Characteristic polynomials

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Outline

- Correlation functions and polynomial ensembles
- Combinatorial evaluation and recursive structure
- Products of complex Gaussian random matrices
- Skew orthogonal polynomials

Correlation functions

Eigenvalues of unitary (symmetry) random matrices form a determinant point process

$$\rho_{(k)}(x_1, \ldots, x_k) = \det[K(x_i, x_j)]_{i,j=1,\ldots,k}$$

For orthogonal and symplectic symmetry, the eigenvalues form a Pfaffian point process

$$\rho_{(k)}(x_1, \ldots, x_k) = \mathrm{Pf}\begin{bmatrix} D(x,y) & S(x,y) \\ -S(y,x) & \tilde{I}(x,y) \end{bmatrix}$$

This is also true of invariant ensembles with complex eigenvalues e.g. the Ginibre ensemble (determinant point process for complex entries, Pfaffian for real or real quaternion entries) and also biorthogonal ensembles (determinantal point process).

Polynomial ensembles

The eigenvalue PDF proportional to

$$\prod_{1\le j<k\le N}(x_j - x_k)\det[g_j(x_k)]_{j,k=1,\dots,N}$$

is called a polynomial ensemble. It is a special class of biorthogonal ensemble. Examples include the squared singular values of products of complex Gaussian random matrices.

For a polynomial ensemble, the correlations are determinantal with kernel

$$K(x,y) = \sum_{l=0}^{N-1}\frac{1}{h_l}p_l(x)q_l(y)$$

Each $p_l(x)$ is a polynomial of degree l, while $\{q_l\}$ are linearly independent functions from Span $\{g_j(x)\}$ with the orthogonality property $\int_{-\infty}^{\infty} p_j(x)q_k(x)\,dx = h_j\delta_{j,k}$.

The polynomials $p_j(x)$ can be expressed as averaged characteristic polynomials,

$$p_j(x) = \left\langle \prod_{l=1}^{j}(x - y_l)\right\rangle$$

Combinatorial evaluation

Let $\mathbf{H}$ be a complex Hermitian matrix. Let the diagonal elements x_{ij} be chosen from a distribution with mean zero. Let the upper triangular elements be chosen with mean zero, variance $1/2$. Use

$$\det(\lambda \mathbf{I}_N - \mathbf{X}) = \sum_{P \in S_N} \varepsilon(P) \prod_{l=1}^{N} (\lambda_{l,P(l)} - x_{l,P(l)})$$

where $\lambda_{i,j} = \lambda$ if $i = j$ and 0 otherwise. After averaging, the only non-zero terms are those consisting entirely of fixed points $(P(j) = j)$ and 2-cycles $(P(j_1) = j_2$ and $P(j_2) = j_1$, $j_1 \neq j_2)$.

Gives the Hermite polynomial,

$$\left\langle \prod_{l=1}^{j} (x - y_l) \right\rangle = \sum_{j=0}^{[N/2]} (-1)^j \binom{N}{2j} \frac{(2j)!}{2^{2j} j!} \lambda^{N-2j}$$

Same approach can be used for chiral random matrices (leading to Laguerre polynomials), and the sum $\mathbf{H} + \mathbf{H}^{(0)}$.

Recursive structure

A β-generalisation of the GOE and GUE can be constructed by taking the viewpoint that both have a recursive structure specified by appending a row and column. Define

$$\mathbf{M}_n = \begin{bmatrix} \mathrm{diag}\,\mathbf{M}_{n-1} & \vec{b} \\ \vec{b}^T & c \end{bmatrix}$$

where c, b_j^2 have distribution $\mathrm{N}[0,1]$, $\Gamma[\beta/2, 1]$.

The corresponding characteristic polynomial satisfies the three-term recurrence

$$C_j(x) = (x - c) C_{j-1}(x) - s_{j-1} C_{j-2}(x),$$

where s_{j-1} has distribution $\Gamma[(j-1)\beta/2, 1]$, and $C_{-1}(x) = 0$, $C_0(x) = 1$. After averaging, this reads $\langle C_j(x) \rangle = x \langle C_{j-1}(x) \rangle - (\beta/2)(j-1) \langle C_{j-2}(x) \rangle$, which is satisfied by $2^{-j}(\beta/2)^{j/2} H_j(\sqrt{2/\beta}x)$.

Products of complex Gaussian random matrices I

The squared singular values of products of complex Gaussian random matrices are an example of a polynomial ensemble. We have

$$p_j(x) = \Big\langle \det(\lambda \mathbf{I}_N - \mathbf{X}_s^\dagger \cdots \mathbf{X}_1^\dagger \mathbf{X}_1 \cdots \mathbf{X}_s) \Big\rangle,$$

Generally,

$$\det(\lambda \mathbf{I}_N - \mathbf{Y}) = \sum_{l=0}^{j} (-1)^l \langle e_l(\mathbf{Y}) \rangle \lambda^{j-l}$$

where $e_l(\mathbf{Y})$ is the l-th elementary symmetric polynomial in the eigenvalues of $\mathbf{Y}$.

We use the invariance $\langle e_l(\mathbf{A}\mathbf{X}^\dagger \mathbf{B}\mathbf{X}) \rangle_{\mathbf{X}} = \langle \langle e_l(\mathbf{A}\mathbf{U}^\dagger \mathbf{X}^\dagger \mathbf{B}\mathbf{X}\mathbf{U}) \rangle_{\mathbf{U}} \rangle_{\mathbf{X}}$, together with the formula

$$\langle e_l(\mathbf{A}\mathbf{U}^\dagger \mathbf{B}\mathbf{U}) \rangle_{\mathbf{U}} = \frac{e_l(\mathbf{A}) e_l(\mathbf{B})}{e_l(1^N)}$$

to deduce that

$$\langle e_l(\mathbf{A}\mathbf{X}^\dagger \mathbf{B}\mathbf{X}) \rangle_{\mathbf{X}} = \frac{e_l(\mathbf{A}) e_l(\mathbf{B})}{(e_l(1^N))^2} \langle e_l(\mathbf{X}^\dagger \mathbf{X}) \rangle_{\mathbf{X}}$$

Products of complex Gaussian random matrices II

The average $\langle e_l(\mathbf{X}^\dagger \mathbf{X}\rangle_{\mathbf{X}}$ is known from the evaluation of the characteristic polynomial for Wishart matrices.

Find that

$$p_N(\lambda) = (-1)^N (N!)^s \, {}_1F_s\Big({-N \atop 1,\dots,1} \Big| \lambda \Big).$$

A noteworthy point is that this satisfies the linear differential equation of degree $s+1$

$$\lambda\Big(\lambda \frac{d}{d\lambda} - N\Big) p = \Big(\lambda \frac{d}{d\lambda}\Big)^{s+1} p$$

An application of this feature relates to the global density $\rho_{(1)}^{\mathrm{G}}$. In general

$$\langle \det(\lambda \mathbf{I}_N - \mathbf{Y}) \rangle_{\mathbf{Y}} = \langle e^{\sum_{l=1}^N \log(\lambda - y_l)} \rangle_{\{y_l\}} \sim e^{N \int_I \log(\lambda - y) \rho_{(1)}^{\mathrm{G}}(y)\, dy}$$

The DE implies, after scaling $\lambda = N^s z$, that the corresponding Green function satisfies

$$z(zG(z) - 1) = (zG(z))^{s+1}.$$

Lagrange inversion

Set $W(z) = zG(z)$ so that $W(z) - 1 = z^{-1}(W(z))^s$. Want to expand this for large z.

Lagrange inversion formula: $\zeta = a + t\phi(\zeta)$ has one solution for t small enough. For $f(\zeta)$ analytic the power series expansion in t is given by

$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{t^n}{n!} \frac{d^{n-1}}{da^{n-1}} (f'(a)(\phi(a))^n).$$

Choose $a = 1$, $t = 1/z$, $\phi(\zeta) = \zeta^p$, $f(\zeta) = \zeta^r$ to get

$$\begin{aligned} w^r &= 1 + \sum_{n=1}^{\infty} \frac{1}{z^n n!} \frac{d^{n-1}}{da^{n-1}} \left(r a^{r-1} a^{np} \right)\Big|_{a=1} \\ &= 1 + r \sum_{n=1}^{\infty} \frac{1}{z^n n!} (np + r - 1)_{n-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{z^n} R_{p,r}(n). \end{aligned}$$

Here $R_{p,r}(n)$ is the Raney number. We have $R_{s+1,1}(k) = C_s(k)$, the k-th Fuss-Catalan number.

Gap probability for Laguerre ensemble

Observe

$$E_{N,\beta}((0,s);x^a e^{-x}) = \frac{1}{\mathcal{C}} \int_s^\infty dx_1 \cdots \int_s^\infty dx_N \prod_{l=1}^N x_l^a e^{-x_l} \prod_{1 \le j < k \le N} |x_k - x_j|^\beta$$

$$= e^{-sN} \Big\langle \prod_{l=1}^N (x_l + s)^a \Big\rangle$$

This is essentially the averaged a-th power of the characteristic polynomial.

In the case $a \in \mathbb{Z}_{\ge 0}$, this can be evaluated in terms of a generalised hypergeometric function of a arguments, based on Jack polynomials,

$$E_{N,\beta}((0,s);x^a e^{-x}) = e^{-\beta Ns/2} \, {}_1F_1^{(\beta/2)}(-N; 2a/\beta; (-s)^a).$$

Similarly for the one-point density with β even

$$\rho_{(1)}^{(N+1)}(r) \propto r^a e^{-r} \Big\langle \prod_{l=1}^N (r - x_l)^\beta \Big\rangle$$

$$\propto r^a e^{-r} \, {}_1F_1^{(\beta/2)}(-N; 2a/\beta + 2; (r)^\beta).$$

Skew orthogonal polynomials

For orthogonal and symplectic symmetry, one encounters an anti-symmetric matrix with entries a skew symmetric inner product $\langle f, g\rangle_{\mathrm{s}} = -\langle g, f\rangle_{\mathrm{s}}$.

It is possible to choose a basis of monic polynomials $\{p_{l-1}(x)\}_{l=1,\dots,N}$ so that this anti-symmetric matrix is block diagonal, with the blocks 2×2 anti-symmetric matrices $\begin{bmatrix} 0 & h_{j-1} \\ -h_{j-1} & 0 \end{bmatrix}$.

In terms of matrix averages, one has

$$p_{2n}(z) = \langle \det(z\mathbb{I}_{2n} - G)\rangle_G$$
$$p_{2n+1}(z) = zp_{2n}(z) + \langle \det(z\mathbb{I}_{2n} - G)\mathrm{Tr}\, G\rangle_G.$$

For orthogonal symmetry, invariance of a single matrix entry under the reflection $G_{jk} \mapsto -G_{jk}$ implies

$$p_{2n}(z) = z^{2n} \qquad \text{and} \qquad p_{2n+1}(z) = z^{2n+1} - \langle \mathrm{Tr}\, G^2\rangle z^{2n-1}.$$

Application to products of real Gaussian matrices I

For the product of m independent standard Gaussian matrices

$$p_{2j}(z) = z^{2j}, \qquad p_{2j+1}(z) = z^{2j+1} - (2j)^m z^{2j-1}$$

with normalisation

$$h_{j-1} = (2\sqrt{2\pi}\Gamma(2j-1))^m.$$

For $m = 1$ the corresponding $N \to \infty$ correlation kernel is

$$\mathbf{K}^{\mathrm{rr}}(x,y) = \begin{bmatrix} \frac{1}{\sqrt{2\pi}}(y-x)e^{-(x-y)^2} & \frac{1}{\sqrt{2\pi}}e^{-(x-y)^2} \\ -\frac{1}{\sqrt{2\pi}}e^{-(x-y)^2} & \frac{1}{2}\mathrm{sgn}\,(x-y)\,\mathrm{erfc}(|x-y|/\sqrt{2}) \end{bmatrix}.$$

Beenakker and co-workers relates this state to the level crossings of so-called Majorana zero modes for a disordered semiconducting wire at a Josephson junction, in a weak magnetic field. Also describes the annihilation process $A + A \to \emptyset$ in the limit $t \to \infty$.

Application to products of real Gaussian matrices II

For a general class of integral operators on $(0, s)$ with difference kernel $K(x, y) = K(x - y, 0)$,

$$\log\det(\mathbb{I} - \xi K_{(0,s)}) \underset{s\to\infty}{\sim} \frac{s}{2\pi}\int_{-\infty}^{\infty} \log(\mathbb{I} - \xi \tilde{K}(u))\, du + O(1),$$

where $\tilde{K}(u) = \int_{-\infty}^{\infty} e^{ixu} K(x, 0)\, dx$. This remains true in the matrix case, implying

$$\log\det(\mathbb{I}_2 - \mathbf{K}^{\mathrm{rr}}_{(0,s)}) \underset{s\to\infty}{\sim} -\frac{s}{2\sqrt{2\pi}}\zeta(3/2)$$

Relevant to gap probability, and probability of no real eigenvalues for N finite.

For general m and N fixed (even), the probability $p^m_{N,N}$ that all eigenvalues are real equals

$$\Big(\prod_{j=1}^{N} \frac{1}{\Gamma(j/2)}\Big)^m \det\Big[[G^{m+1,m}_{m+1,m+1}\Big(\begin{matrix} \frac{3}{2} - j, \dots, \frac{3}{2} - j, 1 \\ 0, k, \dots, k \end{matrix} \Big| 1 \Big)]_{j,k=1,\dots,N/2} \Big]$$

Can use this to show $p^m_{N,N} \to 1$ as $m \to \infty$.

Application to products of real Gaussian matrices III

For $m = 1$, the expected number of real eigenvalues E_N has the large N form $\sqrt{2N/\pi}$, and to leading order the variance is $(2-\sqrt{2})E_N$.

In the global scaling regime

$$\lim_{N\to\infty} N^{m-1}\rho^{c}_{(1)}(N^{m/2}w) = \frac{|w|^{(2/m)-2}}{m\pi}\chi(1 > |w|)$$

and

$$\lim_{N\to\infty} \frac{N^{(m-1)/2}\rho^{r}_{(1)}(N^{m/2}x)}{\mathbb{E}(\#\mathrm{reals})} = \frac{|x|^{(1/m)-1}}{2m}\chi(x^2 < 1)$$

The first result can be understood from free probability theory as the m-th power of the circular law. The mechanism for the universality of the latter result remains to be determined.