix} -F(\mu)^{-1} \\ 0 \end{pmatrix} & \text{if } \mu \in \tilde{\gamma} \end{cases}.$$

Notice: $p = 2$, $\Sigma = \gamma \cup \tilde{\gamma}$

So the associated RH problem Y is of dimension 2×2, normalized at infinity $Y = I + \mathcal{O}\left(\lambda^{-1}\right)$ as $\lambda \to \infty$



So the associated RH problem Y is of dimension 2×2, normalized at infinity $Y = I + \mathcal{O}\left(\lambda^{-1}\right)$ as $\lambda \to \infty$ and with jumps of the form



Bertola (2010) and Bertola-Cafasso (2011) applied to our case gives us that

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det (1 - \mathbb{M}) = \int_{\gamma \cup \tilde{\gamma}} \mathrm{Tr} \left[Y_-^{-1}(\lambda) Y_-'(\lambda) \, \partial_s J(\lambda) J^{-1}(\lambda) \right] \frac{\mathrm{d}\lambda}{2\pi i}$$

Bertola (2010) and Bertola-Cafasso (2011) applied to our case gives us that

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det (1 - \mathbb{M}) = \int_{\gamma \cup \tilde{\gamma}} \mathrm{Tr} \left[Y_-^{-1}(\lambda) Y_-'(\lambda)\, \partial_s J(\lambda) J^{-1}(\lambda) \right] \frac{\mathrm{d}\lambda}{2\pi i}$$

This can be further simplified and after expanding Y at infinity and deforming the contours, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln \det (1 - \mathbb{M}) = \frac{1}{s} \left(Y_1(s) \right)_{2,2}.$$

Conclusions

We studied the smallest particle distribution for the product of r Ginibre matrices and obtained in that

$$\mathbb{P}(x^* > s) = ce^{-K_1 s^{\frac{2}{r+1}} + K_2 s^{\frac{1}{r+1}} + K_3 \ln(s)} \left(1 + o(1)\right), \qquad s \to \infty$$

with c, K_1, K_2, K_3 constants depending on the model.

Conclusions

We studied the smallest particle distribution for the product of r Ginibre matrices and obtained in that

$$\mathbb{P}(x^* > s) = ce^{-K_1 s^{\frac{2}{r+1}} + K_2 s^{\frac{1}{r+1}} + K_3 \ln(s)} \left(1 + o(1)\right), \qquad s \to \infty$$

with c, K_1, K_2, K_3 constants depending on the model.

Similar results hold for other models: product of truncated unitary matrices, and so-called Muttalib-Borodin Laguerre ensembles.

Conclusions

We studied the smallest particle distribution for the product of r Ginibre matrices and obtained in that

$$\mathbb{P}(x^* > s) = ce^{-K_1 s^{\frac{2}{r+1}} + K_2 s^{\frac{1}{r+1}} + K_3 \ln(s)} \left(1 + o(1)\right), \qquad s \to \infty$$

with c, K_1, K_2, K_3 constants depending on the model.

Similar results hold for other models: product of truncated unitary matrices, and so-called Muttalib-Borodin Laguerre ensembles.

The main clue is the double-contour integral representation of the type

$$\mathbb{K}(x, y) = \int_\gamma \frac{\mathrm{d}u}{2\pi i} \int_{\tilde{\gamma}} \frac{\mathrm{d}v}{2\pi i} \frac{F(x, v)}{F(y, u)} \frac{1}{v - u}.$$

Thank you for your attention!