# Global Spectral Distributions and Linear Statistics of Spectra of Product Matrices 

Friedrich Götze

Bielefeld University
www.math.uni-bielefeld.de/~goetze

Joint work with: A. Naumov and A. Tikhomirov
Random Product Matrices
New Developments \& Applications,
Bielefeld, August 26, 2016

## Topics

- Approaches to Universality of Spectra of Products of Matrices


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- The Central Limit Theorem for Linear Singular Value Statistics of Product Matrices


## Spectral Universality for Product Matrices

- Let $\mathbf{X}^{(q)}, q=1, \ldots, m$ be $m$ independent random matrices:

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$$
F_{n}^{\mathrm{W}}(x)=\frac{1}{n} \sum_{k=1}^{n} \mathbb{1}\left(s_{k}^{2} \leq x\right)
$$

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satisfying (G.-Kösters-Tikhomirov (2014)) (hypergeometric function)

$$
1+z s_{m}(z)+(-1)^{m+1} z^{m} s_{m}^{m+1}(z)=0
$$

## Extensions to Singular and Complex Spectral Models

Singular values of sequences of products of $m$ rectangular $p_{l} \times p_{l+1}$ matrices with $\lim _{n \rightarrow \infty} \frac{n}{p_{l}}=y_{l} \in(0,1]$ have a limit df $G_{\mathbf{y}}$ such that

Theorem (G.-Kösters-Tikhomirov (2014))
Let $\mathbf{E} X_{j k}^{(\nu)}=0, \mathbf{E}\left|X_{j k}^{(\nu)}\right|^{2}=1$. Assume Lindeberg: i.e. for any $\tau>0$

$$
L_{n}(\tau):=\max _{\nu=1, \ldots, m} \frac{1}{n^{2}} \sum_{j=1}^{p_{\nu-1}} \sum_{k=1}^{p_{\nu}} \mathbf{E}\left|X_{j k}^{(\nu)}\right|^{2} I_{\left\{\left|X_{j k}^{(\nu)}\right| \geq \tau \sqrt{n}\right\}} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

Then,

$$
\lim _{n \rightarrow \infty} \sup _{x}\left|F_{n}(x)-G_{y}(x)\right|=0
$$

## Elliptical Random Matrix Ensembles

- let $\mathbf{X}_{n}(\omega)=\left\{X_{i j}(\omega)\right\}_{i, j=1}^{n} \quad$ with condition $\mathbf{C 0}$ :
a) $\left(X_{j k}, X_{k j}\right)$ mutually independent for $1 \leq j<k \leq n$;
b) for any $j, k=1, \ldots, n$

$$
\mathbf{E} X_{j k}=0 \text { and } \mathbf{E} X_{j k}^{2}=1 ;
$$

c) for any $1 \leq j<k \leq n$

$$
\mathbf{E}\left(X_{j k} X_{k j}\right)=\rho, \quad|\rho| \leq 1 ;
$$

## Examples of Elliptical Laws



$$
n=3000, \quad \rho=-0.5
$$



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## Examples: Product Laws for Elliptical Matrices


i.i.d. $m=2 \quad n=3000, \rho=0$


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## Product Laws for Elliptical non Hermitian Matrices

- Th. (non i.i.d. product case) (G.-Naumov-Tikhomirov, (2013)). Let $\mathbf{X}_{n}^{(q)}, q \geq 2$ be independent $n \times n$ random matrices, Assume C0 and $|\rho|<1$ and condition:


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(L*) $\quad \sup _{q, j, k} \mathbf{E}\left|X_{j k}^{(q)}\right|^{2} I\left(\left|X_{j k}^{(q)}\right| \geq M\right) \rightarrow 0$ as $n \rightarrow \infty$.
Let $\mathbf{V}=n^{-m / 2} \prod_{q=1}^{m} \mathbf{X}_{n}^{(q)}, \quad m \geq 2$ and
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$\mu_{n}$-empirical spectral measure of the eigenvalues of $\mathbf{V}$.
Then $\quad \mathbf{E} \mu_{n} \rightarrow \mu, \quad$ with density:

$$
g(x, y)= \begin{cases}\frac{1}{\pi m\left(x^{2}+y^{2}\right)^{\frac{m-1}{m}}}, & x, y \in\left\{u, v \in \mathbb{R}: u^{2}+v^{2} \leq 1\right\} \\ 0, & \text { elsewhere } .\end{cases}
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Gaussian case: Akemann, Burda $(2010,2012)$.

## Complex Spectra via Girko's Hermitization

Singular values versus spectral values with emp. df. $\mu_{n}$ :

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|\operatorname{det}(\mathbf{A})|=\prod_{i=1}^{n}\left|\lambda_{i}(\mathbf{A})\right|=\prod_{i=1}^{n} s_{i}(\mathbf{A})
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Complex spectra of smooth functions $\mathbb{F}$ of $m$ independent matrices $\mathbf{X}_{n}$ :
$\mathbb{F}\left(\mathbf{X}_{n}\right): \quad \nu_{\mathbb{F}(\mathbf{X})} \quad$ for matrices $\mathbf{X}_{n}=\left(\mathbf{X}_{n}^{(1)}, \ldots, \mathbf{Y}_{n}^{(m)}\right)$
$\mathbb{F}\left(\mathbf{Y}_{n}\right): \quad \nu_{\mathbb{F}(\mathbf{Y})} \quad$ for i.i.d. Gaussian matrices $\mathbf{Y}_{n}=\left(\mathbf{Y}_{n}^{(1)}, \ldots, \mathbf{Y}_{n}^{(m)}\right)$

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## Lemma ( Bordenave-Chafai (2009))

Weak convergence of $\nu_{\mathbb{F}(\mathbf{X})}$ and $\nu_{\mathbb{F}(\mathbf{Y})}$ to some limit $\nu_{\mathbb{F}}$ : condition (C1): Log-potential of shifted singular distr. $\nu_{\mathbb{F}(\mathbf{X})}$ and $\nu_{\mathbb{F}(\mathbf{Y})}$

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G.-Tikhomirov (2007), Tao and Vu (2010), methods: Rudelson (2006):
$\left(C_{0}\right)$ yields unif. log-integr. for $\quad m=1$ and $\quad \mathbb{F}(\mathbf{X})=\mathbf{X}^{(1)}$. (Circular law)


## Universality of $\mathbb{F}$

Theorem (G.-Kösters-Tikhomirov (2014), RMTA)

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- $\mathbb{F}_{\mathbf{X}}, \mathbb{F}_{\mathbf{Y}}$ satisfy condition (C1)
(uniform log integrability of shifted singular values of $\mathbb{F}$ )

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Universality of the singular values distribution of shifts holds.
Hence universality of (complex) eigenvalue distribution follows. Laws determined by asymptotic freeness and free calculus of $S$-transforms for product type functions $\mathbb{F}$ of $X_{j}, X_{j}^{*}, X_{j}^{-1}$ etc.

## Example: Singular Values of Products of Spherical Matrices

 $m \geq 1, \mathbf{X}^{(\nu)}=\frac{1}{\sqrt{n}}\left(X_{j k}^{(q)}\right)$ independent, independent entries.
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Product function
$\mathbb{F}=\prod_{q=1}^{m} \mathbf{X}^{(2 q-1)}\left(\mathbf{X}^{(2 q)}\right)^{-1} \quad$ and $\quad \mathbf{W}=\mathbb{F F} F^{*}$.
$\mathcal{G}_{n}(x)$ : empirical distribution function of $\mathbf{W}$.
Theorem (G.-Kösters-Tikhomirov (2014))
Assume that $X_{j k}^{(q)}$, for $q=1, \ldots, 2 m$ and $j, k=1, \ldots, n$ satisfy uniform $2 n d$ order moment conditions. Then

$$
\begin{aligned}
\lim _{n \rightarrow \infty} \mathcal{G}_{n}(x) & =G_{m}(x) \quad \text { in probability, } \quad \text { where, } \\
p_{m}(x)=G_{m}^{\prime}(x) & \left.=\frac{1}{\pi} \frac{\sin \frac{\pi m}{m+1}}{x^{\frac{m}{m+1}}\left(x^{\frac{2}{m+1}}-2 x^{\frac{1}{m+1}}\right.} \cos \frac{\pi m}{m+1}+1\right)
\end{aligned} .
$$

Forrester (2014), free multiplicative Levy processes Biane (1998)

## Linear Statistics of Singular Values of Product Matrices

- Let $\mathbf{X}^{(q)}, q=1, \ldots, m$ be $m$ independent random matrices:

$$
\mathbf{X}^{(q)}:=\frac{1}{\sqrt{n}}\left[X_{j k}^{(q)}\right]_{j, k=1}^{n} .
$$

with i.i.d. $X_{j k}^{(q)}$ entries for $1 \leq j, k \leq n$,

- for any $1 \leq j, k \leq n \quad \mathbf{E} X_{j k}^{(q)}=0$ and $\mathbf{E}\left(X_{j k}^{(q)}\right)^{2}=1$;
- $\mathbf{E}\left(X_{j k}^{(q)}\right)^{4}=: \mu_{4}<\infty$.

Let $\quad \mathbf{W}:=\prod_{q=1}^{m} \mathbf{X}^{(q)}$,
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$$
F_{n}^{\mathrm{W}}(x)=\frac{1}{n} \sum_{k=1}^{n} \mathbb{1}\left(s_{k}^{2} \leq x\right)
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Recall $G_{m}(x)$, is confined to $\left[0, K_{m}\right], K_{m}=(m+1)\left(1+\frac{1}{m}\right)^{m}$ and described via its Stieltjes transform:

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satisfying (hypergeometric function)

$$
1+z s_{m}(z)+(-1)^{m+1} z^{m} s_{m}^{m+1}(z)=0
$$

CLT for linear statistics: Wigner and Marcenko-Pastur: Jonsson (82),
Bai-Silverstein (10), Sinai-Soshnikov(98), Anderson-Zeitouni(06), Lytova-Pastur (09), Zheng (12)

## CLT for Linear Statistics of Eigenvalues of $\mathbf{W W}^{\top}$ for $m=2$

Consider functions $f: \mathbf{R} \rightarrow \mathbf{R}$ with $\quad \int_{-\infty}^{\infty}\left(1+|t|^{5}\right)|\widehat{f}(t)| d t<\infty$.

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& \begin{aligned}
& \sigma_{f}^{2}=\frac{\kappa_{4}}{2}\left[\int_{-a}^{a} f\left(\lambda^{2}\right)\left[p(\lambda)+\lambda p^{\prime}(\lambda)\right] d \lambda\right]^{2} \\
&+\frac{1}{2 \pi^{2}} \int_{-a}^{a} \int_{-a}^{a} \frac{\left(f\left(\lambda^{2}\right)-f\left(\mu^{2}\right)\right)^{2}}{(\lambda-\mu)^{2}} \\
& \quad \times \frac{\left[p(\lambda)-p^{\prime}(\lambda)(\lambda-\mu)\right]}{3 p(\mu)} \frac{\left[4 p_{1}(\mu)^{4}+11 p_{1}(\mu)^{2}+4\right]}{4 p_{1}^{2}(\mu)+3} d \lambda d \mu,
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where $\kappa_{4}:=m_{4}-3, \quad p_{1}(\lambda):=\pi p(\lambda), \quad p(\lambda):=|\lambda| P_{2}\left(\lambda^{2}\right)$ is the symmetrized Fuss-Catalan density, and $a:=\sqrt{K_{2}}$.

## The Gaussian Case and Stein-Tikhomirov

Assume W to be Gaussian. Use "linarisation" (Burda-Nowak-Swiech-al (2011)) of $s_{1}^{2}(\mathbf{W}), \ldots, s_{n}^{2}(\mathbf{W})$ constructing a $2 n \times 2 n$ hermitian block matrix $\mathbf{Y}$ with eigenvalues $\lambda_{j}(\mathbf{Y})$ :

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we are done via (Tikhomirov (1980))

$$
Z(x)=1-\sigma_{f}^{2} \int_{\text {Spectral Limits for Products }}^{x} y Z(y) d y
$$

## Integral Equations

Represent $\quad f(\lambda)=\int_{-\infty}^{\infty} \widehat{f}(t) e^{i t \lambda} ; \quad$ expand factor $S_{n}$ in $Z_{n}^{\prime}(x)$ as

$$
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Z_{n}^{\prime}(x)=i \mathbf{E} S_{n} e^{i \times S_{n}} & =\frac{i}{2} \int_{-\infty}^{\infty} \widehat{f}(t) \mathbf{E}\left(\operatorname{Tr} e^{i t Y}-\mathbf{E} \operatorname{Tr} e^{i t Y}\right) e^{i x S_{n}} d t, \\
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Let $p(x)$ denote spectral density of $\mathbf{Y}$. Since $e^{i \boldsymbol{Y} \mathbf{Y}}=\mathbf{I}+i \int_{0}^{t} \mathbf{Y} e^{i s \mathbf{Y}} d s$ and for a standard Gaussian random variable $\xi$ (Stein)

$$
\mathbf{E} \xi f(\xi)=\mathbf{E} \xi^{2} \mathbf{E} f^{\prime}(\xi)
$$

calculations leads to: sub-limiting equation for $Y(x, t):=\lim _{n} Y_{n}(x, t)$

$$
\begin{aligned}
& Y(x, t)+3 \int_{0}^{t} Y(x, s) \hat{p}^{2}(s-t) d s \\
& =-x Z(x) \int_{0}^{t}\left[2 \hat{p}(s) \widehat{f^{\prime} p}(t-s)+\widehat{f^{\prime} p}(s)\right] d s,
\end{aligned}
$$

## Fourier-Laplace Transforms and Volterra-Equations

A Fourier-Laplace transform of this Volterra equation with

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\tilde{p}(z):=i^{-1} \int \exp \{-i z t\} \hat{p}(-t) d t=\int \frac{1}{x-z} p(x) d x=s(z)
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using contour integration with $p_{1}(\mu):=\pi p(\lambda)$, and evaluating $\Im K, \Re K$ in terms of $\Im s, \Re s$.

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Furthermore needed: Gaussian approximations as well as concentration of measure for Lipschitz functions.

## Thank you!

## Multiplicative Free Convolution on $\mathbf{R}_{+}$

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1+M_{\mu}(z) & :=G_{\mu}\left(z^{-1}\right) / z, \quad\left(=1+\sum_{k=1}^{\infty} m_{k} z^{k}, \quad \text { formally }\right) \\
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$\mu_{1}, \mu_{2}$ : distribution on $\mathbf{R}_{+}: \quad S_{\mu_{1} \boxtimes \mu_{2}}=S_{\mu_{1}} S_{\mu_{2}}$.

## Lemma

$A_{n} \in \mathcal{M}(n \times n), n \in \mathbb{N}$ Gaussian complex random matrices, independent entries. Then

$$
\mathbf{A}_{n}:=\left[\begin{array}{cc}
\mathbf{0} & \mathbf{X}_{n} \\
\mathbf{X}_{n}{ }^{*} & \mathbf{0}
\end{array}\right] \quad \text { and } \quad \mathbf{B}_{n}:=\mathbf{J}(\alpha)=\left[\begin{array}{cc}
\mathbf{0} & -\alpha \mathbf{1} \\
-\bar{\alpha} \mathbf{l} & \mathbf{0}
\end{array}\right],
$$

are asymptotically free, where $\alpha=u+i v, u, v \in \mathbf{R}$.

## $R$-Transform of $J(\alpha)$

## $2 n \times 2 n$ block-matrix

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\mathbf{J}(\alpha)=\left(\begin{array}{lc}
\mathbf{0} & -\alpha \mathbf{l} \\
-\bar{\alpha} \mathbf{l} & \mathbf{0}
\end{array}\right)
$$

spectral distribution $\mu(\cdot)=\frac{1}{2} \delta_{|\alpha|}+\frac{1}{2} \delta_{-|\alpha|}$,

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M(z)=\frac{|\alpha|^{2} z^{2}}{1-|\alpha|^{2} z^{2}}
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\begin{gathered}
R_{\alpha}^{-1}(z)=\frac{\sqrt{z(1+z)}}{|\alpha|} . \\
R_{\alpha}^{2}(z)+R_{\alpha}(z)-|\alpha|^{2} z^{2}=0 .
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## $R$-Transform of $J(\alpha)$

$2 n \times 2 n$ block-matrix

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\mathbf{J}(\alpha)=\left(\begin{array}{lc}
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Solving this equation, we obtain

$$
R_{\alpha}(z)=\frac{-1+\sqrt{1+4|\alpha|^{2} z^{2}}}{2}
$$

$$
\mathbf{V}(\alpha):=\mathbf{V}-\mathbf{J}(\alpha)=\left[\begin{array}{cc}
\mathbf{0} & \mathbf{F}(\mathbf{Y})-\alpha \mathbf{I} \\
\mathbf{F}(\mathbf{Y})^{*}-\bar{\alpha} \mathbf{l} & \mathbf{O}
\end{array}\right]
$$

- $V$ has spectral limit $\mu_{V}$ with $R$-transform $R_{V}(z)$ and 1/(a) has limit Sticlties transform $g^{\prime}(z, a)$
- $\mathbf{V}$ and $\mathbf{J}(\alpha)$ are asymptotically free.

Then $g^{\prime}(z, a)$ and $w=W^{\prime}(z, a)$ are determined by the system

$g(z, \alpha)=(1+w g(z, \alpha)) S_{V}(-(1+w g(z, \alpha))$.

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$$
\begin{aligned}
w & =z+\frac{R_{\alpha}(-g(z, \alpha))}{g(z, \alpha)}, \quad z \text { large } \\
g(z, \alpha) & =(1+w g(z, \alpha)) S_{v}(-(1+w g(z, \alpha))
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The $S$-transform needs to be extended to $\mathbf{V}$ where $\mathbf{E}(\mathbf{V})=0$ WITH Two branches $S_{\mathrm{V}}(z)$ AND $\tilde{S}_{\mathrm{V}}(z)$

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## Spectral Limit Density of $\mathbb{F}(\mathbf{Y})$

$$
\alpha=u+i v, \quad u, v \in \mathbf{R}
$$

```
                                    q(y,\alpha) and require decay of tails of }\mp@subsup{\mu}{\textrm{V}}{}\mathrm{ ,
and }J(a)\mathrm{ are acymntotically, free
Let Sv resp. p(u,v) denote the limit S-transform resp. limit density of V for
n->\infty}\mathrm{ . Define
\varkappa(n):=-\mp@subsup{\operatorname{lim}}{\cdots,10}{}ig(ix,a)\geq0,\quad\psi(a)=\mp@subsup{\operatorname{lim}}{xj0}{0}(-i)g(ix,a)(-i)w(ix,a)\geq0.
Then \psi=\psi(\alpha)\geq0 and }\varkappa=\varkappa(\alpha)\geq0\mathrm{ satisfy the equations
\psi(1-\psi)=|\alpha\mp@subsup{|}{}{2}\mp@subsup{\varkappa}{}{2}\quad\mathrm{ and }\quad\varkappa=-i(1-\psi)S}\mp@subsup{S}{V}{}(-(1-\psi))
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```



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## Theorem ( G.-Kösters-Tikhomirov 2014, RMTA)

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$$

Let $Q_{V}(\alpha):=\int(\log |y|) q(y, \alpha) d y$. Then

$$
p(u, v)=\frac{1}{2 \pi} \Delta Q_{v}(\alpha)=\frac{1}{2 \pi|\alpha|^{2}}\left(u \frac{\partial \psi}{\partial u}+v \frac{\partial \psi}{\partial v}\right), \quad \alpha=u+i v,
$$

assuming that $\psi(\alpha)$ is differentiable up to a finite set of values $|\alpha|$.

