Exercises: Set 3

Q1. The invariant measure for $U(N)$ satisfies the recurrence

$$(V_N^\dagger dV_N) = 2^{N-1}(V_{N-1}^\dagger dV_{N-1})(dS_N)$$

where (dS_N) is the volume form of the complex unit $(N-1)$-sphere. Hence deduce that

$$\frac{\mathrm{vol}\,(U(N))}{\mathrm{vol}\,U(N-1)} = 2^N \frac{\pi^N}{\Gamma(N)}.$$

Q2. (a) Use the fact that

$$\frac{1}{(2\pi)^{N/2}} \frac{1}{\pi^{N(N-1)/4}} e^{-\mathrm{Tr}\,\mathbf{X}^2/2}$$

defines a PDF on the space of real symmetric matrices to deduce that

$$\frac{1}{N!}\int_{-\infty}^{\infty} d\lambda_1 \cdots \int_{-\infty}^{\infty} d\lambda_N \prod_{l=1}^{N} e^{-\lambda_l^2/2} \prod_{1\le j<k\le N} |\lambda_k - \lambda_j| = \frac{(2\pi)^{N/2}\pi^{N(N-1)/4}}{\mathrm{vol}\,O(N)/((O(1))^N}$$

$$= (2\pi)^{N/2}\prod_{j=1}^{N}\Gamma(j/2)$$

(b) By identifying the COE as the quotient space $U(N)/O(N)$, explain why we must have

$$\frac{1}{C_N}\int_0^{2\pi} d\theta_1 \cdots d\theta_N \prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|$$

with

$$C_N = N! \frac{\mathrm{vol}\,(U(N)/O(N))}{\mathrm{vol}\,(O(N)/(O(1))^N)}.$$

Q3. In the case $N=1$ the Selberg density is the beta density

$$\frac{t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha,\beta)}, \quad 0<t<1.$$

where

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$$

is the beta integral.

(a) Show that is x, y are random variables with PDFs $x^{\alpha-1}e^{-x}$, $y^{\alpha-1}e^{-y}$, then $x/(x+y)$ has PDF given by the beta density.

(b) Book, Exercises 4.1 q.6.

Q4. (a) Use integration by parts to show that the beta integral satisfies the recurrence

$$(\alpha+\beta-1)B(\alpha,\beta) = (\beta-1)B(\alpha,\beta-1)$$

and thus

$$B(\alpha,\beta+M) = \frac{\Gamma(\alpha+\beta)\Gamma(\beta+M)}{\Gamma(\beta)\Gamma(\alpha+\beta+M)}B(\alpha,\beta).$$

(b) Show that for large β, $B(\alpha,\beta)\sim\beta^{-\alpha}\Gamma(\alpha)$, and thus deduce the evaluation of $B(\alpha,\beta)$ in terms of gamma functions.

Q5. (a) Show that with

$$\sum_{i=1}^{n} \frac{w_i}{a_i - \lambda} = \frac{\prod_{l=1}^{n-1}(a_j - \lambda_l)}{\prod_{l=1, l \neq j}(a_j - a_l)}$$

we have

$$w_j = \frac{\prod_{l=1}^{n-1}(a_j - \lambda_l)}{\prod_{l=1, l \neq j}^{n}(a_j - a_l)}.$$

(b) Deduce from (a) that the Jacobian for the change of variables for $\{w_j\}$ to the $\{\lambda_j\}$ is given by

$$\prod_{j=1}^{n-1} w_j \frac{\prod_{1 \leq j < k \leq n-1}(a_j - a_k)(\lambda_j - \lambda_k)}{\prod_{j,k=1}^{n-1} |a_j - \lambda_k|}$$

where use has been made of the Cauchy double alternate identity.

(c) In the circumstance that $\{w_i\}$ have the Dirichlet distribution

$$\frac{\Gamma(s_1) \cdots \Gamma(s_n)}{\Gamma(s_1 + \cdots + s_n)} \prod_{i=1}^{n} w_i^{s_i - 1}, \quad \sum_{i=1}^{n} s_i = 1$$

deduce that $\{\lambda_l\}$ have the Dixon–Anderson density for their PDF.

Q6. By setting $a_1 = 1$, $a_{N+1} = 0$ in the Dixon–Anderson integral, multiplying both sides by $\prod_{1 \leq j < k \leq n}(a_j - a_k)$, and integrating over $\{a_j\}$, show how a recurrence in N for the Selberg integral can be obtained.

Q7. Verify the identity

$$\sum_{\substack{S \subset \{1,\dots,2N+1\} \\ |S|=N}} \Delta(x_S)\Delta(x_{\{1,\dots,2N+1\}-S}) = 2^N \Delta(x_{\{1,3,\dots,2N+1\}})\Delta(x_{\{2,4,\dots,2N\}}),$$

by first showing that both sides are anti-symmetric under the interchange of x_i and x_{i+2}.

Q8. Book, Exercises 4.4 q.1.

Q9. Book, Exercises 5.5 q.5. & q.6.