A discrete random walk approach to spectral statistics in Bernoulli matrix ensembles

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(joint work with Uzy Smilansky)

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Outline

1. **Introduction:** RMT, Bernoulli ensembles, spectral statistics
2. **Previously:** Random walks and eigenvalue motion
3. **Stein's method:** Illustration and example with random walks.
4. **Stein's method in RMT:** Application to Bernoulli ensembles.
5. **Conclusions:** Discussion and potential developments.

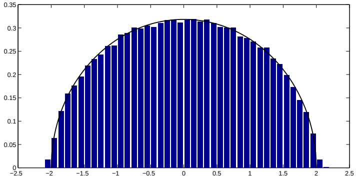
RMT: The Gaussian models

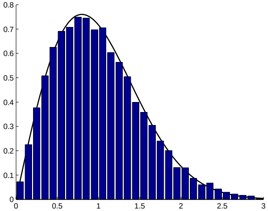
Gaussian orthogonal ensemble (GOE): Matrices of size $N \times N$ with $H_{ij} = H_{ji}$ Gaussian distributed.

$$H = \begin{pmatrix} H_{11} & H_{12} & \cdots \\ H_{21} & H_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad \rightarrow \quad \boldsymbol{\sigma} = (\lambda_1 \leq \ldots \leq \lambda_N)$$

$$P(H) \propto \prod_{i \leq j} e^{-H_{ij}^2/2(1+\delta_{ij})} = e^{-\mathrm{Tr}(H^2)/2} \quad \rightarrow \quad P(\boldsymbol{\sigma}) \propto \prod_{\nu < \mu} |\lambda_\nu - \lambda_\mu| e^{-\sum_\nu \lambda_\nu^2/2}$$

From this we obtain analytical expressions for certain statistics:



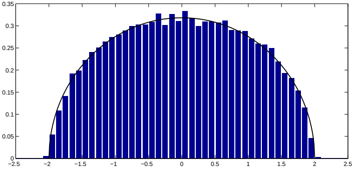


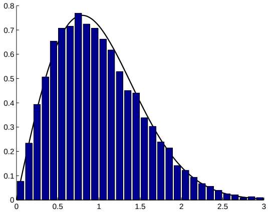
RMT: Universality

What if we take other distributions?

$$B = \begin{pmatrix} +\sqrt{2} & +1 & -1 & \cdots \\ +1 & -\sqrt{2} & -1 & \cdots \\ -1 & -1 & +\sqrt{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \rightarrow \quad \boldsymbol{\sigma} = (\lambda_1 \leq \ldots \leq \lambda_N)\ ?$$

Both local and global measures become close to GOE results as $N \rightarrow \infty$



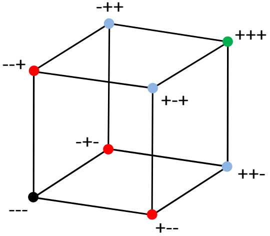


Previously...

Our Bernoulli matrices of dimension N

$$B_{ij} = B_{ji} = \frac{1}{\sqrt{N}} \begin{cases} \pm 1 & i \neq j \\ \pm\sqrt{2} & i = j \end{cases}$$

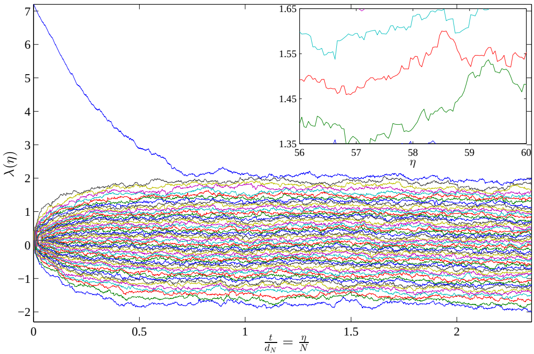
Correspond to hypercube $\mathfrak{B}_N$ of dimension $d_N = N(N-1)/2$.



$$P_{t+1}(B) = \sum_{B' \in \mathfrak{B}_N} \rho(B' \to B) P_t(B') = \sum_{B' : |B' - B| = 1} \frac{P_t(B')}{d_N}$$

If $P_t(B) = |\mathfrak{B}_d|^{-1} = 2^{-d_N}$, then $P_{t+1}(B) = P_t(B)$.

Previously...



In the limit of large N the $\lambda = (\lambda_1, \ldots, \lambda_N)$ motion is described by the Dyson BM Fokker-Planck eqn:

$$\frac{\partial P}{\partial t} = \sum_{\mu} \left[\frac{\partial (F_\mu P)}{\partial \lambda_\mu} + \frac{1}{\beta} \frac{\partial^2 P}{\partial \lambda_\mu^2} \right] \qquad F_\mu(\lambda) = -\lambda_\mu + \sum_{\nu:\nu\neq\mu} \frac{1}{\lambda_\nu - \lambda_\mu}$$

Results

Theorem (CJ, Smilansky '15)

Let $B \in \mathfrak{B}_N$ be our ensemble of Bernoulli matrices and $M \in GOE$, $\mathbb{E}[M_{ij}^2] = N^{-1/2}$ with $\phi(x) \in C_3(\mathbb{R}^k)$, $k > 0$ fixed. Then for $\boldsymbol{t} = (t_1, \dots, t_k)$

$$|\mathbb{E}[\phi(\boldsymbol{t}(B))] - \mathbb{E}[\phi(\boldsymbol{t}(M))]| \leq \|\nabla^3 \phi\| \mathcal{O}(N^{-1/2})$$

where $t_n(M) = Tr(M^n) = \sum_\mu \lambda_\mu^n(M)$ and $||\nabla^3 \phi|| = \sup_{x \in \mathbb{R}^k} \sup_{n,m,r} |\partial^3_{n,m,r} \phi(x)|$.

Comments

- If $\boldsymbol{t} = (t_2, \dots, t_k)$ then $\mathcal{O}(N^{-1/2}) \to \mathcal{O}(N^{-1})$.
- Can probably take $k = \mathcal{O}(N^\alpha)$.
- Very likely that $\|\nabla^3 \phi\| \to \|\nabla^2 \phi\|$.
- Extensions to Wigner matrices.
- Extensions to matrix ensembles with correlations, e.g. Regular tournaments, random regular graphs (with growing degree).

Stein's method: Introduction

Let us introduce the following *Stein operator*

$$\mathcal{A} := \frac{d^2}{dx^2} - x\frac{d}{dx}$$

Lemma (Stein's lemma)

$$\mathbb{E}[\mathcal{A}f(W)] = 0 \ \forall\ f \in C_c^2(\mathbb{R}) \iff W \sim N(0,1)$$

Proof.

Integration by parts yields

$$\int dx\ p(x)\left[\frac{d^2}{dx^2} - x\frac{d}{dx}\right] f(x) = \int dx\ f(x)\left[\frac{d^2}{dx^2} + x\frac{d}{dx}\right] p(x) = 0$$

□

Stein's method: Introduction

Stein's idea: If W "close" to normal dist. then $\mathbb{E}[\mathcal{A}f(W)] \approx 0$. Therefore write

$$\mathcal{A}f(x) = \phi(x) - \mathbb{E}[\phi(G)] \tag{1}$$

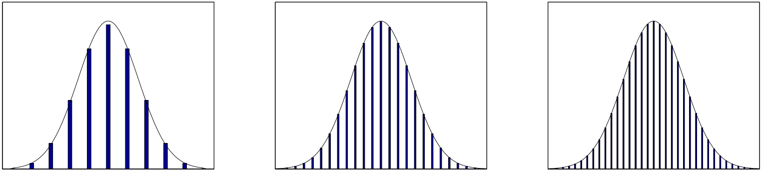
with $G \sim N(0,1)$. Then

- **Step 1.** Determine $|\mathbb{E}[\mathcal{A}f(W)]|$
- **Step 2.** Use this to estimate $|\mathbb{E}[\phi(W)] - \mathbb{E}[\phi(G)]|$ by above
- **Step 3.** Find f in terms of ϕ by solving (1)

Stein's method: Example

Suppose we want to show that Binomial distribution converges to Gaussian, i.e. $W_d \to G \sim N(0,1)$ as $d \to \infty$

$$W_d = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} B_i \qquad\qquad P(B_i = \pm 1) = \frac{1}{2}$$



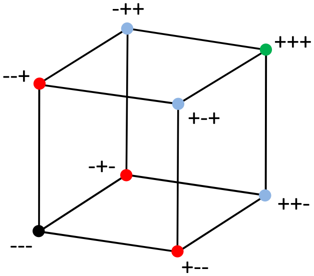
$d = 10$ $\qquad$ $d = 50$ $\qquad$ $d = 100$

For all bounded continuous test functions $\phi(x)$ we require

$$\lim_{d\to\infty} |\mathbb{E}[\phi(W_d)] - \mathbb{E}[\phi(G)]| = 0$$

Stein's method: Example

Take a hypercube $\mathfrak{B}_d$ of dimension d with simple random walk, note $|\mathfrak{B}_d| = 2^d$.

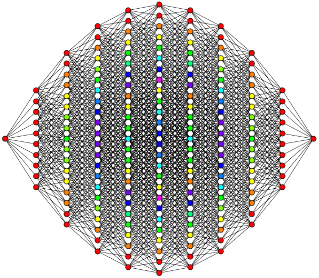
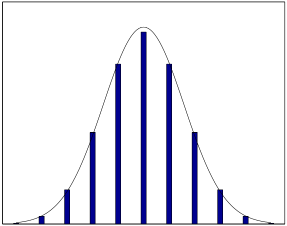


$B_t = B = (+1, -1, +1 \ldots) \to B_{t+1} = B' = (+1, -1, -1 \ldots)$ is position on the hypercube, then

$$W_d(B) = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} B_i \qquad W_d(B') = W_d(B) - 2B_j$$

If $P_t(B) = |\mathfrak{B}_d|^{-1}$, then $P_{t+1}(B) = P_t(B)$.

Stein's method: Example

If $P_t(B)$ our stat. dist. and we write $\delta W = W(B') - W(B) = -B_j$

- **Drift:**

$$\mathbb{E}[\delta W | B] = -2\frac{W(B)}{d}$$

- **Diffusion:**

$$\mathbb{E}[\delta W^2 | B] = \frac{4}{d}$$

Stein's method: Example

$$
\begin{aligned}
0 &= \sum_B f(W(B))[P_{t+1}(B) - P_t(B)] \\
&= \left[\sum_{B,B'} f(W(B'))\rho(B \to B') - \sum_W f(W(B))\right] P(B) \\
&= \sum_B P(B) \left\{ \mathbb{E}[\delta W|B] f'(W) + \frac{1}{2}\mathbb{E}[\delta W^2|B] f''(W) + \dots \right\}
\end{aligned}
$$

Using $\mathbb{E}[\delta W|B] = -\frac{d}{2}W$ and $\mathbb{E}[\delta W^2|B] = \frac{4}{d}$.

$$
0 = \mathbb{E}[\mathcal{A}f(W)] + \frac{d}{2}\sum_B p(B)\frac{\mathbb{E}[\delta W^3|B]}{3!} f'''(W^*)
$$

Therefore using that $\mathcal{A}f(W) = \phi(W) - \mathbb{E}[\phi(G)]$ we have

$$
|\mathbb{E}[\phi(W)] - \mathbb{E}[\phi(G)]| \leq \frac{d}{2}\frac{\mathbb{E}[|\delta W^3||B]}{3!}\|f'''\| = \mathcal{O}(d^{-1/2})\|f'''\|
$$

Stein's method: Example

Stein equation:

$$\mathcal{A}f(x) = \phi(x) - \mathbb{E}[\phi(G)]$$

[Generator approach to Stein's method - Barbour '90] Since $\mathcal{A}$ is the generator for an Ornstein-Uhlenbeck process we know the solution!

$$f(x) = -\int_0^\infty dt \int dy\ p(y)\phi(xe^{-t} + \chi(t)y),$$

where $\chi(t)^2 = 1 - e^{-2t}$. From this one may show, e.g.

$$\|f'''\| \leq C\|\phi''\|$$

Stein's method: Bernoulli matrices

Our random matrix is a multivariate Gaussian with variance of the elements $\mathbb{E}[M_{ij}^2] = (1+\delta_{ij})/N$. Therefore we have the Stein operator

$$\mathcal{A} := \sum_{i \le j} \left[\frac{(1+\delta_{ij})}{N} \frac{\partial^2}{\partial M_{ij}^2} - M_{ij} \frac{\partial}{\partial M_{ij}} \right]$$

i.e. $\mathbb{E}[\mathcal{A}f(W)] = 0 \; \forall \; f \iff W \sim GOE$.

Follow our steps again for Bernoulli matrices B

- **Step 1.** Determine $|\mathbb{E}[\mathcal{A}f(B)]|$
- **Step 2.** Use this to estimate $|\mathbb{E}[\phi(B)] - \mathbb{E}[\phi(M)]|$
- **Step 3.** Find f in terms of ϕ by solving $\mathcal{A}f(x) = \phi(x) - \mathbb{E}[\phi(M)]$

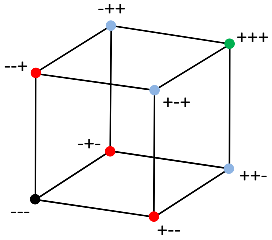
Multivariate exchangeable pairs approach - Chaterjee & Meckes '08, Meckes '09]

Stein's method: Bernoulli matrices

Our Bernoulli matrices of dimension N

$$B_{ij} = B_{ji} = \frac{1}{\sqrt{N}} \begin{cases} \pm 1 & i \neq j \\ \pm\sqrt{2} & i = j \end{cases}$$

Correspond to hypercube $\mathfrak{B}_N$ of dimension $d_N = N(N-1)/2$.



$$P_{t+1}(B') = \sum_{B \in \mathfrak{B}_N} \rho(B \to B') P_t(B) = \sum_{B : |B' - B| = 1} \frac{P_t(B)}{d_N}$$

If $P_t(B) = |\mathfrak{B}_d|^{-1} = 2^{-d_N}$, then $P_{t+1}(B) = P_t(B)$.

Stein's method: Bernoulli Matrices

By swapping the element (p,q) we get a change in the matrix

$$\delta B^{(pq)} = B' - B = \begin{cases} -2B_{pq}[|p\rangle\langle q| + |q\rangle\langle p|] & p \neq q \\ -2B_{pp}\,|p\rangle\langle p| & p = q \end{cases}$$

and thus

$$\delta B_{ij}^{(pq)} = \begin{cases} -2B_{pq}[\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}] & p \neq q \\ -2B_{pp}\delta_{ip}\delta_{jq} & p = q \end{cases}$$

We thus get

- **Drift:**

$$\mathbb{E}[\delta B_{ij}|B] = \frac{1}{d_N}\sum_{p<q}\delta B_{ij}^{(pq)} = -2\frac{B_{ij}}{d_N}$$

- **Diffusion:**

$$\mathbb{E}[\delta B_{ij}\delta B_{kl}|B] = \frac{4}{Nd_N}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$$

Stein's method: Bernoulli Matrices

If $P_t(B) = |\mathfrak{B}_d|^{-1}$ our stat. dist. then

$$\begin{aligned} 0 &= \sum_{B\in\mathfrak{B}_N} f(B)[P_{t+1}(B) - P_t(B)] \\ &= \left[\sum_{B,B'} f(B')\rho(B\to B') - \sum_B f(B)\right] P(B) \\ &= \left[\sum_{B,B'} \rho(B\to B')\left\{f(B) + \sum_{i\leq j} \delta B_{ij}\partial_{ij} f(B) + \dots\right\} - \sum_B f(B)\right] P(B) \end{aligned}$$

where $\delta B_{ij} = B'_{ij} - B_{ij}$ and $\partial_{ij} f = \partial f/\partial B_{ij}$. Then

$$0 = \sum_{B\in\mathfrak{B}_N} P(B)\left\{\sum_{i\leq j}\mathbb{E}[\delta B_{ij}|B]\partial_{ij} f(W) + \frac{1}{2}\sum_{i\leq j}\sum_{k\leq l}\mathbb{E}[\delta B_{ij}\delta B_{kl}|B]\partial^2_{ij,kl} f(B) + \dots\right\}$$

Thus require $\mathbb{E}[\delta B_{ij}|B]$, $\mathbb{E}[\delta B_{ij}\delta B_{kl}|B]$ etc.

Stein's method: Bernoulli Matrices

We replace $\mathbb{E}[\mathcal{A}f(B)] = |\mathbb{E}[\phi(B)] - \mathbb{E}[\phi(M)]|$ and use integral form of remainder in Taylor's theorem

$$|\mathbb{E}[\phi(B)] - \mathbb{E}[\phi(M)]| = \frac{d_N}{2}|\mathbb{E}[R]|$$

where, using the form of $\delta B_{ij}^{(pq)}$ this reduces to

$$\mathbb{E}[R] \leq \frac{C}{d_N}\mathbb{E}\left[\int_0^1 du(1-u)^2 \sum_{i,j} B_{ij}^3 \frac{\partial^3 f(B + u\delta B^{(pq)})}{\partial B_{ij}^3}\right].$$

Suppose, for example, that $f = f(t_n(B))$, $n > 1$, then by chain rule

$$\frac{d_N}{2}|\mathbb{E}[R]| \leq C\int_0^1 du(1-u)^2 \sum_{i,j} \left\|\frac{\partial^3 f}{\partial t_n^3}\right\| \mathbb{E}\left|\left(B_{ij}\frac{\partial t_n}{\partial B_{ij}}\right)^3\right|$$

which requires estimating $\mathbb{E}\left|\left(B_{ij}\frac{\partial t_n}{\partial B_{ij}}\right)^3\right|$

Using $\mathbb{E}[|X|] \leq \sqrt{\mathbb{E}[X^2]}$ we have

$$\mathbb{E}\left|\left(B_{ij}\frac{\partial t_n}{\partial B_{ij}}\right)^3\right| \leq \sqrt{\mathbb{E}\left(B_{ij}\frac{\partial t_n}{\partial B_{ij}}\right)^6}$$

Since $t_n = \sum_{i_i,\ldots,i_n} B_{i_1 i_2}\ldots B_{i_n i_1}$ we find

$$\mathbb{E}\left(B_{ij}\frac{\partial t_n}{\partial B_{ij}}\right)^6 = \mathcal{O}(N^{-3n})\#\{\gamma : |\gamma| = n-2 \text{ retracing between } i \text{ and } j\}^3 = \mathcal{O}(N^{-6})$$

Hence

$$\begin{aligned} |\mathbb{E}[\phi(t_n(B))] - \mathbb{E}[\phi(t_n(M))]| &\leq \mathcal{O}(N^{-3})\int_0^1 du(1-u)^2 \sum_{i,j}\left\|\frac{\partial^3 f}{\partial t_n^3}\right\| \\ &\leq \mathcal{O}(N^{-1})\left\|\frac{\partial^3 f}{\partial t_n^3}\right\| \qquad (2) \end{aligned}$$

Stein's method: Step 3

Stein equation:

$$\mathcal{A}f(X) = \phi(X) - \mathbb{E}[\phi(M)] \qquad \mathcal{A} := \sum_{i\leq j} \frac{(1+\delta_{ij})}{N}\partial_{ij}^2 - M_{ij}\partial_{ij}$$

Since $\mathcal{A}$ is generator MD Ornstein-Uhlenbeck process so we can solve above

$$f(\boldsymbol{t}(B)) = -\int_0^\infty du \int dM\; p(M)\phi(\boldsymbol{t}(Be^{-u} + \chi(u)M)),$$

where $\chi(u)^2 = 1 - e^{-2u}$ and $p(M)$ is JPDF for GOE. From this one may show, e.g.

$$|\mathbb{E}[\phi(t_n(B))] - \mathbb{E}[\phi(t_n(M))]| \leq \mathcal{O}(N^{-1}) \left\| \frac{\partial^3\phi}{\partial t_n^3} \right\|.$$

RMT Universality: Global statistics

Our result:

$$|\mathbb{E}[\phi(\boldsymbol{t}(B))] - \mathbb{E}[\phi(\boldsymbol{t}(M))]| \leq \|\nabla^3\phi\| \mathcal{O}(N^{-1/2})$$

For $t = (1_1, \ldots, t_k)$.

- Johansson '98: For GOE,GUE,GSE, $Y_n = \mathrm{Tr}(T_n(M)) - \alpha_k$

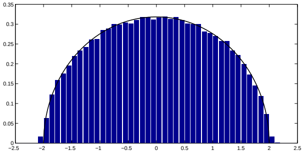
  $$(Y_1, Y_2, \ldots, Y_k) \to \frac{1}{2}\sqrt{\frac{2}{\beta}}(Z_1, \sqrt{2}Z_2, \ldots, \sqrt{k}Z_k) \quad \text{as } N \to \infty.$$

  with Z_i iid standard normal.
- (Moment method) Sinai and Soshnikov '98, Schenker and Schulz-Baldes '07

  $$\mathrm{Tr}(M^k) - \mathbb{E}[\mathrm{Tr}(M^k)] \to \text{Normal distribution}$$

- (Stein's Method with Poincaré inequality) Chatterjee '09, as above with rates of convergence but $k = O(\log n)$.

Universality: Local statistics



Take functions $f(\lambda) = \sum_i g(\lambda_i)$, with g on scale of MLS.

Two methods:

- (Heat-flow method) Erdős, Peche, Ramirez, Schlein, Yau, Yin, Knowles
- (Element swapping)Tao, Vu

Theorem (Tao-Vu '11)

If Wigner matrices W_{ij} and M_{ij} have moments matching to order 4, $\phi : \mathbb{R}^k \to \mathbb{R}$ and $|\nabla^j \phi(\lambda)| < \mathcal{O}(N^c)$, for $j = 0, \ldots 5$ then

$$|\mathbb{E}[\phi(\lambda(W))] - \mathbb{E}[\phi(\lambda(M))]| < \mathcal{O}(N^{-c})$$