Exercises: Set 2

Q1. Book, exercises 1.9, q.3.

Q2. Book, exercises 1.1 q.1.

Q3. In the Harish-Chandra/ Itzykson–Zuber integral, show that in the limit $\mathbf{B} \to \mathbb{I}$, the RHS is proportional to $e^{-\mathrm{Tr}\,\mathbf{A}}$.

Q4. For $\mathbf{S}_1 \in$ COE, and thus diagonalisable according to $\mathbf{S}_1 = \mathbf{V}\mathbf{D}\mathbf{V}^T$, for $\mathbf{V} \in O(N)$, show that

$$\mu(d\mathbf{S}_1) = \prod_{1 \le j < k \le N} |e^{i\theta_k} - e^{i\theta_j}| \prod_{l=1}^{N} d\theta_l (\mathbf{V}^T d\mathbf{V}).$$

Q5. Let $\mathbf{U} \in \mathrm{C}\beta\mathrm{E}$, and define the Hermitian matrix $\mathbf{H} = i(\mathbb{I}_N - \mathbf{U})(\mathbb{I}_N + \mathbf{U})^{-1}$. Write the invariant measure $\mu(d\mathbf{U})$ in terms of the eigenvalues and eigenvectors of $\mathbf{H}$ to show

$$d\mu(\mathbf{U}) = 2^{N(\beta(N-1)/2+1)} \Big(\det(\mathbb{I}_N + \mathbf{H}^2) \Big)^{-\beta(N-1)/2-1}$$

Q6. Consider $\mathbf{R} \in O^+(2n)$.

(a) Show that the eigenvalues occur in complex conjugate pairs $e^{\pm i\theta_j}$, $(j = 1, \dots, n)$, as do the corresponding eigenvectors.

(b) In the diagonalisation formula $\mathbf{R} = \mathbf{U}\mathbf{L}\mathbf{U}^\dagger$, explain why there must be $2n(n-1)$ independent elements in $\mathbf{U}^\dagger d\mathbf{U}$). Note that $(\mathbf{U}^\dagger d\mathbf{U})_{2j-1,2j} = 0$, $(j = 1, \dots, n)$, so suggesting that the strictly upper triangular entries of $\mathbf{U}^\dagger d\mathbf{U}$ excluding these are independent (but not their individual real and imaginary parts).

(c) From (b) show that

$$(\mathbf{R}^T d\mathbf{R}) = \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 |e^{i\theta_j} - e^{-i\theta_k}|^2 \prod_{l=1}^{n} d\theta_l \, (\mathbf{U}^\dagger d\mathbf{U}).$$

(d) Change variables $y_j = \cos\theta_j$ in the eigenvalue portion of (c) to obtain the Jacobi unitary ensemble with parameters $(n, -1/2, -1/2)$.

Q7. Book Exercises 2.6, q.1.

Q8. Book Exercises 3.2, q.6.

Q9. We know from the notes that for $\mathbf{A}, \mathbf{B}$ positive definite, with $\mathbf{B}$ random and having complex entries with distribution proportional to $e^{-\mathrm{Tr}\,\mathbf{B}}(\det \mathbf{B})^{n-N}(d\mathbf{B})$, the matrix product $\mathbf{C} = \mathbf{A}^{1/2}\mathbf{B}\mathbf{A}^{1/2}$ has distribution proportional to

$$(\det \mathbf{A})^{-n} (\det \mathbf{C})^{n-N} \int e^{-\mathrm{Tr}\,\tilde{\mathbf{A}}^{-1}\mathbf{U}\tilde{\mathbf{C}}\mathbf{U}^\dagger} (\mathbf{U}^\dagger d\mathbf{U}),$$

where $\tilde{\mathbf{A}}^{-1} = \mathrm{diag}\,(a_1^{-1}, \dots, a_N^{-1})$ and $\tilde{C} = \mathrm{diag}\,(c_1, \dots, c_N)$.

(a) Let the positive definite matrix $\mathbf{A}$ have eigenvalue PDF

$$\prod_{l=1}^{N} g(a_l) \prod_{1 \le j < k \le N} (a_k - a_j)^2.$$

By making use of the Harish-Chandra/ Itzykson–Zuber integral, show that the joint distribution of $\{a_j\}$ and $\{c_j\}$ is proportional to

$$\prod_{l=1}^{N} a_l^{-n} g(a_l) c_l^{n-N} \prod_{1 \le j < k \le N} (a_k - a_j)(c_k - c_j) \det \Big[e^{-(c_j/a_i)} \Big]_{i,j=1,\dots,N}$$

(b) Integrate over $\{a_j\}$ to show that the PDF for $\{c_j\}$ is of the form of a polynomial ensemble

$$\prod_{l=1}^{N} c_l^{n-N} \prod_{1 \le j < k \le N} (c_k - c_j) \det \Big[\int_0^\infty \frac{g(a)}{a^n} a^{i-1} e^{-c_j/a} \, da \Big]_{i,j=1,\dots,N}$$

Q10. Denote by $q(\{\lambda_l\}); \{y_l\})$ the conditional PDF

$$\prod_{j=1}^{N} \lambda_j^{n-N} e^{-\lambda_j} \prod_{l=1}^{N-1} y_l^{-(n-N+2)+1} e^{-y_l} \frac{\prod_{1 \le j < k \le N} (\lambda_j - \lambda_k)}{\prod_{1 \le j < k \le N-1} (y_j - y_k)},$$

and denote by R the region

$$x_1 > \lambda_1 > x_2 > \lambda_2 > \cdots > \lambda_{N-1} > x_N > 0.$$

Show that the recurrence

$$p_N(\{\lambda_l\}_{l=1}^N\} = \int_R q(\{\lambda_l\}_{l=1}^N; \{y_l\}_{l=1}^{N-1}) p_{N-1}(\{y_l\}_{l=1}^{N-1}\} \, d\vec{y}$$

is satisfied by p_N proportional to

$$\prod_{j=1}^{N} \lambda_j^{n-N} e^{-\lambda_j} \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2$$