Exercises: Set 4

Q1. Suppose $n \geq p$ and consider the chiral random matrix

$$K = \begin{bmatrix} 0_{n\times n} & X_{n\times p} \\ (X^\dagger)_{p\times n} & 0_{p\times p} \end{bmatrix}$$

(a) Show that K has $n-p$ zero eigenvalues, with the remaining eigenvalues given by $\pm$ the positive square root of the eigenvalues of $X^\dagger X$.

(b) Suppose $X = [x_{ij}]$, where each x_{ij} is independently distributed with mean zero and $\langle |x_{ij}|^2 \rangle = 1$. Show that

$$\langle \det(\lambda \mathbb{I}_{n+p} - K) \rangle = p! \lambda^{n-p} L_p^{n-p}(\lambda^2), \qquad L_m^a(x) := \sum_{k=0}^m (-1)^k \binom{m+a}{m-k} \frac{x^k}{k!}$$

Q2. Let $\Pi = \mathbb{I}_N - \vec{v}\vec{v}^\dagger$, where $\vec{v}$ is a unit vector such that $|v_i|^2 = \mu_i$. Define the recursive matrix structure

$$M_{N-1} = \Pi \operatorname{diag} M_N \, \Pi$$

Let $q_N(x) = \prod_{l=1}^N (x - \lambda_l)$ where $\{\lambda_l\}$ is the set of non-zero eigenvalue of M_N

(a) Show that

$$\det(M_{N-1} - \lambda \mathbb{I}) = -\lambda \det(A - \lambda I) \operatorname{Tr}\left((A - \lambda \mathbb{I})^{-1} \vec{v}\vec{v}^\dagger \right)$$

(b) Read off from (a) that

$$\frac{q_{N-1}(x)}{q_N(x)} = \sum_{j=1}^N \frac{\mu_j}{x - \lambda_j}$$

Q3. Let $\vec{w}$ be an $1 \times N$ column vector with $q_i = |w_i|^2$. Define the recursive matrix structure

$$M_{N+1} = \begin{bmatrix} \operatorname{diag} M_N & \vec{w} \\ \vec{w}^\dagger & 1 \end{bmatrix}$$

(a) With $q_N(x) = \det(x\mathbb{I}_N - M_N)$, show that

$$\frac{q_{N+1}(x)}{q_N(x)} = x - a - \sum_{i=1}^N \frac{q_i}{x - a_i}$$

(b) Suppose the distribution of q_i is given by $\Gamma[1,1]$, and in (b) above choose $\mu_j = q_i / (\sum_{j=1}^N q_j)$.

(c) Substitute the result of Q2.(b) in (a) to obtain the 3 term recurrence for the GUE

$$q_{N+1}(x) = (x-a) q_N(x) - (\sum_{j=1}^N q_j) p_{N-1}(x).$$