A discrete random walk approach to spectral statistics in Bernoulli matrix ensembles

Chris Joyner

(joint work with Uzy Smilansky)

August 10th, 2016

Queen Mary University of London

The Leverhulme Trust

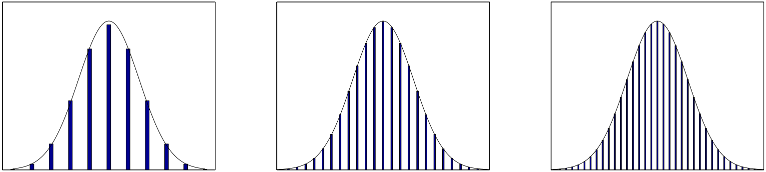
Outline

- Classic Central Limit Theorem
- RMT Central Limit Theorem
- Conclusions and outlook

Central Limit Theorem

- **Sums of iid rv:** Let B_i be iid random Bernoulli rv.

$$S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} B_i \qquad\qquad P(B_i = \pm 1) = \frac{1}{2}$$



$N = 10$ $\qquad$ $N = 50$ $\qquad$ $N = 100$

- **Convergence:** For all bounded continuous test functions $\phi(x)$

$$\lim_{N\to\infty} |\mathbb{E}[\phi(S_N)] - \mathbb{E}[\phi(Z)]| = 0 \qquad Z \sim N(0,1).$$

A random walk approach

- **Stein's Method:** [Stein '72] Alternative proof of the CLT provided by Stein
  - Easy to deal with correlations between elements
  - Rates of convergence emerge naturally

- [Barbour '90, Götze '91] - Dynamical interpretation and multidimensional Gaussians

- [Chaterjee & Meckes '08, Reinert & Röllin '09, Meckes '09] - Exchangeable pairs approach to multidimensional Gaussians

Introducing dynamics

- **Markov dynamics:** At time $t+1$ we choose a random entry $B_i \to -B_i$

$$S' = \frac{1}{\sqrt{N}}(B_1 + B_2 + \ldots - B_i + \ldots + B_N)$$

Therefore $\delta S^{(i)} = S' - S = -2B_i/\sqrt{N}$.

- **Expected change:** For $B = (B_1, \ldots, B_N)$

$$\mathbb{E}[\delta S|B] = \frac{1}{N}\sum_{i=1}^{N} \delta S^{(i)} = -\frac{1}{N}\sum_{i=1}^{N} \frac{2B_i}{\sqrt{N}} = \frac{2}{N}(-S)$$

$$\mathbb{E}[\delta S^2|B] = \frac{1}{N}\sum_{i=1}^{N} (\delta S^{(i)})^2 = \frac{1}{N}\sum_{i=1}^{N} \frac{4}{N} = \frac{2}{N}(2)$$

Introducing dynamics

- **Motion:** let $\delta f := f(S') - f(S)$

$$\begin{aligned}\mathbb{E}[\delta f|S] &= \frac{1}{N}\sum_{i=1} f(S+\delta S^{(i)}) - f(S) \\ &= \left(\mathbb{E}[\delta S|S]\frac{\partial f}{\partial S} + \frac{1}{2}\mathbb{E}[\delta S^2|S]\frac{\partial^2 f}{\partial S^2}\right) + \mathcal{R}(S),\end{aligned}$$

$$\boxed{\frac{N}{2}\mathbb{E}[\delta f|S] = \mathcal{A}f(S) + \frac{N}{2}\mathcal{R}(S),} \quad (1)$$

- **Stein operator:**

$$\mathcal{A}f(x) := \left(-x\frac{d}{dx} + \frac{d^2}{dx^2}\right) f(x)$$

Stein's Lemma

- **Taking expectations:**

$$0 = \mathbb{E}[\mathcal{A}f(S)] + \frac{N}{2}\mathbb{E}[\mathcal{R}(S)], \tag{2}$$

- **Stein's Lemma:** What if $\mathbb{E}[\mathcal{R}(S)] = 0$?

$$\mathbb{E}[\mathcal{A}f(Z)] = 0 \ \forall \ f \in C^2(\mathbb{R}) \iff Z \sim N(0,1) \tag{3}$$

- **Proof:** Integration by parts yields

$$\int dx \ p(x) \left[\frac{d^2}{dx^2} - x\frac{d}{dx}\right] f(x) = \int dx \ f(x) \left[\frac{d^2}{dx^2} + \frac{d}{dx}x\right] p(x) = 0$$

Stein's equation

- **Stein's equation:** What if $\mathbb{E}[\mathcal{A}f(S)] \approx 0$. Write

$$\boxed{\mathcal{A}f(x) = \phi(x) - \mathbb{E}[\phi(Z)]} \tag{4}$$

$Z \sim N(0,1)$.

- **Solve:** The solution to (4) is given by

$$f(x) = -\int_0^\infty dt \int dy\ p(y)\phi(xe^{-t} + \chi(t)y),$$

where $\chi(t)^2 = 1 - e^{-2t}$. From this one may show, e.g.

$$\left|\frac{d^k f(x)}{dx^k}\right| \le C_k \left|\frac{d^{k-1}\phi(x)}{dx^{k-1}}\right| \tag{5}$$

Establishing convergence

- **Putting together:** From (2) and (4)

$$|\mathbb{E}[\phi(S_N)] - \mathbb{E}[\phi(Z)]| = |\mathbb{E}[\mathcal{A}f(S_N)]| = \frac{N}{2}|\mathbb{E}[\mathcal{R}(S_N)]|$$

- **Error term:** From Taylor's Theorem

$$|\mathbb{E}[\mathcal{R}(S_N)]| \leq C\mathbb{E}[|\delta S_N^3|]\left\|\frac{d^3 f}{dx^3}\right\| = \mathcal{O}(N^{-\frac{3}{2}})\left\|\frac{d^3 f}{dx^3}\right\|$$

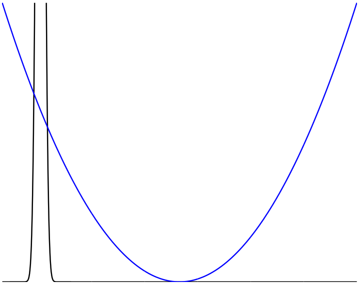
- **Convergence:** Using the derivative bound (5)

$$|\mathbb{E}[\phi(S_N)] - \mathbb{E}[\phi(Z)]| \leq \mathcal{O}(N^{-\frac{1}{2}})\left\|\frac{d^2 \phi}{dx^2}\right\|$$

The meaning

- **Ornstein-Uhlenbeck Process:** Heat flow in a harmonic trap

$$\frac{\partial P(x;t)}{\partial t} = \mathcal{A}^* P(x,t) := \frac{\partial (xP)}{\partial x} + \frac{\partial^2 P}{\partial x^2}$$



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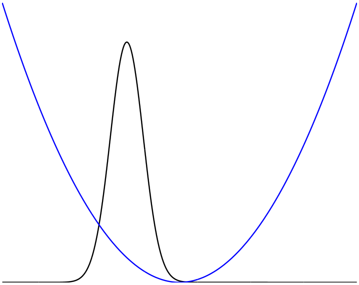

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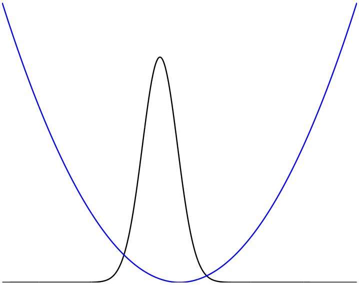

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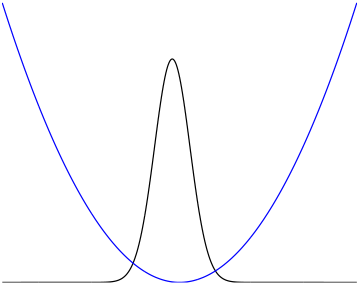

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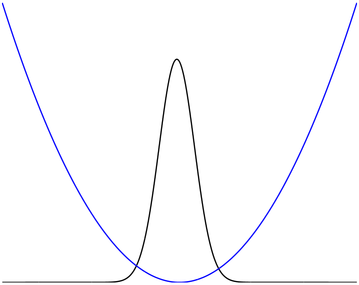

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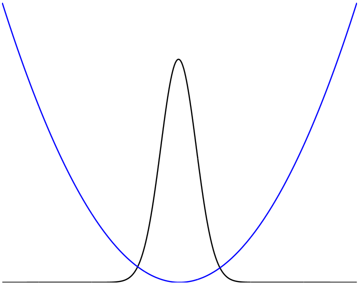

Bernoulli matrices

- **What are Bernoulli matrices?**

$$B = \begin{pmatrix} 0 & +1 & -1 & \cdots \\ +1 & 0 & -1 & \cdots \\ -1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- **What are they good for?**
    - Related to the adjacency matrices of graphs/networks
    - Erdős-Réyni models, percolation

- **What are their spectral properties?** Take $P(B_{ij} = \pm 1) = \frac{1}{2}$.

$$\lambda(B) = (\lambda_1(B) \leq \ldots \leq \lambda_N(B)) \,?$$

Universality

- **Linear Statistics:** M a random matrix: How is $\Phi_f(M)$ distributed as $N \to \infty$?

$$\Phi_f(M) := \mathrm{Tr}(f(M)) - \mathbb{E}[\mathrm{Tr}(f(M))] := \sum_{\mu} f(\lambda_\mu(M)) - \mathbb{E}[f(\lambda_\mu(M))]$$

e.g. $f(x) = x^3$ then $\mathrm{Tr}(f(M)) = \mathrm{Tr}(M^3)$.

- **Theorems:** Results concerning linear statistics
  - [Johansson '98] Unitary invariant ensembles, $\Phi_f(M) \xrightarrow{D} Z_f$
  - [Schenker & Schulz-Baldes '03] Wigner + weak correlations, $\Phi_f(M) \xrightarrow{D} Z_f$, f polynomial.
  - [Sosoe & Wong '15] Many M, optimal conditions on f.

CLT in RMT

- **Chebyshev polynomials:** $T_n(x) = \cos(n \arccos(x))$, for an $N \times N$ matrix X

$$T_n(X) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} d_r^{(n)} X^{n-2r} \qquad \mathrm{Tr}(T_n(X)) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} d_r^{(n)} \mathrm{Tr}(X^{n-2r})$$

- **Random variable:** Take the quantity

$$Y_n(B) = \mathrm{Tr}\left(T_n\left(\frac{B}{\sqrt{4N}}\right)\right) - \mathbb{E}\left[\mathrm{Tr}\left(T_n\left(\frac{B}{\sqrt{4N}}\right)\right)\right]$$

- **Equivalence:** The following statements are equivalent
  - $\Phi_f(B) = \mathrm{Tr}(f) - \mathbb{E}[\mathrm{Tr}(f)] \xrightarrow{D} Z_f$ for polynomial f of degree k
  - $(Y_3, Y_4, \ldots, Y_k) \xrightarrow{D} (Z_3, Z_4 \ldots, Z_k)$ independent with $Z_i \sim N(0, \frac{n}{2})$.

Introducing dynamics

- **Markov process:** Choose random element and change sign $B_{ij} \to -B_{ij}$

$$B' = \begin{pmatrix} 0 & B_{12} & -B_{13} & \cdots \\ B_{12} & 0 & B_{23} & \cdots \\ -B_{13} & B_{23} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- **Matrix change:** $\delta B^{pq} = B' - B = -2B_{pq}(|p\rangle\langle q| + |q\rangle\langle p|)$

- **Expected change:** $\delta Y_n^{pq} = Y_n(B + \delta B^{pq}) - Y_n(B)$, then

$$\mathbb{E}[\delta Y_n | B] = \frac{1}{d_N} \sum_{p<q} \delta Y_n^{pq} = \left(\frac{2n}{d_N}\right)(-Y_n(B)) + R_n(B)$$

$$\mathbb{E}[\delta Y_n \delta Y_m | B] = \frac{1}{d_N} \sum_{p<q} (\delta Y_n^{pq})^2 = 2\left(\frac{2n}{d_N}\right)\sigma_n^2 \delta_{nm} + R_{nm}(B)$$

Result

- **Result:** Due to deviations in first and second moments we have

$$|\mathbb{E}[\phi(Y)] - \mathbb{E}[\phi(Z)]| \leq \mathcal{O}(N^{-1})\|\phi\| + \mathcal{O}(N^{-1/2})\|\phi'\| + \mathcal{O}(N^{-1})\|\phi''\|$$

- **Error scaling:** This comes from using
  - $\mathbb{E}|R_n(B)| = \mathcal{O}(N^{-1})$
  - $\mathbb{E}|R_{nm}(B)| = \mathcal{O}(N^{-1/2})$
  - $\mathbb{E}|R_{nml}(B)| = \mathbb{E}[|\delta Y_n \delta Y_m \delta Y_l|] = \mathcal{O}(N^{-1})$

Conclusions and outlook

- Classical CLT via Stein's method

- RMT CLT for polynomial functions via Stein's method

- **Motivations:** Matrices with strong correlations: Regular tournaments, $H_{pq} = \pm i, H_{pq} = -H_{qp}$ and $\sum_q H_{pq} = 0$