Evaluation of some multi-dimensional integrals

Peter Forrester,

M&S, University of Melbourne

Outline

- Classical groups
- Selberg and Dixon–Anderson integrals
- Superimposed and decimated ensembles
- Singular values

Volumes for the classical groups

A study of the invariant measure for $SO(N)$ reveals the factorisation

$$(V_N^T dV_N) = 2^{(N-1)/2}(V_{N-1}^T dV_{N-1})(dS_N),$$

where (dS_N) is the volume form for the unit $(N-1)$-sphere. Hence

$$\frac{\mathrm{vol}\,(SO(N))}{\mathrm{vol}\,(SO(N-1))} = 2^{(N-1)/2}\mathrm{vol}\,(S_N) = \frac{2^{(N+1)/2}\pi^{N/2}}{\Gamma(N/2)}.$$

A similar formula holds in the unitary case.

Knowledge of $\mathrm{vol}\,O(N)$ and $\mathrm{vol}\,U(N)$ allows the integrals

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \, e^{-\sum_{l=1}^N x_l^2} \prod_{1\le j<k\le N} |x_k - x_j|^\beta$$

and

$$\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \prod_{1\le j<k\le N} |e^{i\theta_k} - e^{i\theta_j}|^\beta$$

to be evaluated for $\beta = 1, 2$ (and $\beta = 4$ from vol $Sp(2N)$).

Selberg density and Selberg integral

The Selberg density is

$$\frac{1}{S_n(\alpha,\beta,\tau)} \prod_{l=1}^{n} t_l^{\alpha_1-1}(1-t_l)^{\alpha_2-1} \prod_{1\le j<k\le n} |t_j-t_k|^{2\tau}, \qquad 0<t_l<1$$

where

$$\begin{aligned} S_n(\alpha_1,\alpha_2,\tau) &:= \int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha_1-1}(1-t_i)^{\alpha_2-1} \prod_{1\le j<k\le n} |t_j-t_k|^{2\tau}\, dt_1\cdots dt_n \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha_1+j\tau)\Gamma(\alpha_2+j\tau)\Gamma(1+(j+1)\tau)}{\Gamma(\alpha_1+\alpha_2+(n+j-1)\tau)\Gamma(1+\tau)}, \end{aligned}$$

is the Selberg integral. The integrals on the previous slide are a special case.

Dixon-Anderson integral

Question: What is the PDF for the density of zeros of the random rational function

$$\sum_{i=1}^{n} \frac{w_i}{a_i - \lambda}$$

where $(w_1, \ldots, w_n) \sim D_n(s_1, \ldots, s_n)$ (Dirichlet distribution)?
Answer: (Dixon (1905), Anderson (1991)) The zeros have PDF

$$\frac{\Gamma(s_1 + \cdots + s_n)}{\Gamma(s_1) \cdots \Gamma(s_n)} \frac{\prod_{1 \le j < k \le n-1} (\lambda_j - \lambda_k)}{\prod_{1 \le j < k \le n} (a_j - a_k)^{s_j + s_k - 1}} \prod_{j=1}^{n-1} \prod_{p=1}^{n} |\lambda_j - a_p|^{s_p - 1}$$

supported on

$$a_1 > \lambda_1 > a_2 > \lambda_2 > \cdots > \lambda_{n-1} > a_n.$$

Dixon/ Anderson density is a PDF in $\{\lambda_j\}$, so integrating out these variables must give one.

Hence with X the region $a_1 > \lambda_1 > \cdots > \lambda_N > a_{N+1}$,

$$\int_X d\lambda_1 \cdots d\lambda_N \prod_{1 \le j < k \le N} (\lambda_j - \lambda_k) \prod_{j=1}^{N} \prod_{p=1}^{N+1} |\lambda_j - a_p|^{s_p - 1}$$

$$= \frac{\prod_{i=1}^{N+1} \Gamma(s_i)}{\Gamma(\sum_{i=1}^{N+1} s_i)} \prod_{1 \le j < k \le N+1} |a_k - a_j|^{s_j + s_k - 1}.$$

Can use this to deduce a recurrence for the Selberg integral in N:

$$S_{N+1}(\lambda_1, \lambda_2, \lambda) = \frac{(N+1)\Gamma((N+1)\lambda)\Gamma(1+\lambda_1)\Gamma(1+\lambda_2)}{\Gamma(\lambda)\Gamma(2+\lambda_1+\lambda_2+N\lambda)}$$

$$\times S_N(\lambda_1 + \lambda, \lambda_2 + \lambda, \lambda)$$

(iterate with $S_0 = 1$ to reclaim Selberg integral evaluation). Another consequence is the inter-relation

$$\text{even}\, \mathrm{OE}_{2N+1}(x^{(a-1)/2}(1-x)^{(b-1)/2}) = \mathrm{SE}_N(x^{a+1}(1-x)^{b+1}).$$

Superimposed orthogonal symmetry ensembles

Let $S = \{l_1, \ldots, l_N\}$ and let $\Delta(x_S) = \prod_{1 \le j < k \le N}(x_{l_j} - x_{l_k})$. One can verify the identity

$$\sum_{\substack{S \subset \{1,\ldots,2N+1\} \\ |S|=N}} \Delta(x_S)\Delta(x_{\{1,\ldots,2N+1\}-S}) = 2^N \Delta(x_{\{1,3,\ldots,2N+1\}})\Delta(x_{\{2,4,\ldots,2N\}}),$$

of a type introduced into random matrix theory by Gunson. The LHS relates to the superposition $\mathrm{MO}_N(f) \cup \mathrm{MO}_{N+1}(f)$.

Using the Dixon-Anderson integral we can read off that

$$\mathrm{even}\Big(\mathrm{OE}_N(f) \cup \mathrm{OE}_{N+1}(f)\Big) = \mathrm{UE}_N(g)$$

for $f(x) = x^{(a-1)/2}(1-x)^{(b-1)/2}$ and $g(x) = x^a(1-x)^b$.

Generalized Dixon-Anderson

Theorem

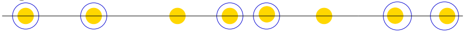
The Dixon-Anderson PDF is the $r = 1$ case of the family of conditional PDFs

$$\frac{1}{\hat{C}} \frac{\prod_{1 \le j < k \le r(n-1)} (\lambda_j - \lambda_k)^{2/(r+1)}}{\prod_{1 \le j < k \le n} (a_j - a_k)^{r(s_j + s_k - 2/(r+1))}} \prod_{j=1}^{r(n-1)} \prod_{p=1}^{n} |\lambda_j - a_p|^{s_p - 1} \chi_{A_r}$$

where A_r is the interlaced region

$$a_j > \lambda_{r(j-1)+1} > \lambda_{r(j-1)+2} > \cdots > \lambda_{r(j-1)+r-1} > a_{j+1} \quad (j = 1, \ldots, n-1).$$

Consequence, e.g.



$$D_3(\mathrm{ME}_{2/3,3N+2}(e^{-x^2}) = \mathrm{ME}_{6,N}(e^{-3x^2})$$

The proof

Set

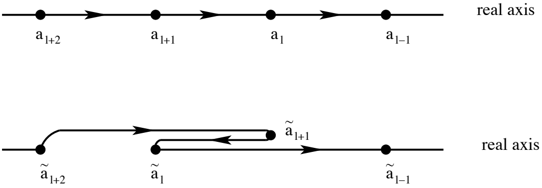
$$L_{r,n}(\{a_p\}) := \int_{A_r} d\lambda_1 \cdots d\lambda_{r(n-1)}$$

$$\times \prod_{1\le j<k\le r(n-1)} (\lambda_j - \lambda_k)^{2/(r+1)} \prod_{j=1}^{r(n-1)} \prod_{p=1}^{n} |\lambda_j - a_p|^{s_p - 1}$$

We proceed by proving

$$L_{r,n}(\{a_p\})\Big|_{\substack{a_l \leftrightarrow a_{l+1} \\ s_l \leftrightarrow s_{l+1}}} = e^{-\pi i r(s_l + s_{l+1} - 2/(r+1))} L_{r,n}(\{a_p\})$$

i.e. interchanging the order of the $\{a_p\}$ on the real line via analytic continuation, and making a corresponding change to the order of $\{s_p\}$, gives back the same integral representation, up to a phase.



This relies on a cancellation effect:

Lemma

Consider an arrangement of r $0's$ and q $1's$ $(1 \leq q \leq r)$ in a line.

Let this be considered as the sequence $\mathcal{A} = (n_j)_{j=1,\ldots,r+q}$ with each $n_j = 0$ or 1. Further let

$$K(n_j) = \begin{cases} 0, & n_j = 0 \\ \#0's \text{ to the right of } n_j, & n_j = 1, \end{cases}$$

and use this to specify the statistic

$$K(\mathcal{A}) = \sum_{j=1}^{r+q} K(n_j) = \sum_{j=1}^{r+q} n_j(r+q-j)$$

One has

$$\sum_{\mathcal{A}} e^{-2\pi i K(\mathcal{A})/(r+1)} = 0.$$

Evenness symmetry

Elementary row and column operations give

$$\det[a_{i-j}]_{i,j=1,\dots,2N} = \det[a_{i-j} + a_{i+j-1}]_{i,j=1,\dots,N} \det[a_{i-j} - a_{i+j-1}]_{i,j=1,\dots,N}.$$

Using this, and an analogous identity in the odd size case, Rains '03 showed

$$|U(n)| = O^+(n+1) \cup O^-(n+1).$$

A generalisation is that provided w_2 is even

$$|\mathrm{UE}_n(w_2)| = \mathrm{chUE}_{\lceil n/2 \rceil}(w_2) \cup \mathrm{chUE}_{\lfloor n/2 \rfloor}(x^2 w_2)$$

Here $\mathrm{chUE}_N(g)$ is specified by the PDF on $x_l > 0$ proportional to

$$\prod_{l=1}^{N} g(x_l) \prod_{j<k} (x_k^2 - x_j^2).$$

For example

$$|\mathrm{GUE}_n| = \mathrm{aGUE}_n \cup \mathrm{aGUE}_{n+1}$$

Decimation of singular values

Proposition: We have, for certain pairs of (w_1, w_2)

$$\mathrm{even}\,|\mathrm{OE}_{2n}(w_1)| = \mathrm{chUE}_n(w_2)$$

The proof of this result relies on a variant of the Gunson identity.

Lemma Suppose $x_{2n} < \cdots < x_1$. Let $\Delta(\{y_j\})$ denote the product of differences in $\{y_j\}$. We have

$$\sum_{\epsilon \in \{\pm\}^{2n}} |\Delta(\{\epsilon(j) x_j\})| = 2^{2n} \Delta(\{x_{2j-1}^2\}) \prod_{j=1}^{n} x_{2j} \Delta(\{x_{2j}^2\})$$

Now integrate over $\{x_{2j-1}^2\}$. In the Jacobi case this can be done using the Dixon-Anderson integral in squared variables.

Similarly, $\mathrm{even}\,|\mathrm{OE}_{2n+1}(w_1)| = \mathrm{chUE}_n(x^2 w_2)$.

Circular case I

In the Cauchy case, the singular value decimation result for an even number of eigenvalues reads

$$\mathrm{even}\left|\mathrm{OE}_{2n}\Big(\frac{1}{(1+x^2)^{(2n+a+1)/2}}\Big)\right| = \mathrm{chUE}_n\Big(\frac{x^2}{(1+x^2)^{n+a}}\Big)$$

With $a = 0$, this is equivalent (apply stereographic projection) to

$$\mathrm{even}\,|\mathrm{COE}_{2n}| = O^+(2n+1), \qquad \mathrm{odd}\,|\mathrm{COE}_{2n}| = O^-(2n+1),$$

Taking the limit $a \to -1$ this is equivalent to

$$\mathrm{even}\,|\mathrm{COE}_{2n-1}| = O^+(2n), \qquad \mathrm{odd}\,|\mathrm{COE}_{2n-1}| = O^-(2n-1).$$

Circular case II

Combine

$$\text{even}\,|\mathrm{OE}_{2n}(w_1)| = \mathrm{chUE}_n(w_2)$$

and

$$\text{even}\,|\mathrm{OE}_{2n+1}(w_1)| = \mathrm{chUE}_n(x^2 w_2)$$

with

$$|\mathrm{UE}_{2n}(w_2)| = \mathrm{chUE}_n(w_2) \cup \mathrm{chUE}_n(x^2 w_2)$$

to get

$$|\mathrm{UE}_{2n}(w_2)| = \text{even}\,|\mathrm{OE}_{2n}(w_1)| \cup \text{even}\,|\mathrm{OE}_{2n+1}(w_1)|$$

Similarly for an odd number of eigenvalues on LHS. Thus, e.g. for all n

$$|\mathrm{GUE}_n| = \text{even}\,|\mathrm{GOE}_n| \cup \text{even}\,|\mathrm{GOE}_{n+1}|$$

In the Cauchy case, after stereographic projection, these imply

$$|\mathrm{CUE}_n| = \text{even}\,|\mathrm{COE}_n| \cup \text{odd}\,|\mathrm{COE}_n|$$