

Outline

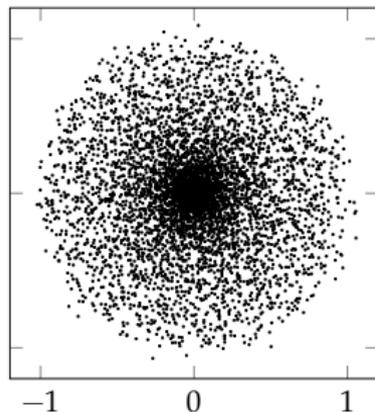
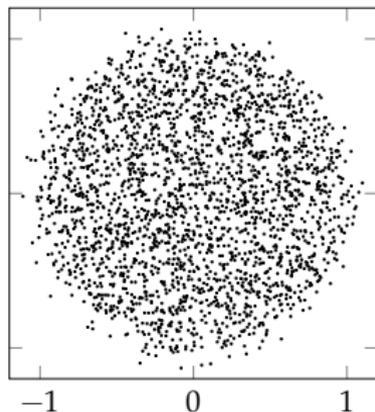
- What have learned from products of complex and quaternionic matrices?
- Products of real matrices?
 - 1 The eigenvalue JPDF
 - 2 Generating functions and skew polynomials
 - 3 How many eigenvalues are real?
 - 4 Global and local densities

What have learned from products of complex and quaternionic matrices?

Complex matrices

$$\Pi_m = X_m \cdots X_1$$

- Matrices are i.i.d. $N \times N$
- Entries are i.i.d. complex Gaussians



Complex matrices ($m = 1$)

$$P_{\text{jpdf}}(\{z\}) = \frac{1}{Z} |\Delta(\{z\})|^2 \prod_{k=1}^N e^{-|z_k|^2}$$

- Determinantal point process

$$\det[K_N(z_i, z_j)]_{ij}$$

- Orthogonal polynomials

$$p_k(z) = z^k$$

- Global density: the circular law

$$\rho(z) = \frac{1}{\pi} \mathbb{1}_{|z| < 1}$$

- Universal local correlation: bulk + edge

Complex matrices (general m)

$$P_{\text{jpdf}}^m(\{z\}) = \frac{1}{Z} |\Delta(\{z\})|^2 \prod_{k=1}^N w_m(|z_k|)$$

- Determinantal point process

$$\det[K_N(z_i, z_j)]_{ij}$$

- Orthogonal polynomials

$$p_k(z) = z^k$$

- Global density: the circular law

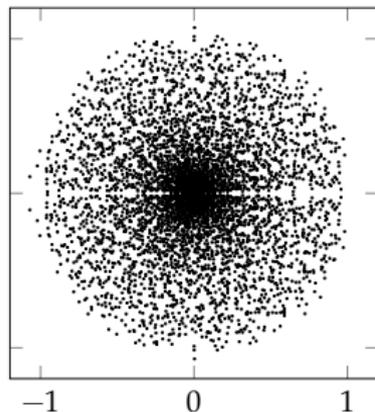
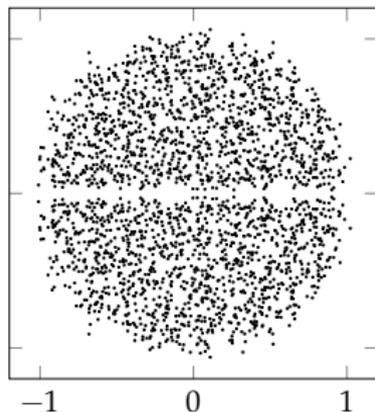
$$\rho(z) = \frac{|z|^{2(1/m-1)}}{m\pi} \mathbb{1}_{|z|<1}$$

- New local correlations near the origin

Quaternionic matrices

$$\Pi_m = X_m \cdots X_1$$

- Matrices are i.i.d. $N \times N$
- Entries are i.i.d. quaternionic Gaussians



Quaternionic matrices ($m = 1$)

$$P_{\text{jpdf}}(\{x \pm iy\}) = \frac{1}{Z} |\Delta(\{x \pm iy\})| \prod_{k=1}^N 2y_k e^{-(x_k^2 + y_k^2)}$$

- Pfaffian point process

$$\text{Pf}[K_N(z_i, z_j)]_{ij}$$

- Skew-orthogonal polynomials

$$p_{2k+1}(z) = z^{2k+1} \quad \text{and} \quad p_{2k}(z) = \sum_{j=1}^k \frac{(2k)!!}{(2j)!!} z^{2j}$$

- Global density: the circular law

$$\rho(z) = \frac{1}{\pi} \mathbb{1}_{|z| < 1}$$

- Universal local correlation: bulk + edge + real axis

Quaternionic matrices (general m)

$$P_{\text{jpdf}}(\{x \pm iy\}) = \frac{1}{Z} |\Delta(\{x \pm iy\})| \prod_{k=1}^N w_m(x_k, y_k)$$

- Pfaffian point process

$$\text{Pf}[K_N(z_i, z_j)]_{ij}$$

- Skew-orthogonal polynomials

$$p_{2k+1}(z) = z^{2k+1} \quad \text{and} \quad p_{2k}(z) = \sum_{j=1}^k \left(\frac{(2k)!!}{(2j)!!} \right)^m z^{2j}$$

- Global density: the circular law

$$\rho(z) = \frac{|z|^{2(1/m-1)}}{m\pi} \mathbb{1}_{|z|<1}$$

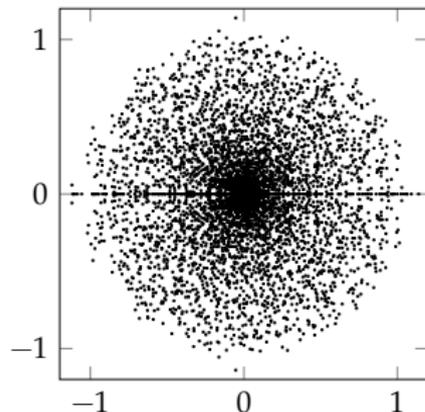
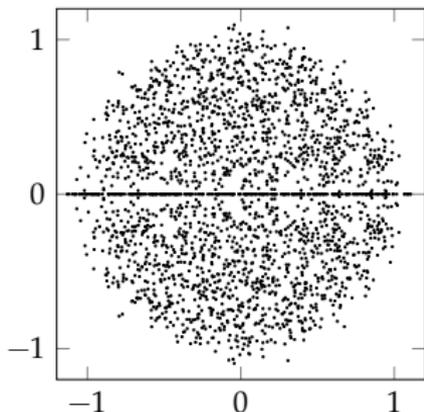
- New local correlations near the origin

Products of real matrices?

Real matrices

$$\Pi_m = X_m \cdots X_1$$

- Matrices are i.i.d. $N \times N$
- Entries are i.i.d. real Gaussians



Part 1

Eigenvalue JPDF

Joint densities

Eigenvalue JPDF for a single matrix assuming k real eigenvalues

$$P_{\text{jpdf}}(\{\lambda\} \cup \{x \pm iy\}) = \frac{1}{Z_{k,N}} |\Delta(\{\lambda\} \cup \{x \pm iy\})| \\ \times \prod_{j=1}^k e^{-\lambda_j^2/2} \prod_{j=1}^{(N-k)/2} 2e^{y_j^2 - x_j^2} \operatorname{erfc}(\sqrt{2}y_j)$$

We might guess that the JPDF for general m reads

$$P_{\text{jpdf}}(\{\lambda\} \cup \{x \pm iy\}) = \frac{1}{Z_{k,N}} |\Delta(\{\lambda\} \cup \{x \pm iy\})| \\ \times \prod_{j=1}^k w_m^r(\lambda_j) \prod_{j=1}^{(N-k)/2} w_m^c(x_j, y_j)$$

Generalised Schur decomposition

$$X_j = U_j T_j U_{j-1}^{-1}, \quad j = 1, \dots, m \quad (U_0 = U_m)$$

$$T_j = \begin{bmatrix} \lambda_1^{(j)} & * & & \dots & & * \\ 0 & \ddots & & & & \\ & & \lambda_k^{(j)} & * & & \vdots \\ \vdots & & 0 & A_1^{(j)} & & \\ & & & & \ddots & * \\ 0 & \dots & & 0 & A_{(N-k)/2}^{(j)} & \end{bmatrix}$$

- $\lambda^{(j)}$: real numbers
- $A^{(j)}$: two-by-two matrices

Weight functions

One-point weights

$$w_1^r(\lambda) = e^{-\lambda^2/2}, \quad w_m(\lambda) = (w_1^r)^{*m}(\lambda)$$

Two-by-two matrix weights

$$W_1(A) = e^{-\text{Tr}AA^T/2}, \quad W_m(A) = (W_1)^{*m}(A)$$

Two-point weights

$$w_m^c(x, y) = 2\pi \int_{-\infty}^{\infty} d\delta \frac{|\delta|}{\sqrt{\delta^2 + 4y^2}} W_m\left(\begin{bmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{bmatrix}\right),$$

with

$$\mu_{\pm} = \frac{1}{2} \left(\pm |\delta| + [\delta^2 + 4(x^2 + y^2)]^{1/2} \right)$$

Example of two-point weight

For $m = 2$ the two-point weight reads

$$w_{m=2}^c(x, y) = 4 \int_0^\infty \frac{dt}{t} \exp\left(-2(x^2 - y^2)t - \frac{1}{4t}\right) \\ \times K_0(2(x^2 + y^2)t) \operatorname{erfc}(2\sqrt{t}y)$$

for higher m the weight becomes even more complicated.

Rule of thumb:

Avoid use of the two-point weight whenever possible

Part 2

Generating functions and skew polynomials

Generating function

$$Z_N[u, v] = \sum_{k=0}^N Z_{k, (N-k)/2}[u, v],$$

where

$$Z_{k, (N-k)/2}[u, v] = \prod_{j=1}^k \int d\lambda_j u(\lambda_j) \prod_{l=1}^{(N-k)/2} \int dx_l dy_l v(x_l, y_l) \\ \times P_{j\text{pdf}}(\{\lambda\} \cup \{x \pm iy\}).$$

Pfaffian

The partition function has a Pfaffian structure

$$Z_{k,(N-k)/2}[u, v] = [\zeta^{k/2}] \text{Pf}[\zeta \alpha_{j,l} + \beta_{j,l}]_{j,l},$$

where

$$\alpha_{j,k} = \int_{-\infty}^{\infty} dx u(x) w_m^r(x) \int_{-\infty}^{\infty} dy u(y) w_m^r(y) \text{sign}(y-x) \\ \times p_{j-1}(x) p_{k-1}(y)$$

$$\beta_{j,k} = 2i \int_{\mathbb{R} \times \mathbb{R}_+} dx dy v(x, y) w_m^c(x, y) \\ \times \left(p_{j-1}(x+iy) p_{k-1}(x-iy) - p_{k-1}(x+iy) p_{j-1}(x-iy) \right),$$

and $p_j(z)$ are monic polynomials

Skew-orthogonal polynomials

It is unknown to find the skew-orthogonal polynomials for general ζ (and general u and v), but for

$$\zeta = 1 \quad (\text{and } u = v = 1)$$

we can find the polynomials. This is sufficient to answer several questions of interest.

$$m=1: p_{2j}(x) = x^{2j}, \quad p_{2j+1}(x) = x^{2j+1} - (2j)x^{2j-1}$$

$$m=2: p_{2j}(x) = x^{2j}, \quad p_{2j+1}(x) = x^{2j+1} - (2j)^2 x^{2j-1}$$

General m ?

$$p_{2j}(x) = x^{2j}, \quad p_{2j+1}(x) = x^{2j+1} - (2j)^m x^{2j-1}$$

How to find the skew orthogonal polynomials

$$\begin{aligned}p_{2n}(z) &= \langle \det(z \mathbb{1} - X) \rangle_X \\ p_{2n+1}(z) &= zp_{2n}(z) + \langle \det(z \mathbb{1} - X) \operatorname{Tr} X \rangle_X,\end{aligned}$$

Use that all entries in all matrices are independent with zero mean and unit variance

Part 3

How many eigenvalues are real?

Number of real eigenvalues

Probability of finding k real eigenvalues

$$p_{N,k}^m = [\zeta^{k/2}] \det[(\zeta \alpha_{2j-1,2l} + \beta_{2j-1,2l})|_{u=v=1}]_{j,l},$$

where

$$\begin{aligned} & (\zeta \alpha_{2j-1,2l} + \beta_{2j-1,2l}) \Big|_{u=v=1} \\ &= (\zeta - 1) \alpha_{2j-1,2l} \Big|_{u=v=1} + (\alpha_{2j-1,2l} + \beta_{2j-1,2l}) \Big|_{u=v=1} \\ &= (\zeta - 1) \alpha_{2j-1,2l} \Big|_{u=v=1} + \langle p_{2j-1}, p_{2l} \rangle_S \end{aligned}$$

Examples

$$p_{6,0}^{m=2} = 1 - \frac{3821355}{8388608} \pi + \frac{873624317}{17179869184} \pi^2 - \frac{64011585}{68719476736} \pi^3$$
$$\approx 0.0419$$

$$p_{6,2}^{m=2} = \frac{3821355}{8388608} \pi - \frac{873624317}{8589934592} \pi^2 + \frac{192034755}{68719476736} \pi^3$$
$$\approx 0.5140$$

$$p_{6,4}^{m=2} = \frac{873624317}{17179869184} \pi^2 - \frac{192034755}{68719476736} \pi^3$$
$$\approx 0.4152$$

$$p_{6,6}^{m=2} = \frac{64011585}{68719476736} \pi^3$$
$$\approx 0.0289$$

Expected number of real eigenvalues

$$\mathbb{E}[k] = 2 \sum_{j=0}^{N-2} (-1)^j \left(\frac{2^j}{\sqrt{\pi} j!} \right)^m a_{\lceil j/2+1 \rceil, \lfloor j/2+1 \rfloor}$$

with

$$a_{j,k} = 2^{-(j+k-1/2)m} \alpha_{2j-1, 2k}$$

Example

$$\mathbb{E}[k] \Big|_{N=6} = \frac{3821355}{4194304} \pi \approx 2.8622$$

Expected number of real eigenvalues

$$\mathbb{E}[k] = c(m)\sqrt{N} + o(N^{1/2})$$

$m = 1$:

$$\mathbb{E}[k] = \sqrt{\frac{2N}{\pi}} + O(1)$$

$$\frac{\log P_{N,0}^{m=1}}{N^{1/2}} = -\frac{\zeta(3/2)}{\sqrt{2\pi}} + o(1) \quad \frac{\log P_{N,N}^{m=1}}{N^2} = -\frac{\log 2}{4} + O(N^{-1})$$

Asymptotics for $m > 1$ is mainly open problems

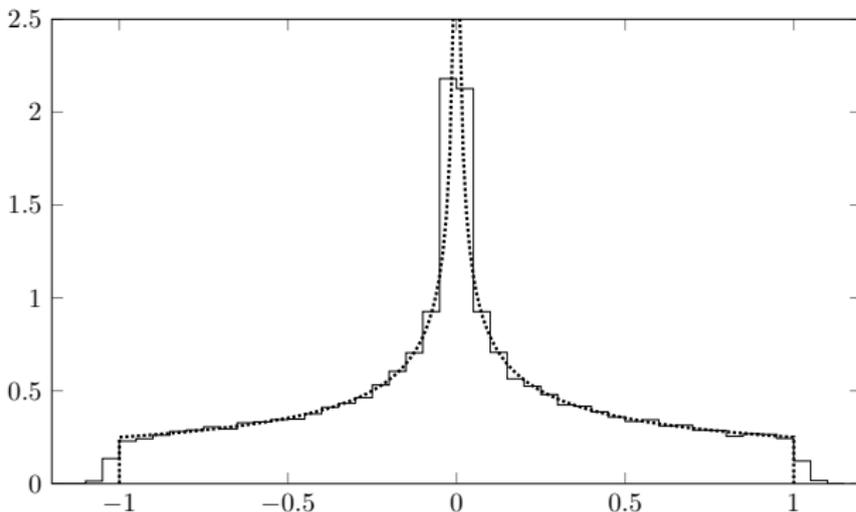
Part 4

Global and local densities

Global densities

complex :
$$\lim_{N \rightarrow \infty} N^{m-1} \rho^c(N^{m/2} z) = \frac{|z|^{2(1/m-1)}}{m\pi} \mathbb{1}_{|z| < 1}$$

real :
$$\lim_{N \rightarrow \infty} \frac{N^{(m-1)/2} \rho^r(N^{m/2} x)}{\mathbb{E}(k)} = \frac{|x|^{1/m-1}}{2m} \mathbb{1}_{|x| < 1}$$



Local densities near the origin

Complex density

$$\frac{2y w_m^c(x, y)}{(2\sqrt{2\pi})^m} G_{0,m}^{1,0}(\bar{\cdot}, 0 \mid -|z|^2)$$

Real density

$$\int_{-\infty}^{\infty} dv |x-v| w_m^r(x) w_m^r(v) G_{0,m}^{1,0}(\bar{\cdot}, 0 \mid -xv)$$

Thanks for your attention.

References

Forrester & J.R.I., to appear in LAA (2016) arXiv:1608.04097

J.R.I. & Kieburg, PRE 80, 032106 (2014) arXiv:1310.4154

Forrester, J. Phys. A 47, 065202 (2014) arXiv:1309.7736