Exercises: Set 1

Q1. Matrices from $U(2)/(U(1))^2$ can be parameterised

$$U = \begin{bmatrix} \cos\phi & \sin\phi \\ -e^{-i\alpha}\sin\phi & e^{-i\alpha}\cos\phi \end{bmatrix}$$

where $0 < \phi < \pi/2$, $0 < \alpha < 2\pi$.

(a) Explain why the first entries in each column have been chosen to be real and positive.

(b) Deduce that $U^\dagger dU = -dU^\dagger U$ and thus conclude $U^\dagger dU$ is equal to i times an Hermitian matrix.

(c) Show that

$$U^\dagger dU = -i \begin{bmatrix} \sin^2\phi\, d\alpha & i\gamma \\ -i\bar{\gamma} & \cos^2\phi\, d\alpha \end{bmatrix}$$

where $\gamma = d\phi + i\sin\phi\cos\phi\, d\alpha$.

(d) From your answer to (c) deduce that $(U^\dagger dU) = \sin\phi\cos\phi\, d\phi\, d\alpha$, clearly stating the meaning you have given to $(U^\dagger dU)$.

Q2. According to Euler, an element of SO(3) can be decomposed

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $0 \le \theta \le \pi$, $0 \le \phi, \psi < 2\pi$.

(a) With the 3 columns denoted $\vec{q}_1$, $\vec{q}_2$, $\vec{q}_3$, show that

$$\vec{q}_1 = \begin{bmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi \\ -\cos\psi\sin\phi - \cos\phi\cos\theta\sin\psi \\ \sin\theta\sin\psi \end{bmatrix}, \vec{q}_2 = \begin{bmatrix} \cos\theta\cos\psi\sin\phi + \cos\phi\sin\psi \\ \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi \\ -\cos\psi\sin\theta \end{bmatrix}, \vec{q}_3 = \begin{bmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{bmatrix}$$

(b) Explain why the invariant measure is equal to $\vec{q}_1^T d\vec{q}_3 \wedge \vec{q}_2^T d\vec{q}_3 \wedge \vec{q}_1^T d\vec{q}_2$.

(c) Observe that $\vec{q}_3$ does not depend on ψ, and thus $\vec{q}_1^T d\vec{q}_3 \wedge \vec{q}_2^T d\vec{q}_3$ does not contain $d\psi$, implying that all dependence on $d\psi$ comes from $\vec{q}_1^T d\vec{q}_2$. In particular, only the term proportional to $d\psi$ is relevant to $\vec{q}_1^T d\vec{q}_2$. Show that this term is equal to

$$\begin{bmatrix} -\cos\theta\sin\psi\sin\phi + \cos\phi\cos\psi \\ -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi \\ \sin\psi\sin\theta \end{bmatrix} d\psi$$

Take the dot product with $\vec{q}_1^T$ and simplify (using computer algebra) to down to $d\psi$.

(d) Noting that

$$d\vec{q}_3 = \begin{bmatrix} \cos\theta\sin\phi \\ \cos\theta\cos\phi \\ -\sin\theta \end{bmatrix} d\theta + \begin{bmatrix} \sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ 0 \end{bmatrix} d\phi,$$

with the help of computer algebra show

$$\vec{q}_1^T d\vec{q}_3 = \cos\psi d\theta + \sin\psi\sin\theta d\phi$$
$$\vec{q}_2^T d\vec{q}_3 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi$$

(e) From the above working conclude $\vec{q}_1^T d\vec{q}_3 \wedge \vec{q}_2^T d\vec{q}_3 \wedge \vec{q}_1^T d\vec{q}_2 = \sin\theta d\theta d\phi d\psi$.

Q3. For $U \in U(N)$, the eigenvalue/ eigenvector decomposition reads $U = VDV^\dagger$ where $D = \mathrm{diag}\,(e^{i\theta_1}, \dots, e^{i\theta_N})$, and V is the matrix of eigenvectors.

(a) Show that

$$V^\dagger dU\, V = V^\dagger\, dV\, D + dD - DV\, dV^\dagger,$$

and from this read off that the element in position (jk) of $V^\dagger dU\, V$ is equal to

$$(e^{i\theta_k} - e^{i\theta_j})\vec{v}_j d\vec{v}_k$$

while the element in position (jj) is $ie^{i\theta_j} d\theta_j$.

(b) Use (a) to deduce that

$$\mathrm{Tr}\, dU dU^\dagger = \sum_{j=1}^N (d\theta_j)^2 + 2\sum_{j<k} |e^{i\theta_k} - e^{i\theta_j}|^2 ((\vec{v}_j^\dagger d\vec{v}_k)^{\mathrm{r}})^2 + ((\vec{v}_j^\dagger d\vec{v}_k)^{\mathrm{i}})^2$$

Q4. Show that for $A, X, Y \in \mathrm{GL}_N(\mathbb{R})$, and with $Y = AX$,

$$(dY) = (\det A)^N (dX)$$

Q5. (a) Let $M \in GL_N(\mathbb{R})$. In the singular value decomposition $M = O_1 D O_2^T$, note that O_2 can be interpreted as the matrix of eigenvectors of $M^T M$ and thus as a member of $O(N)/\{-1, 1\}$.

(b) Use the fact that $O_1^T dM\, O_2 = O_1^T dO_1\, D + dD - DO_2^T\, dO_2$, and the anti-symmetry of $O_i^T dO_i$ to deduce that

$$(dM) = 2^{-N} (O_1^T dO_1)(O_2^T dO_2) \prod_{1 \le j < k \le N} (\sigma_j^2 - \sigma_k^2) d\sigma_1 \cdots d\sigma_N$$

Q6. Define

$$I(t) = \int_0^R dy_1 \cdots \int_0^R dy_N\, \delta\Big(t - \prod_{l=1}^N y_l\Big) \prod_{1 \le j < k \le N} |y_j^2 - y_k^2|$$

(a) Show that

$$\int_0^\infty t^{s-1} I(t)\, dt = 2^{-N} R^{Ns + N(N-1)} \int_0^1 dx_1 \cdots \int_0^1 dx_N \prod_{l=1}^N x_l^{s/2-1} \prod_{1 \le j < k \le N} |x_k - x_j|$$

(b) By using the Selberg integral, deduce that for a suitable c

$$I(1) = 2^{-N} \frac{R^{N(N-1)}}{2\pi i} \int_{c-i\infty}^{c+i\infty} R^{Ns} \prod_{j=0}^{N-1} \frac{\Gamma((s+j)/2)}{\Gamma((s+N+1+j)/2)}\, ds$$

Q7. With $s_1, s_2 > 0$, consider the decomposition

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} s_1 & r \\ 0 & s_2 \end{bmatrix} Q, \qquad Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(a) Show that

$$dA\, Q^T = \begin{bmatrix} ds_1 & dr + s_1 d\theta \\ -s_2 d\theta & ds_2 + r d\theta \end{bmatrix}$$

and thus $(dA) = s_2 ds_1 ds_2 dr d\theta$.

(b) Use (a) to show $\int_0^{2\pi} d\theta \int_0^\infty ds_2\, \delta(1 - s_1 s_2)\, (dA) = 2\pi \frac{ds_1 dr}{s_1^2}$

Q8. (a) Let $\vec{b}_1$, $\vec{b}_2$ be linearly independent vectors in $\mathbb{R}^2$ such that $|\vec{b}_1| \leq |\vec{b}_2|$. Show that the inequality $2|\vec{b}_1 \cdot \vec{b}_2| \leq |\vec{b}_1|^2$ is equivalent to the inequality $\vec{b}_2 + n\vec{b}_1| \geq |\vec{b}_2$ for all $n \in \mathbb{Z}$.

(b) Suppose $\vec{b}_1, \vec{b}_2$ are as in (a). Let $\vec{u} = n_1\vec{b}_1 + n_2\vec{b}_2$, $n_1, n_2 \in \mathbb{Z}$. Show that for $(n_1, n_2) \neq (0, 0)$, $|\vec{u}| \geq |\vec{b}_1|$, and for $n_1 \neq 0$, $|\vec{u}| \geq |\vec{b}_2|$.

Q9. Consider the un-normalised measure $2\pi dr_{11} dr_{12}$ restricted to the region $r_{12}^2 + r_{22}^2 \geq r_{11}^2$ and $2|r_{12}| \leq r_{11}$.

(a) Show that the volume of this region is equal to $\pi/3$.

(b) Show that the PDF of the distribution of the variable r_{11} is

$$\frac{12}{\pi}\Big(\frac{s}{2} - \chi_{s>1}(s^2 - 1/s^2)^{1/2}\Big), \quad 0 < s < (4/3)^{1/4}$$