

Products of Independent Bi-Invariant Random Matrices

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(joint work with Mario Kieburg)

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Overview

- 1 Introduction
- 2 Old Results: Bi-Invariant Random Matrices
- 3 New Results: Polynomial Ensembles of Derivative Type
- 4 Summary and Open Problems

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Introduction

Let \mathbf{X}_n be a random matrix with values in $\mathrm{GL}(n, \mathbb{C})$. (why \mathbb{C} ?)

Definition:

\mathbf{X}_n is called *bi-invariant (bi-unitarily invariant) (isotropic)*
if $\mathbf{V}_n \mathbf{X}_n \mathbf{W}_n^* \stackrel{d}{=} \mathbf{X}_n$ for any (fixed) unitary matrices $\mathbf{V}_n, \mathbf{W}_n$.

Examples:

- Ginibre matrices
- truncated unitary matrices
- products of *independent* bi-invariant random matrices (or their inverses)

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Starting Point:

Products of independent Ginibre matrices (or their inverses)
have *determinantal* singular value and eigenvalue distributions.

Akemann–Burda (2012), Akemann–Strahov (2013), Akemann–Kieburg–Wei (2013), Akemann–Ipsen–Kieburg (2013),
Adhikari–Reddy–Reddy–Saha (2013), Ipsen–Kieburg (2014), Forrester (2014), Akemann–Ipsen–Strahov (2014),
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Introduction

Ginibre Matrix

$$f_{\text{SqSV}}(a) \propto |\Delta_n(a)|^2 \prod_{j=1}^n e^{-a_j} \propto \det(K_{\text{SqSV}}(a_j, a_k))_{jk=1,\dots,n}$$

$$f_{\text{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n e^{-|z_j|^2} \propto \det(K_{\text{EV}}(z_j, \bar{z}_k))_{jk=1,\dots,n}$$

Product of p Ginibre Matrices

Akemann–Kieburg–Wei (2013) Akemann–Burda (2012)

$$f_{\text{SqSV}}(a) \propto \Delta_n(a) \det [(-a_j \partial_{a_j})^{k-1} w_{p,0}(a_j)] \propto \det(K_{\text{SqSV}}^{(p)}(a_j, a_k))_{jk=1,\dots,n}$$

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$$\text{where } w_{p,0}(x) = (e^{-x})^{\otimes p} = G_{0,p}^{p,0} \left(\begin{smallmatrix} & \\ 0, \dots, 0 & \end{smallmatrix} \middle| x \right) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma^p(s) x^{-s} ds$$

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The same is true for products of independent truncated unitary matrices
(or their inverses).

Adhikari–Reddy–Reddy–Saha (2013), Ipsen–Kieburg (2014), Akemann–Burda–Kieburg–Nagao (2014),
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Moreover, it is possible to consider “mixed products”.

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Main Question:

Are there more examples of *bi-invariant* random matrices such that ...

- the singular value and eigenvalue distributions are *determinantal*,
- this structure is preserved when taking independent products ?

(Comment: HCIZ-type integrals)

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Harmonic Analysis on Matrix Spaces

Univariate Situation

X : r.v. with values in \mathbb{C}

and a **rotation-invariant** density f_X

dist. of $X \longleftrightarrow$ dist. of $|X|^2$

$|X|^2$: r.v. with values in \mathbb{R}_+

and a density $f_{|X|^2}$

Mellin Transform

$$\begin{aligned}\mathcal{M}_X(s) &= \int_{\mathbb{R}_+} f_{|X|^2}(y) y^s \frac{dy}{y} \\ &= \int_{\mathbb{C}} f_X(x) |x|^{2s} \frac{dx}{|x|^2}\end{aligned}$$

for suitable $s \in \mathbb{C}$

\mathcal{M}_X is defined on $1 + i\mathbb{R}$

\mathcal{M}_X determines the dist. of X

X_1, X_2 ind. $\Rightarrow \mathcal{M}_{X_1 X_2} = \mathcal{M}_{X_1} \cdot \mathcal{M}_{X_2}$

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Multivariate Situation

\mathbf{X} : r.v. with values in $GL(n, \mathbb{C})$
and a **bi-invariant** density $f_{\mathbf{X}}$

dist. of $\mathbf{X} \longleftrightarrow$ dist. of $\mathbf{X}^* \mathbf{X}$

$\mathbf{X}^* \mathbf{X}$: r.v. with values in $Pos(n, \mathbb{C})$
and a **conjugation-invariant** density $f_{\mathbf{X}^* \mathbf{X}}$

Spherical Transform

$$\begin{aligned}S_{\mathbf{X}}(s) &= \int_{Pos(n, \mathbb{C})} f_{\mathbf{X}^* \mathbf{X}}(y) \varphi_s(y) \frac{dy}{(\det y)^n} \\ &= \int_{GL(n, \mathbb{C})} f_{\mathbf{X}}(x) \varphi_s(x^* x) \frac{dx}{|\det x|^{2n}}\end{aligned}$$

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QR Decomposition I

Let \mathbf{X}_n be a random matrix with values in $\mathrm{GL}(n, \mathbb{C})$.

QR Decomposition (Gram decomposition, Iwasawa decomposition)

Any matrix $\mathbf{X}_n \in \mathrm{GL}(n, \mathbb{C})$ has a unique decomposition $\mathbf{X}_n = \mathbf{Q}_n \mathbf{R}_n$ with \mathbf{Q}_n unitary and \mathbf{R}_n upper-triangular with positive diagonal elements.

(This is essentially the Gram–Schmidt orthonormalization theorem.)

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Proposition (QR Decomposition of Bi-Invariant Random Matrices)

If \mathbf{X}_n is bi-invariant, then

- \mathbf{Q}_n and \mathbf{R}_n are independent,
- dist. of \mathbf{Q}_n = normalized Haar measure,
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- dist. of $\mathbf{X}_n \longleftrightarrow$ dist. of $\mathrm{diag}(\mathbf{R}_n) = (R_{11}, \dots, R_{nn})$.

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(non-symmetric in s_1, \dots, s_n)

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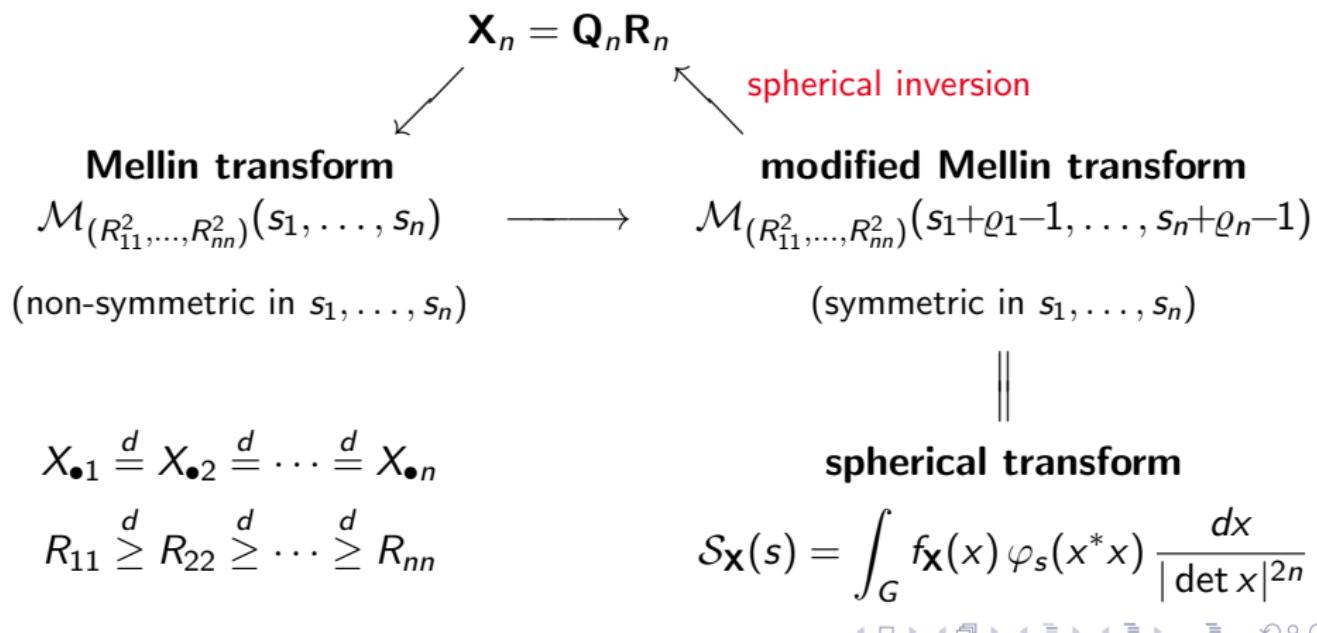
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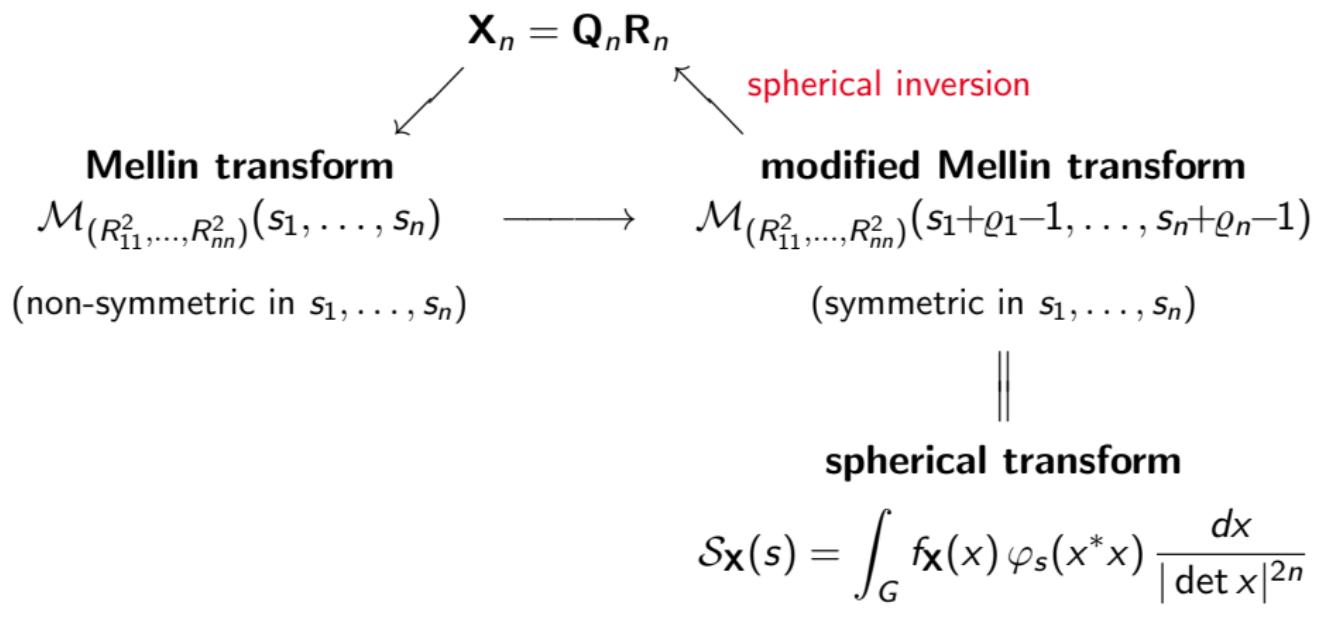
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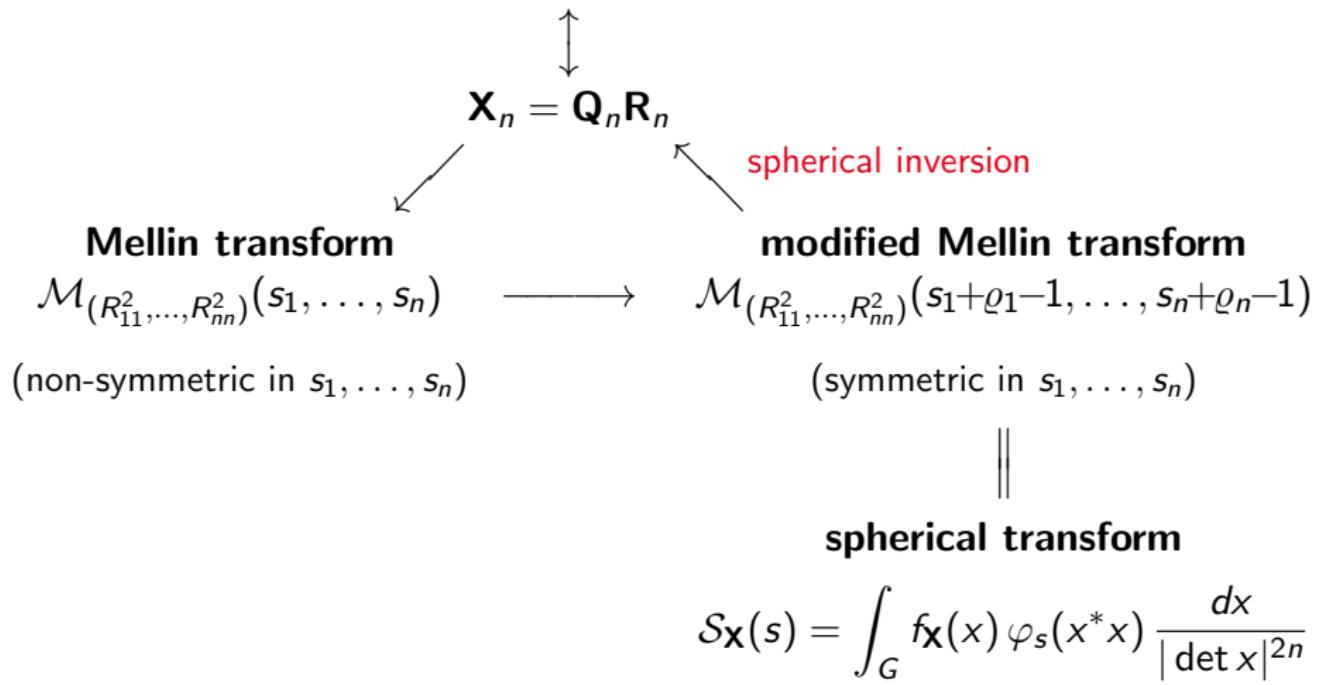
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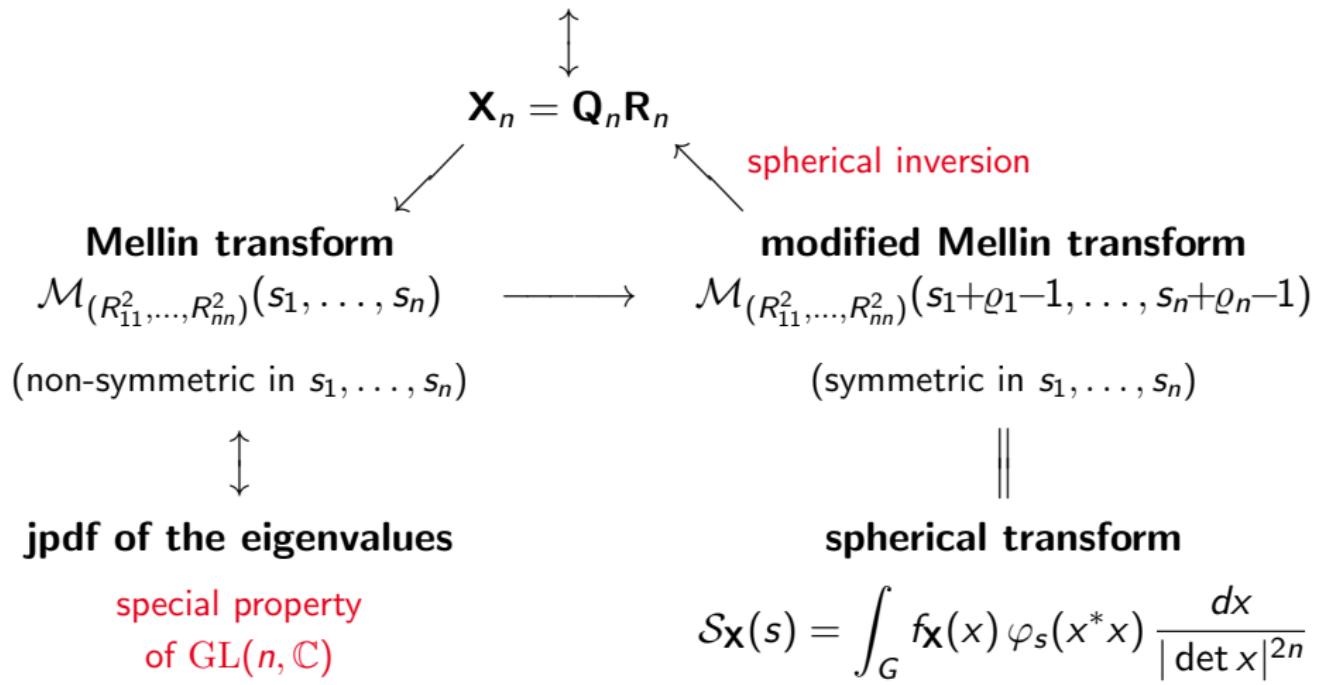
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Theorem (Transfer Law)

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If $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$ and $\mathbf{Y}_n \sim \text{Ginibre}$ be independent.

Then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \circledast w_G, \dots, w_n \circledast w_G)$, where $w_G(x) := e^{-x}$.

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Kieburg–Kösters (2016)

$$f_{\text{SqSV}}(a) \propto \Delta_n(a) \det ((-a_k \frac{d}{da_k})^{j-1} w_0(a_k))_{j,k=1,\dots,n}$$

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\mathbf{X}_n is from a polynomial ensemble of derivative type if it is bi-invariant and

$$f_{\text{SqSV}}(a) \propto \Delta_n(a) \det \left((-a_k \frac{d}{da_k})^{j-1} w_0(a_k) \right)_{j,k=1,\dots,n}$$

for some weight function w_0 (with suitable properties).

Abbreviation: $\mathbf{X}_n \sim DPE_n(w_0)$.

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Examples

- induced Wishart–Laguerre ensemble: $w_0(a) = a^\nu e^{-a}$
- induced Jacobi ensemble: $w_0(a) = a^\nu (1-a)^{\mu-1} \mathbf{1}_{(0,1)}(a)$
- induced Cauchy–Lorentz ensemble: $w_0(a) = a^\nu (1+a)^{-\nu-\mu-1}$

- products of such random matrices: $w_0(a) = \text{Meijer-G-function}$

- Muttalib–Borodin ensemble (of Wishart–Laguerre type)

Muttalib (1995), Borodin (1999), Cheliotis (2014), Forrester–Liu (2014), Forrester–Wang (2015), Zhang (2015), ...

- (a) $f_{\text{SqSV}}(a) \propto \Delta_n(a) \Delta_n(a^\theta) (\det a)^\nu e^{-\text{tr } a^\theta}$ $w_0(a) = a^\nu e^{-\alpha a^\theta}$
- (b) $f_{\text{SqSV}}(a) \propto \Delta_n(a) \Delta_n(\ln a) (\det a)^\nu e^{-\text{tr}(\ln a)^2}$ $w_0(a) = a^\nu e^{-\alpha(\ln a)^2}$

Main Results

Theorem 1 (Spherical Transform)

Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE_n(w_0)$. Then $\mathcal{S}_{\mathbf{X}_n}(s) \propto \prod_{k=1}^n (\mathcal{M}w_0)(s_k - \frac{n-1}{2})$.

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Proof: $Df(x) := (-x)f'(x)$

$$\mathcal{S}_{\mathbf{X}}(s) = \int_G f_{\mathbf{X}}(x) \varphi_s(x^* x) \frac{dx}{|\det x|^{2n}} = \int_{(0,\infty)^n} f_{\text{SqSV}}(\lambda) \varphi_s(\lambda) \frac{d\lambda}{(\det \lambda)^n}$$

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Spherical Functions for $\text{GL}(n, \mathbb{C})$ Gelfand–Naimark (1950)

$$\varphi_s(x^* x) = \Delta_n(\varrho) \frac{\det((\lambda_j(x^* x))^{s_k + (n-1)/2})}{\Delta_n(s) \Delta_n(\lambda(x^* x))}, \quad s \in \mathbb{C}^n, x \in \text{GL}(n, \mathbb{C}).$$

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Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE_n(w_0)$. Then $\mathcal{S}_{\mathbf{X}_n}(s) \propto \prod_{k=1}^n (\mathcal{M}w_0)(s_k - \frac{n-1}{2})$.

Theorem 2 (Joint Density of Eigenvalues)

Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE_n(w_0)$. Then $f_{\text{EV}}(z) \propto |\Delta_n(z)|^2 \prod_{j=1}^n w_0(|z_j|^2)$.

Main Results

Theorem 3 (Transfer Law) Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE_n(w_0(x))$. Then $\mathbf{X}_n^{-1} \sim DPE_n(w_0(x^{-1})x^{-n-1})$.

Theorem 4 (Transfer Law) Kieburg–K. (2016)

Let $\mathbf{X}_n \sim DPE_n(w_0)$ and $\mathbf{Y}_n \sim DPE_n(v_0)$ be independent.

Then $\mathbf{X}_n \mathbf{Y}_n \sim DPE_n(w_0 \circledast v_0)$, where

$$(w_0 \circledast v_0)(x) = \int_0^\infty w_0(xy^{-1})v_0(y) dy/y, \quad x > 0.$$

Remark compare Kuijlaars–Stivigny (2014), Kuijlaars (2015), Claeys–Kuijlaars–Wang (2015)

Theorem 4 admits the following generalization:

Let $\mathbf{X}_n \sim PE(w_1, \dots, w_n)$ and $\mathbf{Y}_n \sim DPE_n(v_0)$ be independent.

Then $\mathbf{X}_n \mathbf{Y}_n \sim PE(w_1 \circledast v_0, \dots, w_n \circledast v_0)$.

Main Results

Theorem 5 (Determinantal Point Processes) Kieburg-K. (2016)

Let $\mathbf{X}_n \sim DPE_n(w_0)$. Then the point processes of the singular values and of the eigenvalues are determinantal with the correlation kernels

$$K_{\text{SqSV}}(a_j, a_k) = \sum_{l=0}^{n-1} p_l(a_j) q_l(a_k)$$

and

$$K_{\text{EV}}(z_j, \bar{z}_k) = \sqrt{w_0(|z_j|^2) w_0(|z_k|^2)} \sum_{l=0}^{n-1} \frac{(z_j \bar{z}_k)^l}{\pi \mathcal{M} w_0(l+1)},$$

respectively, for certain polynomials $p_l(a)$ and functions $q_l(a)$ satisfying the bi-orthogonality relations $\int_0^\infty p_i(a) q_j(a) da = \delta_{ij}$.

- $p_l(a) = \frac{1}{2} \int_0^\infty \int_{-\pi}^{+\pi} (ae^{i\varphi} - r)^l K_{\text{EV}}(\sqrt{r}, \sqrt{a}e^{-i\varphi}) d\varphi dr$
- $q_l(a) = \frac{1}{2l!} \left(-\frac{d}{da}\right)^l \int_{-\pi}^{+\pi} e^{i\varphi} K_{\text{EV}}(\sqrt{a}, \sqrt{a}e^{-i\varphi}) d\varphi$

Application

The products of p Ginibre matrices and q inverse Ginibre matrices lead to **heavy-tailed** 1-point densities (in the limit as $n \rightarrow \infty$):

$$g_{\text{SV}}(a) \asymp a^{-\frac{q+3}{q+1}} \quad (a \rightarrow \infty) \quad g_{\text{EV}}(z) \asymp |z|^{-\frac{2q+2}{q}} \quad (z \rightarrow \infty)$$

Is it possible to interpolate between these ensembles of product matrices?

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$$w_{p,q}(x) = G_{q,p}^{p,q} \left(\begin{array}{c} -n, \dots, -n \\ 0, \dots, 0 \end{array} \middle| x \right) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma^p(s) \Gamma^q(1+n-s) x^{-s} ds$$

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Proposition (Interpolating Ensembles) Kieburg-K. (2016)

For general $p, q > 0$, $DPE_n(w_{p,q})$ is a probability measure if and only if

$$(p \in \mathbb{N} \text{ or } p > n-1) \text{ and } (q \in \mathbb{N} \text{ or } q > n-1).$$

Furthermore, $DPE_n(w_{p_1,q_1}) \circledast DPE_n(w_{p_2,q_2}) = DPE_n(w_{p_1+p_2,q_1+q_2}).$

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Thus, in general, one can't let $n \rightarrow \infty$ for fixed $p, q > 0$.

(However, one can use Muttalib-Borodin ensembles ($\tilde{w}_\theta(x) = e^{-x^\theta}$) instead.)



Characterization

Proposition (Characterization)

Kieburg–K. (2016)

Let \mathbf{X}_n have a bi-invariant density $f \in L^1(\mathrm{GL}(n, \mathbb{C}))$, and let $\mathbf{X}_n = \mathbf{Q}_n \mathbf{R}_n$. Then the following are equivalent:

- (i) \mathbf{X}_n is from a polynomial ensemble of derivative type.
- (ii) The diagonal elements R_{11}, \dots, R_{nn} are independent, and R_{nn}^2 has a density in $L_{[1,n]}^{1,n-1}(\mathbb{R}_+)$.

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- Ginibre ensemble (“multivariate gamma distribution”):
 R_{11}, \dots, R_{nn} are independent with $R_{jj}^2 \sim \Gamma(1, n-j+1)$
- truncated unitary ensemble (“multivariate beta distribution”):
 R_{11}, \dots, R_{nn} are independent with $R_{jj}^2 \sim B(n-j+1, m-n)$
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Corollary

compare Kostlan (1990), Akemann–Strahov (2013)

The eigenvalue *radii* may be regarded as independent random variables.

Overview

1 Introduction

2 Old Results: Bi-Invariant Random Matrices

3 New Results: Polynomial Ensembles of Derivative Type

4 Summary and Open Problems

Summary and Open Problems

• Summary

- the **spherical transform** is a useful tool for investigating products of independent bi-invariant random matrices
- special class: **polynomial ensembles of derivative type**
 - the singular value and eigenvalue distributions are determinantal
 - this structure is preserved when taking independent products

• Open Problems (partly work in progress)

- which functions w_0 define polynomial ensembles of derivative type ?
- limiting spectral distributions at the *global* and *local* level ?
- beyond free probability: refined convergence results
- power-law decay: new applications of random matrix theory ?
- extension to real and / or quaternionic matrices ?

Thank you very much for your attention!