Eigenvalue PDFs

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Outline

- Hermitian matrices with real, complex or real quaternion elements
- Circular ensembles and classical groups
- Products of random matrices
- Matrices constructed by recurrence

Dyson's 3 fold way

Dyson (1961) identified Hermitian matrices $\mathbf{X}$ with real, complex and real quaternion entries as models for chaotic quantum mechanical systems with time reversal symmetry $T^2 = 1$ (think of as complex conjugation K), no time reversal symmetry, and time reversal symmetry $T^2 = -1$ (even dimensional, odd number of spin 1/2 particles, $T = \mathbf{Z}_{2N} K$).

The least familiar is the latter. Note that this constrains the $2N \times 2N$ matrix $\mathbf{X}$ to have the property $\mathbf{X} = \mathbf{Z}_{2N} \bar{\mathbf{X}} \mathbf{Z}_{2N}^{-1}$. This implies $\mathbf{X}$ can be viewed as an $N \times N$ matrix with elements consisting of 2×2 blocks of the form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix},$$

which is the 2×2 matrix representation of a real quaternion.

Diagonalisation

$\mathbf{X}$ real symmetric $\mathbf{X} = \mathbf{RLR}^T$, where $\mathbf{R}$ is real orthogonal.

$\mathbf{X}$ complex Hermitian $\mathbf{X} = \mathbf{ULU}^\dagger$, where $\mathbf{U}$ is complex unitary.

$\mathbf{X}$ real quaternion Hermitian $\mathbf{X} = \mathbf{ULU}^\dagger$, where $\mathbf{U}$ is symplectic unitary equivalent. Eigenvalues are doubly degenerate.

Note that the matrix of eigenvectors have real, complex and real quaternion elements respectively.

The diagonalisation formulae can be deduced by making use of Householder transformations, for example:

$$\mathbf{R}_N \mathbf{A}_N \mathbf{R}_N^T = \begin{bmatrix} \lambda & \vec{\beta}_{N-1}^T \\ \vec{0}_{N-1} & \mathbf{A}_{N-1} \end{bmatrix}$$

By symmetry $\vec{\beta}_{N-1} = \vec{0}$.

Factorisation of the volume form

For $\mathbf{X}$ real symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$) or real quaternion Hermitian ($\beta = 4$), the diagonalisation formulae imply the factorisations of the volume forms

$$(d\mathbf{X}) = \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta \prod_{l=1}^{N} d\lambda_l \, (\mathbf{U}^\dagger d\mathbf{U}).$$

Impose a PDF $q(\mathbf{X})$ on the matrices $\mathbf{X}$ with the property that $q(\mathbf{X}) = q(\mathbf{U}^\dagger \mathbf{X} \mathbf{U})$ where $\mathbf{U}$ is from the subset of unitary matrices which diagonalises $\mathbf{X}$. Then $q(\mathbf{X})$ is a function of the eigenvalues only, and the probability $q(\mathbf{X})(d\mathbf{X})$ factorises into eigenvalue and eigenvector parts:

$$q(\lambda_1, \ldots, \lambda_N) \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta \prod_{l=1}^{N} d\lambda_l \, (\mathbf{U}^\dagger d\mathbf{U})$$

The Gaussian case

Consider e.g. the real case, and suppose

$$q(\mathbf{X}) = \frac{1}{(2\pi)^{N/2}} \frac{1}{\pi^{N(N-1)/4}} e^{-\mathrm{Tr}\,\mathbf{X}^2/2} = \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-x_{jj}^2/2} \prod_{j<k}^{N} \frac{1}{\sqrt{\pi}} e^{-x_{jk}^2}$$

The Gaussian case has the distinguishing feature among the invariant ensembles as deriving from independently distributed elements.

The shifted mean Gaussian case is also special. The PDF is no longer unitary invariant, being proportional to $e^{-\mathrm{Tr}\,(\mathbf{X}-\mathbf{X}^{(0)})^2/2}$, and so there is need to compute an integral over the invariant measure for the eigenvectors. In the complex case, one has the Izykson-Zuber/ Harish-Chandra integral

$$\int e^{-\mathrm{Tr}\,\mathbf{AUBU}^\dagger} (\mathbf{U}^\dagger d\mathbf{U}) \propto \frac{\det[e^{-a_i b_j}]_{i,j=1,\dots,N}}{\prod_{1\le j<k\le N}(a_j - a_k)(b_j - b_k)}$$

where $\{a_i\}$, $\{b_i\}$ are the eigenvalues of the Hermitian matrices $\mathbf{A}$, $\mathbf{B}$.

Circular ensembles I

Matrices in the Gaussian ensembles can be written

$$\mathbf{H} = \frac{1}{2}(\mathbf{X}^T + \mathbf{X}) \qquad (\beta = 1)$$

$$\mathbf{H} = \frac{1}{2}(\mathbf{X}^\dagger + \mathbf{X}) \qquad (\beta = 2)$$

$$\mathbf{H} = \frac{1}{2}(\mathbf{X}^D + \mathbf{X}) \qquad (\beta = 4)$$

where $\mathbf{X}^D = \mathbf{Z}_{2N}\mathbf{X}^T\mathbf{Z}_{2N}^{-1}$, $\mathbf{X}$ real ($\beta = 1$) etc.

In the cases $\beta = 1$ and $\beta = 4$, Dyson used the operations of T and D to define ensembles of unitary matrices, beginning with elements of $\mathbf{U} \in U(N)$:

$$\mathbf{S}_1 = \mathbf{U}^T\mathbf{U}, \qquad \mathbf{S}_4 = \mathbf{U}^D\mathbf{U}$$

These ensembles, referred to as COE and CSE, permit the diagonalisations

$$\mathbf{S}_1 = \mathbf{R}\mathbf{L}\mathbf{R}^T, \qquad \mathbf{S}_4 = \mathbf{V}\mathbf{L}\mathbf{V}^D.$$

Circular ensembles II

For Dyson's circular ensembles, the invariant measure factorises in an analogous way to the Hermitian ensembles,

$$\mu(d\mathbf{S}) = \prod_{1\leq j<k\leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta \prod_{l=1}^{N} d\theta_l \,(\mathbf{V}^\dagger d\mathbf{V}),$$

where $\mathbf{V}$ denotes the matrix of eigenvectors.

Associated with the circular ensembles are ensembles of Hermitian matrices defined by the Cayley transformation

$$\mathbf{H} = i\frac{\mathbb{I}_N - \mathbf{U}}{\mathbb{I}_N + \mathbf{U}}$$

The matrices $\mathbf{H}$ has PDF proportional to

$$\left(\det(\mathbb{I}_N + \mathbf{H}^2)\right)^{-\beta(N-1)/2-1}.$$

This is sometimes referred to as the Cauchy ensemble.

Classical groups I

In the case $\beta = 2$, Dyson's circular ensemble is denoted CUE, and coincides with matrices from the classical group $U(N)$ with Haar measure. But the COE and CSE are distinct from matrices from the classical groups $O(N)$ and $Sp(2N)$ with Haar measure. Instead the COE is the symmetric space $U(N)/O(N)$, while the CSE is the symmetric space $U(2N)/Sp(2N)$.

For the decomposition of the invariant measure, there are in fact 4 distinct cases associated with $O(N)$: $O^{\pm}(2n+1)$ and $O^{\pm}(2n)$. Note that in all cases, if $e^{i\theta}$ is an eigenvalue, then so is $e^{-i\theta}$.

Perhaps surprisingly, the eigenvalue PDF for all the classical groups has Dyson exponent $\beta = 2$.

Classical groups II

Introduce the eigenvalue PDF supported on $(-1, 1)$ proportional to

$$\prod_{l=1}^{n} (1 + x_l)^b (1 - x_l)^a \prod_{1 \le j < k \le n} (x_k - x_j)^2,$$

referred to as the Jacobi unitary ensemble with parameters (n, a, b).

In terms of the variable $y_j = \cos \theta_j$, the eigenvalue PDF for the eigenvalues in $(0, \pi)$ for matrices from $O(N)$ is given by

$$(n, a, b) = \begin{cases} (N/2, -1/2, -1/2) & \textit{for matrices in } O^+(N),\ N \textit{ even}, \\ ((N-1)/2, 1/2, -1/2) & \textit{for matrices in } O^+(N),\ N \textit{ odd}, \\ ((N-1)/2, -1/2, 1/2) & \textit{for matrices in } O^-(N),\ N \textit{ odd}, \\ (N/2 - 1, 1/2, 1/2) & \textit{for matrices in } O^-(N),\ N \textit{ even}. \end{cases}$$

$Sp(2N)$ has the same eigenvalue PDF as $O^-(2N)$.

Classical groups III

Applying the Cayley transformation to matrices from $\mathbf{R} \in O^+(N)$ gives

$$i\mathbf{A} = \frac{\mathbb{I}_N - \mathbf{R}}{\mathbb{I}_N + \mathbf{R}}$$

where the matrix A is Hermitian with pure imaginary entries only. Thus $i\mathbf{A}$ is an anti-symmetric real matrix.

This can be used to show that the eigenvalue factor in the Jacobian for the matrices $\mathbf{A}$ is proportional to

$$\prod_{1 \le j < k \le N/2} (\lambda_k^2 - \lambda_j)^2, \; N \,\mathrm{even}; \qquad \prod_{l=1}^{(N-1)/2} \lambda_l^2 \prod_{1 \le j < k \le (N-1)/2} (\lambda_k^2 - \lambda_j)^2, \; N \,\mathrm{odd}$$

Applying the Cayley transformation to matrices from $\mathbf{S} \in Sp(2N)$ leads to the ensemble of Hermitian real quaternion matrices with pure imaginary elements, and shows the eigenvalue PDF is proportional to that for $(2N+1) \times (2N+1)$ real anti-symmetric matrices.

Products of random matrices I

Products of random matrices have already been seen in the construction of the COE and CSE.

Let X be of size $n \times N$, with $n \geq N$. The block matrix

$$\begin{bmatrix} \mathbf{0}_n & \mathbf{X} \\ \mathbf{X}^\dagger & \mathbf{0}_N \end{bmatrix}$$

then has $n - N$ zero eigenvalues, and nonzero eigenvalues given by $\pm$ the square root of the eigenvalues of $\mathbf{X}^\dagger \mathbf{X}$.

The positive square roots of the eigenvalues of $\mathbf{X}^\dagger \mathbf{X}$ are equal to the singular values of $\mathbf{X}$. Recall the singular value decomposition

$$\mathbf{X} = \mathbf{U}_1 D \mathbf{U}_2^\dagger$$

where $\mathbf{D} = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_N)$

Products of random matrices II

Consideration of the singular value decomposition gives a factorisation of the volume form, with the portion dependent on the singular values as proportional to

$$\prod_{j=1}^{N} \sigma_j^{\beta\alpha} \prod_{1 \le j < k \le N} |\sigma_k^2 - \sigma_j^2|^\beta, \qquad \alpha = n - N + 1 - 1/\beta.$$

An important result (due to Wishart) is that the volume form for the matrix product $\mathbf{B} = \mathbf{X}^\dagger \mathbf{X}$ is proportional to $(\det \mathbf{B})^{\beta\alpha - 1}(d\mathbf{B})$.

Changing variables to the eigenvalues and eigenvectors in the latter gives the eigenvalue PDF as proportional to

$$\prod_{j=1}^{N} \lambda_j^{\beta(n-N+1-2/\beta)/2} \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta,$$

This is equivalent to the singular value PDF with $\sigma_k^2 = \lambda_k^2$.

Products of random matrices III

Let $\mathbf{A}$ be positive definite, and consider the matrix product $\mathbf{C} = \mathbf{A}^{1/2}\mathbf{B}\mathbf{A}^{1/2}$, where $\mathbf{B}$ has distribution proportional to $e^{-\mathrm{Tr}\,\mathbf{B}}(\det \mathbf{B})^{n-N}(d\mathbf{B})$.

Since $(d\mathbf{C}) = (\det \mathbf{A})^N(d\mathbf{B})$, $\mathbf{C}$ has distribution proportional to

$$e^{-\mathrm{Tr}\,\mathbf{A}^{-1}\mathbf{C}}(\det \mathbf{A})^{-n}(\det \mathbf{C})^{n-N}(d\mathbf{C}).$$

Changing variables to the eigenvalues and eigenvectors of $\mathbf{C}$, our task is to compute the matrix integral

$$\int e^{-\mathrm{Tr}\,\tilde{\mathbf{A}}^{-1}\mathbf{U}\tilde{\mathbf{C}}\mathbf{U}^\dagger}(\mathbf{U}^\dagger d\mathbf{U}),$$

where $\tilde{\mathbf{A}}^{-1} = \mathrm{diag}\,(a_1^{-1},\dots,a_N^{-1})$ and $\tilde{C} = \mathrm{diag}\,(c_1,\dots,c_N)$. This we recognise as the Harish-Chandra/ Itzykson–Zuber integral. Simplest case when the eigenvalues of $\mathbf{A}$ form a polynomial ensemble.

Eigenvalue PDFs from recurrences I

Non-zero eigenvalues of $\mathbf{A}^{1/2}\mathbf{X}^\dagger\mathbf{X}\mathbf{A}^{1/2}$ are the same as those for $\mathbf{X}\mathbf{A}\mathbf{X}^\dagger$ (recall $\mathbf{X}$ is $n \times N$).

Observe $\mathbf{X}^{(N)}\mathbf{A}^{(N)}(\mathbf{X}^{(N)})^\dagger = \mathbf{X}^{(N-1)}\mathbf{A}^{(N-1)}(\mathbf{X}^{(N-1)})^T + a_N\vec{x}\vec{x}^\dagger$

The RHS is a rank 1 perturbation.

Denote the non-zero eigenvalues of $\mathbf{X}^{(N-1)}\mathbf{A}^{(N-1)}(\mathbf{X}^{(N-1)})^\dagger$ by $y_1 > y_2 > \cdots > y_{N-1}$.

Denote the non-zero eigenvalues of $\mathbf{X}^{(N)}(\mathbf{X}^{(N)})^\dagger$ by $\lambda_1 > \lambda_2 > \cdots > \lambda_N$.

Must have that $x_1 > y_1 > x_2 > y_2 > \cdots > y_{N-1} > x_N > 0$, with $\{x_j\}$ determined by the secular equation

$$0 = 1 - a_N\frac{q_0}{\lambda} - a_N\sum_{j=1}^{N-1}\frac{q_j}{\lambda - y_j}$$

where $q_0 \overset{\mathrm{d}}{=} \Gamma[(n-N+1),1]$ and $q_j \overset{\mathrm{d}}{=} \Gamma[1,1]$.

Eigenvalue PDFs from recurrences II

It is possible to change variables from the residues $\{q_j\}_{j=0}^N$ to the zeros of the secular equation. This gives the conditional PDF for $\{\lambda_j\}$ as proportional to

$$\prod_{j=1}^{N} \lambda_j^{n-N} e^{-\lambda_j/b_N} \prod_{l=1}^{N-1} y_l^{-(n-N+2)+1} e^{-y_l} \frac{\prod_{1 \le j < k \le N} (\lambda_j - \lambda_k)}{\prod_{1 \le j < k \le N-1} (y_j - y_k)}$$

Denote this conditional PDF by $q(\{\lambda_l\}); \{y_l\})$. Then the PDF $p_N(\{\lambda_l\}_{l=1}^N\}$ must satisfy

$$p_N(\{\lambda_l\}_{l=1}^N\} = \int_R q(\{\lambda_l\}_{l=1}^N; \{y_l\}_{l=1}^{N-1}) p_{N-1}(\{y_l\}_{l=1}^{N-1}\} \, d\vec{y}$$

Note that for $b_N = 1$ the solution must be proportional to $\prod_{j=1}^N \lambda_j^{n-N} e^{-\lambda_j} \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2$.