

Variations on the Catalan theme: the census of Hausdorff moments

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Outline

1) Basic PDF related to Binomial numbers

a) Marchenko-Pastur

b) Arcsin

c) Wigner's semicircle

2) Solutions of Hausdorff moment problem
via inverse Mellin transform

a) Mellin convolutions (double, triple...)

b) New features?

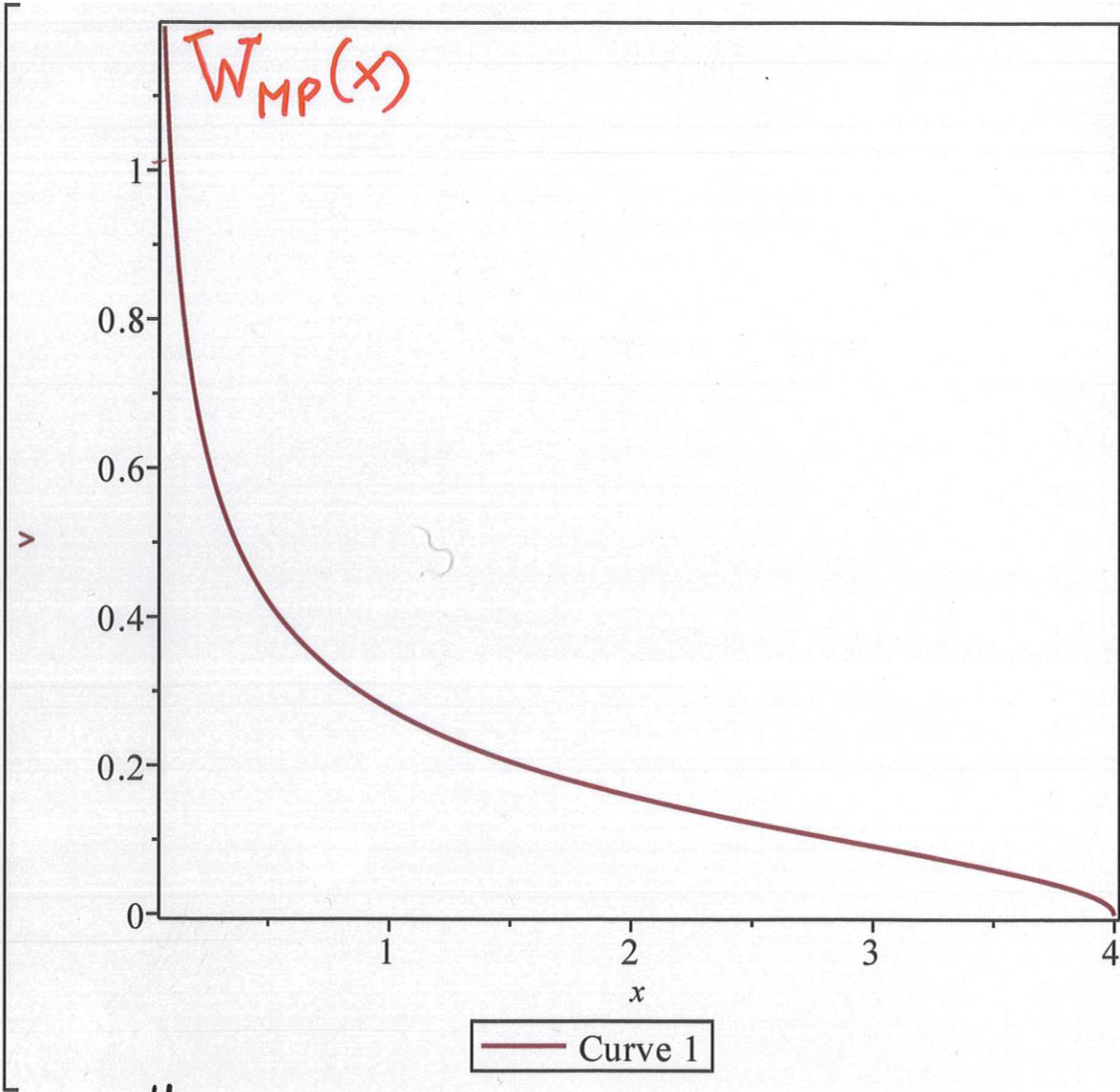
3) General Meijer G results:

a) Speicher - Mingo - Nica numbers
b) Constellation numbers (Tutte,
Bousquet - Méliot...)

... Expository character / many results known ...

Collaborators: K. Zyczkowski, M. Nowak, A. Horzela
K. Górska (Gronow), W. Młotkowski
(Wrocław), G.H.E. Duchamp,
Ph. Flajolet, S. Smith (Paris)...

Marchenko-Pastur distribution



$$\int_0^4 x^n W_{MP}(x) dx = \frac{1}{n+1} \binom{2n}{n}, \quad n=0,1,\dots$$

\nearrow

Catalan numbers $C(n)$

$A000108(n) = 1, 1, 2, 5, 14, 42, 132, \dots$

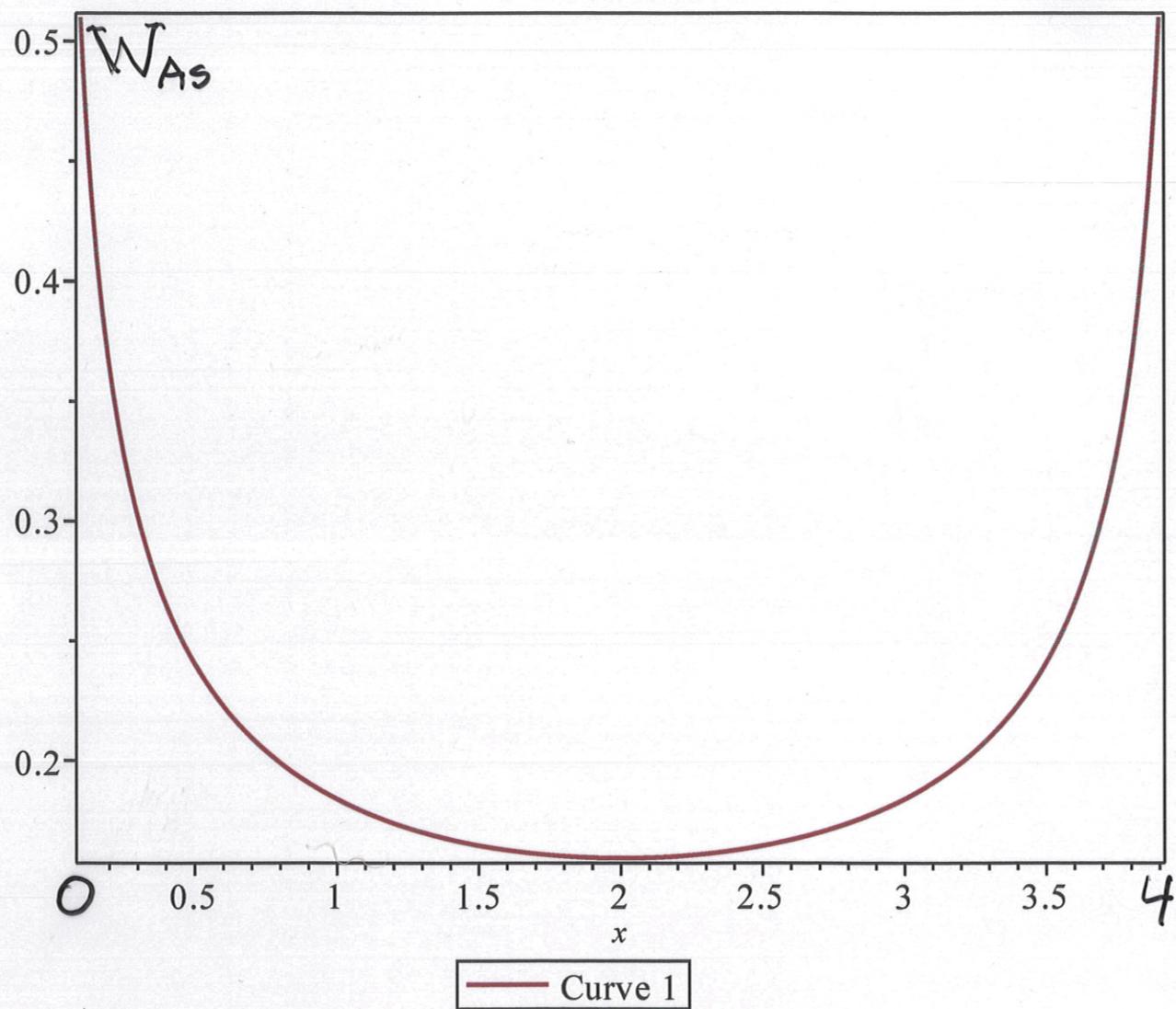
OEIS ... largest entry

Informations about the Catalan numbers: $\frac{1}{n+1} \binom{2n}{n}$

- 1) Online Encyclopedia of Integer Sequences (**OEIS**), by N.J.A. Sloane
<https://oeis.org>
 $\text{Catalan}(n) \implies A000108(n)$
- 2) **T. Koshy**, "Catalan Numbers with Applications", Oxford U.P. (2015)
- 3) **R.P. Stanley**, "Catalan Numbers"
Cambridge U.P. (2015)

Arc-sin Distribution

$$W_{AS}(x) = \frac{1}{\pi} \sqrt{\frac{1}{x(4-x)}}$$



$$\int_0^4 x^n W_{AS}(x) dx = \binom{2n}{n}, \quad n=0,1,\dots$$

Central binomial coeffs.

1, 2, 6, 20, 70, 252, 924

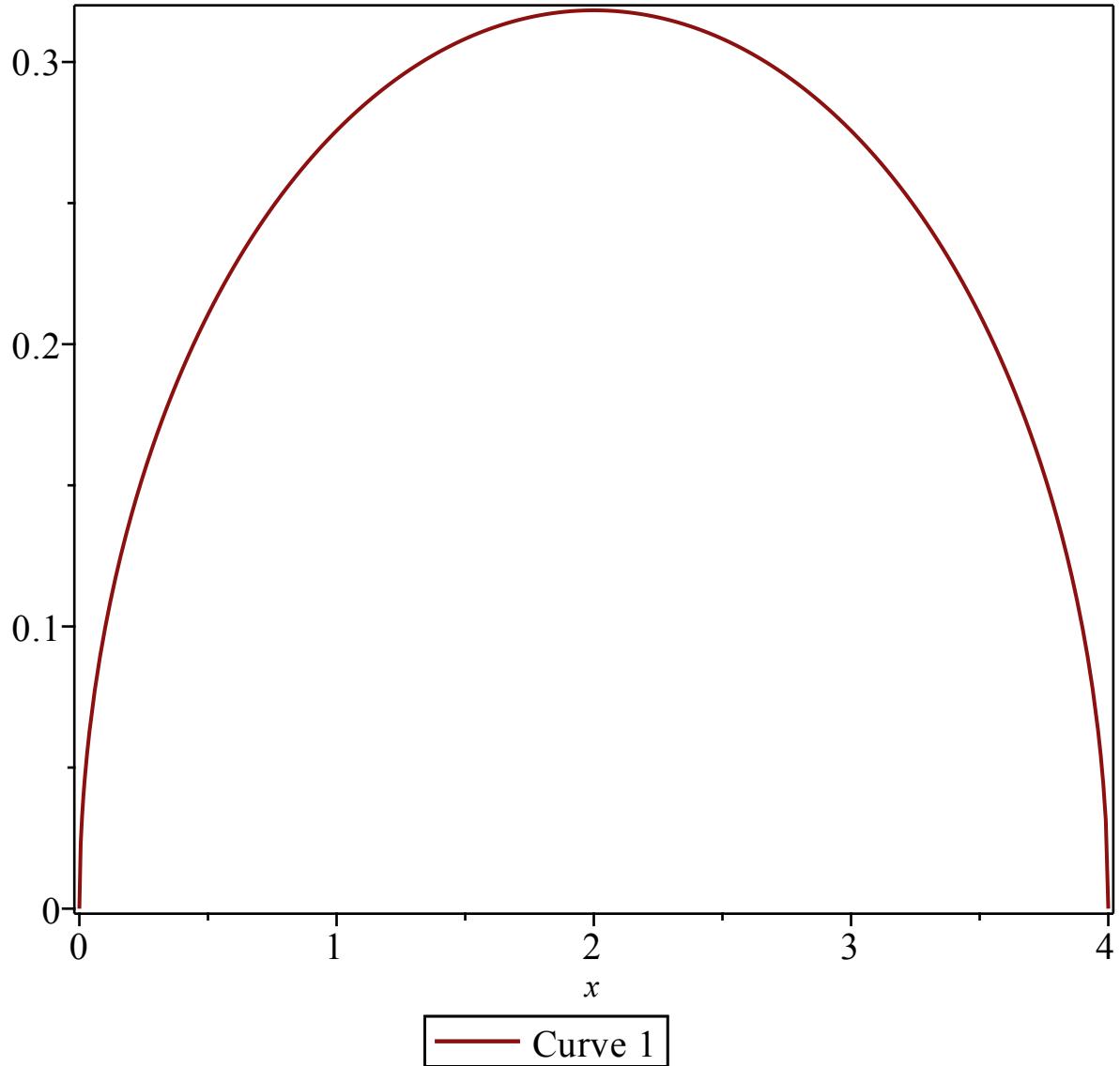
$\underbrace{A000984(n)}_{\text{OEIS}}$

```
[> # Wigner Semicircle law
```

```
> seq(int(sqrt(x*(4-x))*x^n/(2*Pi),x=0..4),n=0..8);  
1, 2, 5, 14, 42, 132, 429, 1430, 4862  
(1)
```

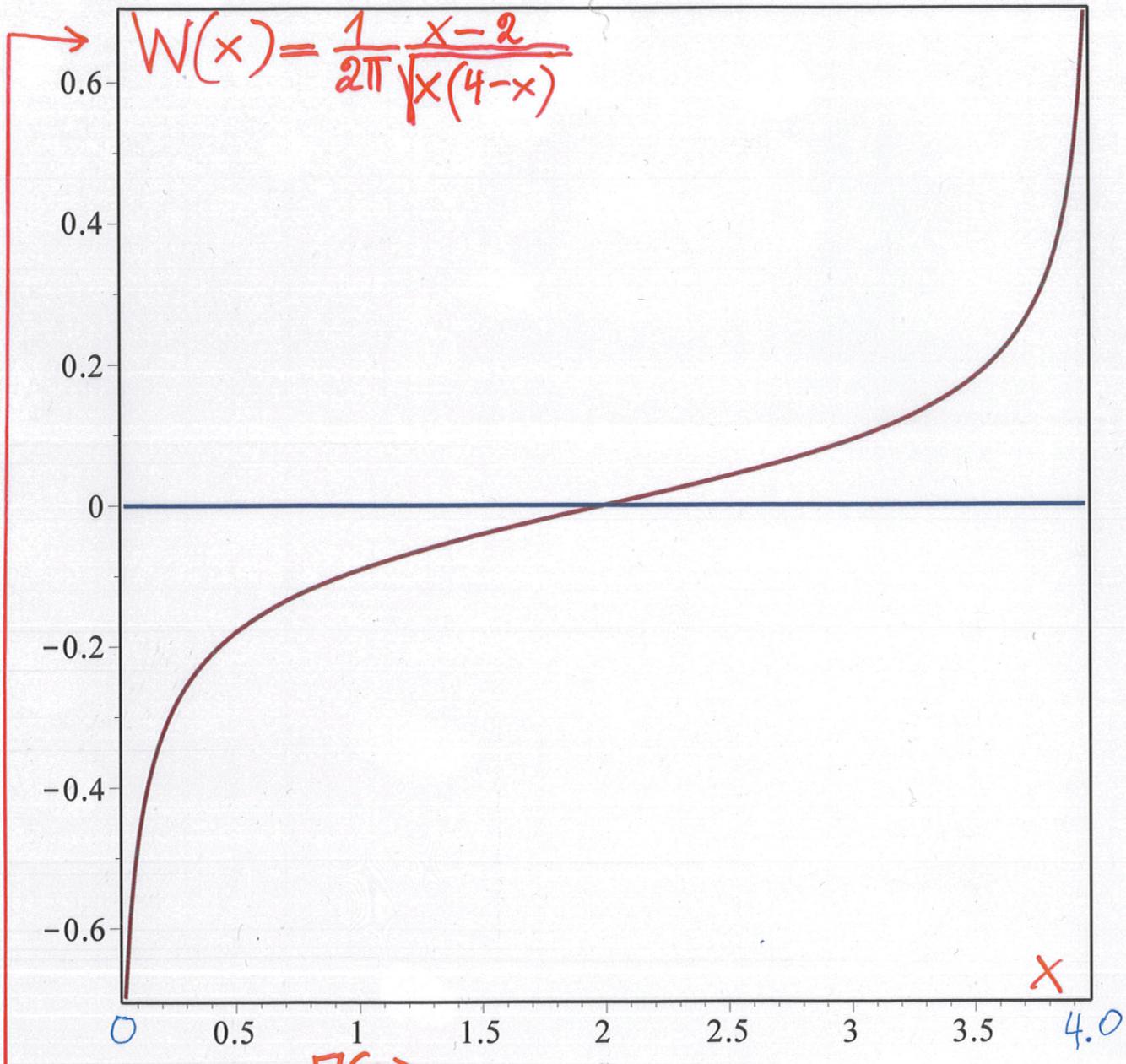
```
> # moments=Catalan(n+1)  
1, 2, 5, 14, 42, 132, 429, 1430  
(2)
```

```
> plot(sqrt(x*(4-x))/(2*Pi),x=0..4,axes=boxed);
```



Non-positive solution with moments

$$\varrho(n) = \frac{1}{n+1} \binom{2n}{n} \cdot n = \binom{2n}{n-1} = 0, 1, 4, 15, 56, \dots$$
$$= A001791(n)$$



$$n = s-1 = \frac{\Gamma(s)}{\Gamma(s-1)} ;$$

$$\varrho(s) = \frac{4^s}{4\sqrt{\pi}} \frac{\Gamma(s-\frac{1}{2}) \Gamma(s)}{\Gamma(s+1) \Gamma(s-1)}$$

$$W(x) = \frac{1}{4\sqrt{\pi}} \text{MeijerG}\left(\left[\left[\right], \left[1, -1\right]\right], \left[\left[0, -\frac{1}{2}\right], \left[-\right]\right], \frac{x}{4}\right)$$

Hausdorff moment problem

Given an infinite set of $\varrho(n)$, $n=0, 1, \dots$
such that

$$\int_0^R x^n W(x) dx = \varrho(n), \quad n=0, 1, \dots$$

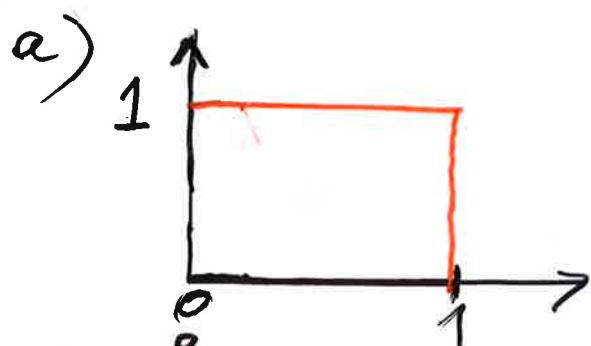
"signed", condition?

Find $W(x) \rightarrow$

$W(x) > 0$
(Hausdorff (1921))

Hausdorff conditions are difficult to verify. Support $(0, R)$.

Remarks:



$$\int_0^1 x^n \cdot 1 \cdot dx = \frac{1}{n+1}$$

b)

$$\int_0^R x^n W(x) dx = \frac{1}{(n+1)!}, \quad \text{No solution at all (?)}$$

c) How do we know if it is Hausdorff?

$$R = \lim_{n \rightarrow \infty} [\varrho(n)]^{\frac{1}{n}}$$

$\rightarrow \leftarrow \infty, \text{Hausdorff}$
 $\rightarrow \infty, \text{Stieltjes}$

How do we know from $\underline{g(n)}$ that it is Hausdorff?

$$\Rightarrow R = \lim_{n \rightarrow \infty} [g(n)]^{\frac{1}{n}}$$

→ ∞ (Stieltjes)
 → $< \infty$ (Hausdorff)

Examples:

$$\text{Catalan}(n) : \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} \binom{2n}{n} \right]^{\frac{1}{n}} = 4$$

$$\binom{2n}{n} : \lim_{n \rightarrow \infty} \left[\binom{2n}{n} \right]^{\frac{1}{n}} = 4$$

————— * —————
 Ordinary generating function (ogf)

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} z^n = G(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

$z < \frac{1}{4}$

$a \equiv$ Radius of convergence

$$\text{Conjecture : } R(\text{support}) = \frac{1}{a(\text{r. converg.})}$$

Always true....

Mellin transform

$$f^*(s) = \int_0^\infty x^{s-1} f(x) dx \equiv M[f(x); s]$$

$$f(x) \equiv M^{-1}[f^*(s); x]$$

Applied to the moment problem :

$$\int_0^\infty x^n W(x) dx = g(n) \Rightarrow W(x) = M^{-1}[g(s-1); x]$$

Mellin (multiplicative) convolution

$$f^*(s) = M[f(x); s] \quad \left. \right\} \text{Known}$$

$$g^*(s) = M[g(x); s]$$

$$\Rightarrow M^{-1}[f^*(s)g^*(s); x] =$$

$$= \int_0^\infty f(t) g\left(\frac{x}{t}\right) \frac{1}{t} dt$$

Conserves the positivity !!!

Inverse Mellin transform for moments

Catalan numbers

$$C(n) = \frac{1}{n+1} \binom{2n}{n} \xrightarrow{n \rightarrow s-1} \frac{\Gamma(2s-1)}{\Gamma(s)\Gamma(s+1)}$$

Gauss-Legendre

$$\frac{4^s}{4\sqrt{\pi}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s+1)} = \text{ratio of } \Gamma's$$

$$= \int_0^\infty x^{s-1} W(x) dx$$

$$W(x) = \frac{1}{4\sqrt{\pi}} \mathcal{M}^{-1} \left[\frac{4^s \Gamma(s-\frac{1}{2})}{\Gamma(s+1)} ; x \right]^{(*)}$$

$$= \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} \equiv W_{MP}(x) \equiv$$

$(*) \rightarrow$ Tables of inverse Mellin
Meijer G functions

$$= \frac{1}{4\sqrt{\pi}} \text{MeijerG}\left(\left[\left[\right], \left[1\right]\right], \left[\left[-\frac{1}{2}\right], \left[\right]\right], \frac{x}{4}\right)$$

\uparrow \uparrow \uparrow

Inverse Mellin transform for moments

Central binomial coeffs

$$\binom{2n}{n} = \frac{(2n)!}{n! n!} \xrightarrow{n \rightarrow s-1} \frac{\Gamma(2s-1)}{\Gamma(s)\Gamma(s)}$$

$\xrightarrow{\text{Gauss-Legendre}}$ $\frac{4^s}{4\sqrt{\pi}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} = \text{ratio of } \Gamma's$

$$= \int_0^\infty x^{s-1} W(x) dx$$

$$W(x) = \frac{1}{4\sqrt{\pi}} \mathcal{M}^{-1} \left[\frac{4^s \Gamma(s-\frac{1}{2})}{\Gamma(s)}; x \right] \stackrel{(*)}{=}$$

$$= \frac{1}{\pi} \frac{1}{\sqrt{x(4-x)}} = W_{AS}(x) \equiv$$

$\xrightarrow{(*)}$ Tables of inverse Mellin
 \rightarrow Meijer G functions

$$\equiv \frac{1}{4\sqrt{\pi}} \text{Meijer } G \left(\left[\left[\right], \left[\circ \right] \right], \left[\left[-\frac{1}{2} \right], \left[\right] \right], \frac{x}{4} \right)$$


```
> seq( (binomial(2*n,n))^2, n=0..10);
1, 4, 36, 400, 4900, 63504, 853776, 11778624, 165636900, 2363904400, 34134779536
```

(1)

Warning, premature end of input, use <Shift> + <Enter> to avoid this message.

```
> W:=proc(x) EllipticK(sqrt(1-x/16))/(2*Pi^2*sqrt(x));end;
W := proc(x) 1/2 * EllipticK(sqrt(1 - 1/16*x)) / (Pi^2 * sqrt(x)) end proc
```

(2)

```
> W(x);
```

$$\frac{1}{2} \frac{\text{EllipticK}\left(\frac{1}{4} \sqrt{16-x}\right)}{\pi^2 \sqrt{x}}$$

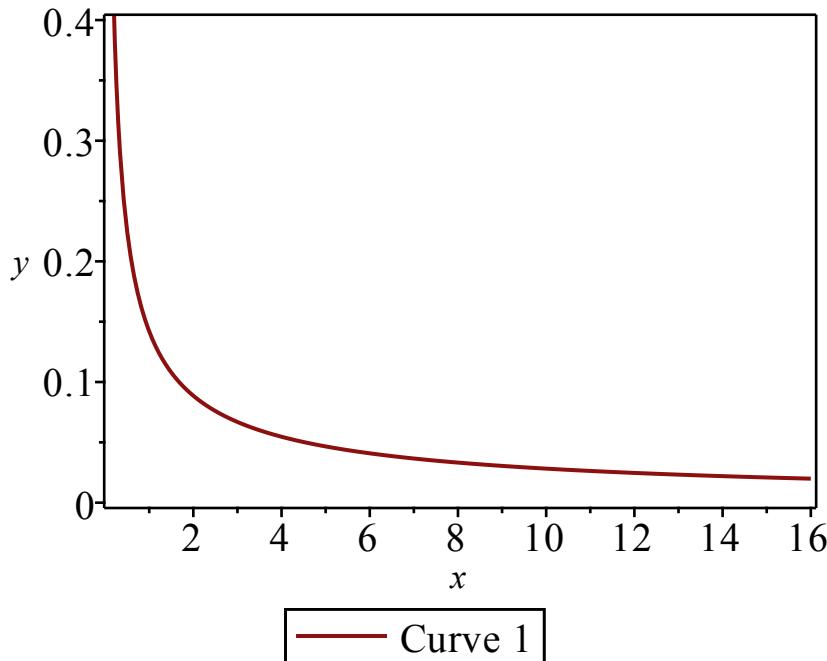
(3)

```
> W(16);
```

$$\frac{1}{16 \pi}$$

(4)

```
> plot(W(x), x=0.1..16, y=0..0.4, axes=boxed);
```



$$\frac{1}{2} \frac{4 \ln(2) - \frac{1}{2} \ln(x)}{\pi^2 \sqrt{x}} + O(\sqrt{x})$$

(5)

Speicher-Mingo-Nica (~ 2007)

Numbers of annular permutations

$$C(K, n) = \frac{2K\textcircled{n}}{K+n} \binom{2K-1}{K} \binom{2n-1}{n} \quad \rightarrow \text{integers}$$

$K = 1, 2, \dots ; n = 0, 1, \dots$ ($K \rightarrow$ parameters)

Integral representation as Hausdorff moments:

$$\int_0^4 x^n W_K(x) dx = C(K, n), n = 0, 1, \dots$$

$C(K, 0) = 0 \Rightarrow W(x)$ "signed" functions

Exact solution:

Jacobi Polynomials

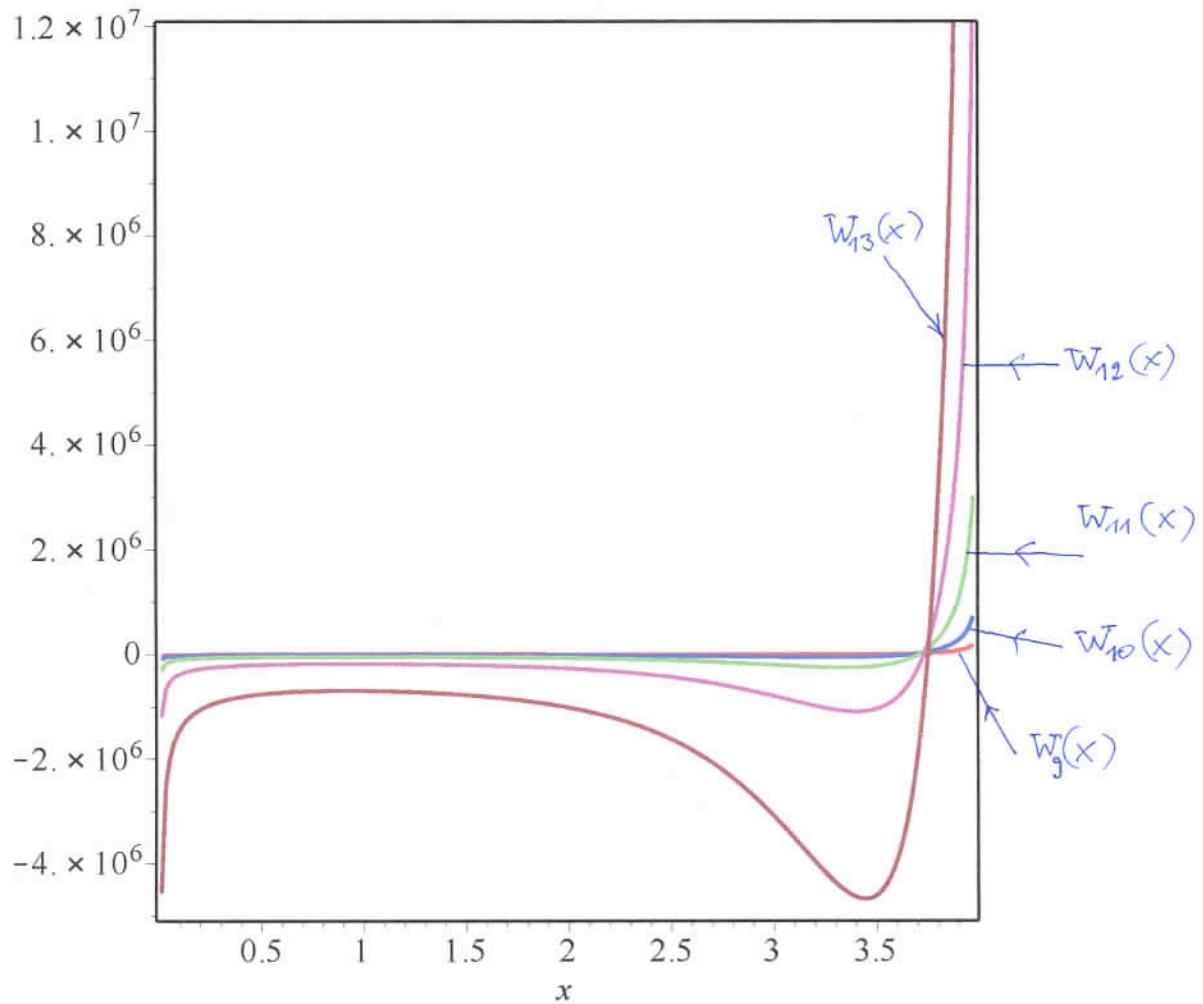
$$W_K(x) = K \cdot K! \binom{2K-1}{K} \frac{P_K^{(-\frac{1}{2}, \frac{1}{2}-K)} \left(\frac{x}{2}-1\right)}{\left(\frac{1}{2}\right)_K \pi \sqrt{x(4-x)}}$$

Example:

$$W_3(x) = \frac{3}{2} \frac{x^3 - 2x^2 - 2x - 4}{\pi \sqrt{x(4-x)}}$$

$W_k(x)$ - signed weight functions in the moment representations
of $c(K, n) = \frac{2kn}{K+n} \binom{2K-1}{K} \binom{2n-1}{n}$, i.e. $\int_0^4 x^n W_k(x) dx = c(K, n)$

```
> plot([w(9,x), w(10,x), w(11,x), w(12,x), w(13,x)], x=0.009...3.97, axes=boxed, color=[red, blue, green, magenta, brown]);
```



For all $K = 1, 2, \dots$, $\int_0^4 W_K(x) dx = 0$.

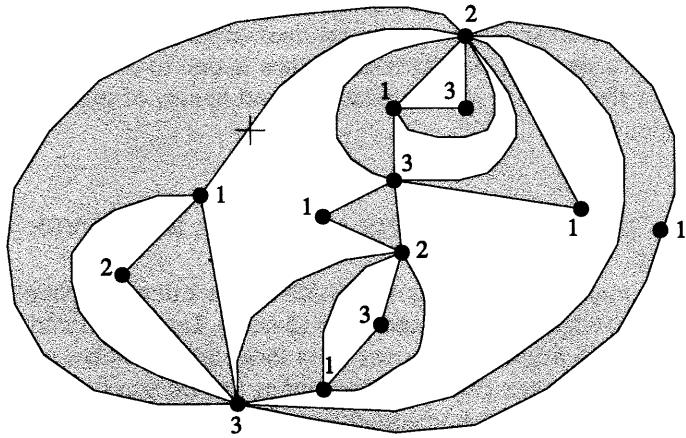


FIG. 1. A rooted 3-constellation and its canonical labelling.

- the group generated by $\sigma_1, \dots, \sigma_m$ acts transitively on $\{1, 2, \dots, n\}$,
- $\sum_{i=0}^m c(\sigma_i) = n(m-1) + 2$, where $\sigma_0 = \sigma_1 \sigma_2 \cdots \sigma_m$,

and rooted m -constellations formed of n polygons, labelled from 1 to n in such a way that the root polygon has label 1. Moreover, if the constellation has d_i white faces of degree mi , then σ_0 has d_i cycles of length i .

Proof. Let C be a rooted m -constellation formed of n polygons labelled from 1 to n . Recall that there is a canonical labelling (by $1, 2, \dots, m$) of the vertices of C . For $1 \leq i \leq m$, each m -gon is adjacent to exactly one vertex of label i : hence, turning clockwise around vertices of label i defines a permutation of the n polygons, denoted σ_i , which we identify with a permutation of \mathfrak{S}_n .

As the constellation is connected, the group generated by $\sigma_1, \dots, \sigma_m$ acts transitively on $\{1, 2, \dots, n\}$.

Moreover, let W be a white face of degree mi : it has exactly i vertices of label m . Let B_1, B_2, \dots, B_i denote the i black faces adjacent to W by an edge labelled $(1, m)$, arranged in counterclockwise order around W . Then

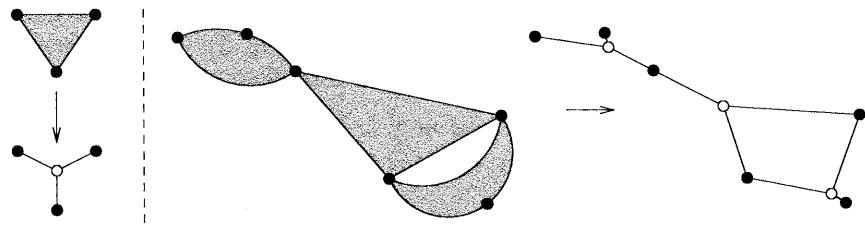


FIG. 2. How constellations appear.

M. Bousquet-Mélou & Gilles Schaeffer

Adv. Appl. Math. 34, 337–368 (2000)

$$C_p(n) = \frac{(p+1) \cdot p^{n-1}}{[(p-1)n+1][(p-1)n+2]} \binom{pn}{n} \quad (*)$$

$p \geq 2$, integer

Combinatorics : $n \geq 1$

Moment problem : $n=0,1,\dots$

$$\int_0^{\infty} x^n W_p(x) dx = C_p(n), \quad \boxed{\alpha_p = \frac{p^{p+1}}{(p-1)^{p-1}}}$$

Explicit and exact solutions for $W_p(x)$
for all p in terms of **Meijer G-functions**
obtained via Mellin transform.

$p=2, 3 \rightarrow$ elementary functions
 $p > 3 \rightarrow$ hypergeometric functions
KAP + W. Młotkowski
(2014)

"Fractional" - Constellation numbers

$p = K/l$, $K > l$, relativ prime integers

$$C(K, l, n) = \frac{\left(\frac{K}{l} + 1\right)\left(\frac{K}{l}\right)^{n-1}}{\left[\left(\frac{K}{l} - 1\right)n + 1\right]\left[\left(\frac{K}{l} - 1\right)n + 2\right]} \binom{\frac{Kn}{l}}{n}$$

no combinatorics ...

$\omega(K, l)$

$$\int_0^1 x^n W(J(K, l, x)) dx = C(K, l, n)$$

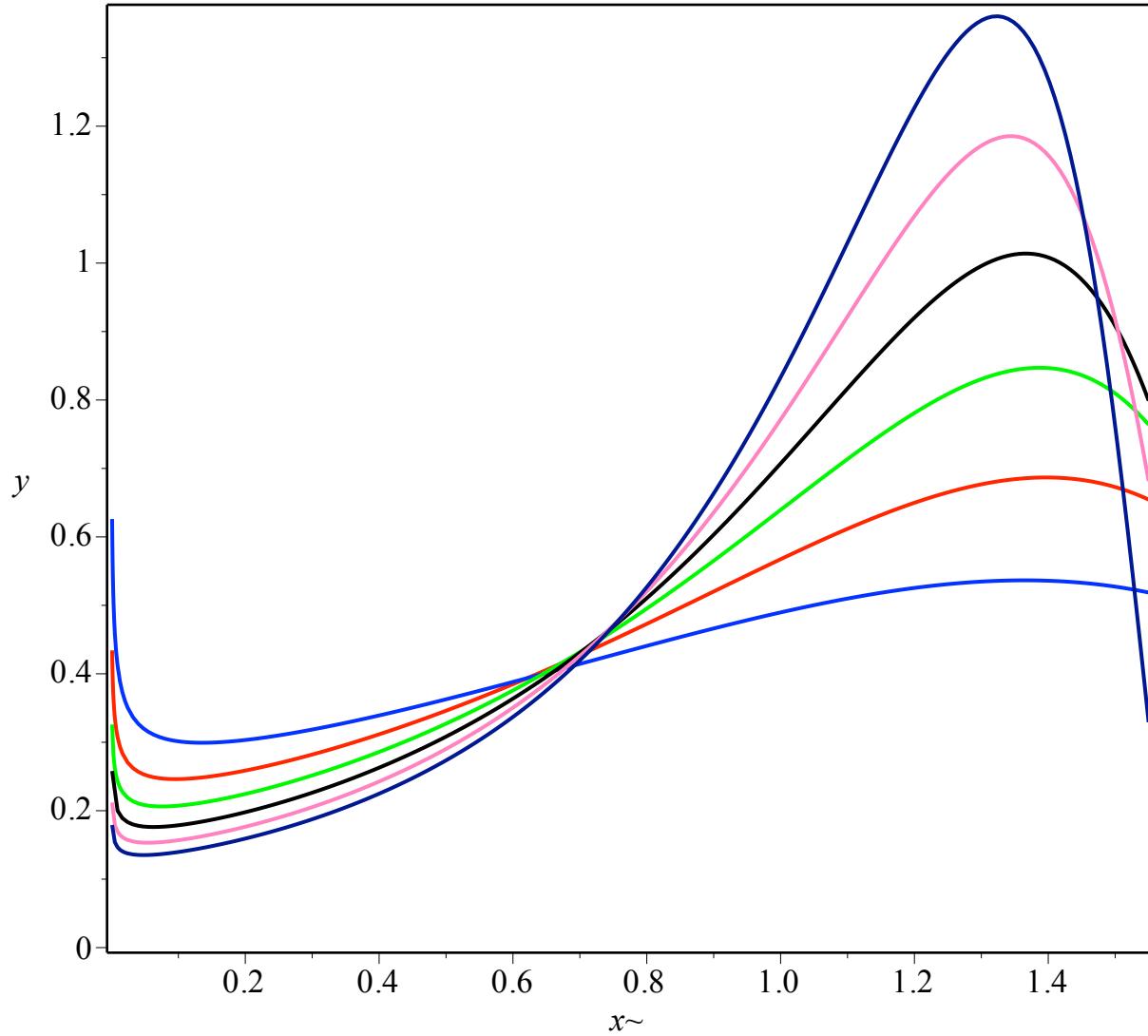
$$\omega(K, l) = \left(0, \frac{K^{(K/l+1)}}{(K-l)(K-l-1)l^2}\right)$$

Exact solution for any K, l .

```

> # Constellation weights:
Blue->5/4, Red->6/5, Green->7/6, Black->8/7, Pink->
9/8, BlueNavy->10/9
> plot([wratcon(6,5,x),wratcon(5,4,x),wratcon(7,6,x),wratcon(8,7,x),
wratcon(9,8,x),wratcon(10,9,x)],x=0.001..1.55,y=0..1.37,color=[red,
blue,green,black,"HotPink","Navy"],axes=boxed);

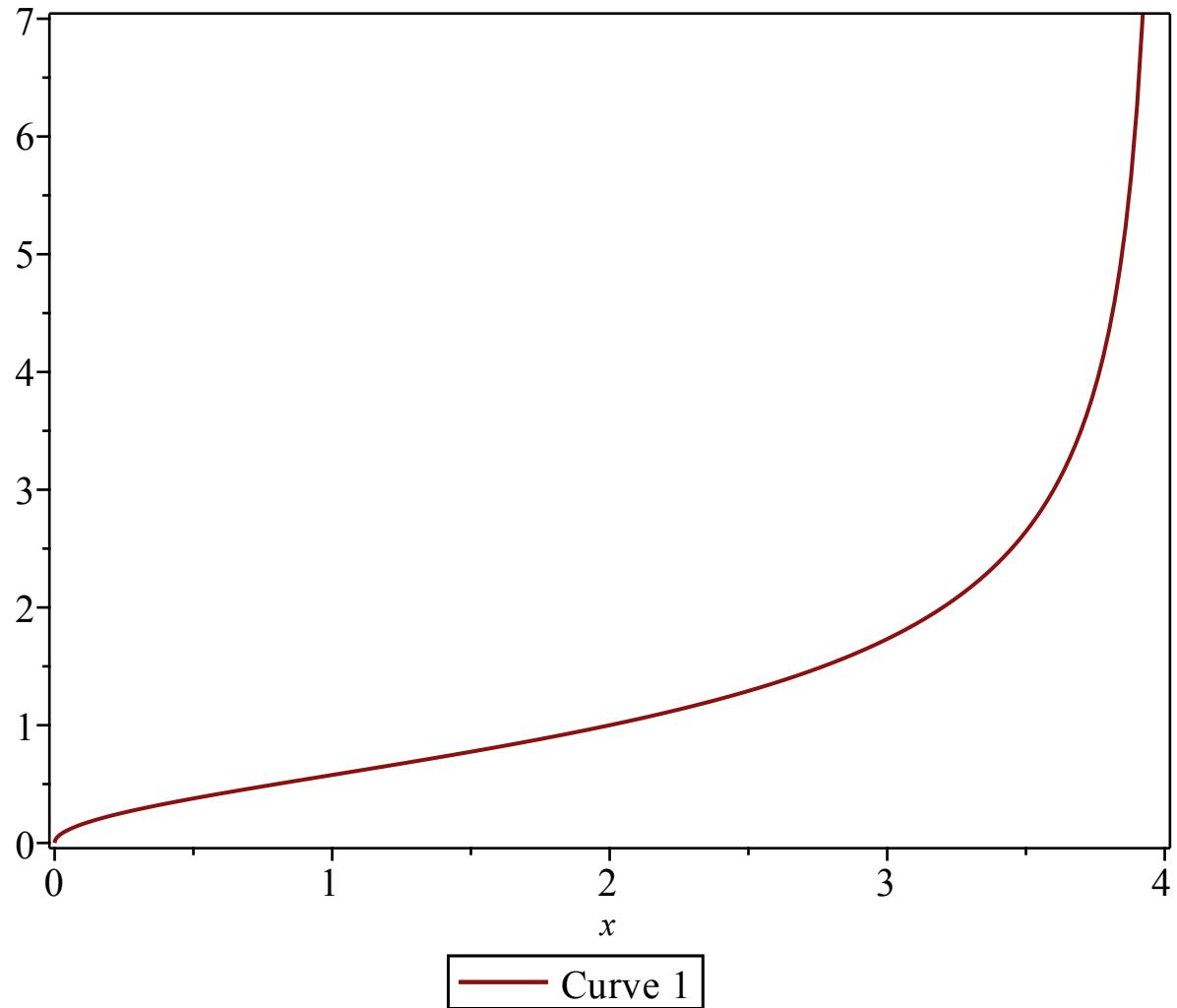
```



```

> # BINOMIAL(2*n+1,n)
> seq(binomial(2*n+1,n),n=0...7);
1, 3, 10, 35, 126, 462, 1716, 6435
> # This was A001700
> plot(sqrt(x/(4-x)),x=0..4,axes=boxed);

```



```

> sum(binomial(2*n+1,n)*z^n,n=0..infinity)=rationalize((1-sqrt(1-4*z))
)/(2*sqrt(1-4*z)*z));

$$\sum_{n=0}^{\infty} \text{binomial}(2n+1, n) z^n = \frac{1}{2} \frac{(-1 + \sqrt{1 - 4z}) \sqrt{1 - 4z}}{z (-1 + 4z)}$$


```

Orthogonal Polynomials with respect
to Marchenko-Pastur distribution

$$R_n(x) \equiv (-1)^n U_{2n}\left(\frac{\sqrt{x}}{2}\right)$$

$$\int_0^4 R_n(x) R_{n'}(x) \left[\frac{1}{2\pi} \sqrt{\frac{4-x}{x}} \right] dx = \delta_{nn'}$$

$\underbrace{\qquad\qquad\qquad}_{W_{MP}(x)}$

Orthogonal Polynomials with respect
to "reversed" Marchenko-Pastur
distribution

$$S_n(x) = U_n(-1 + \frac{x}{2}) - U_{n-1}(-1 + \frac{x}{2})$$

$$\int_0^4 S_n(x) S_{n'}(x) \left[\frac{1}{2\pi} \sqrt{\frac{x}{4-x}} \right] dx = \delta_{nn'}$$

$\underbrace{\qquad\qquad\qquad}_{W_{MP}^{(r)}(x)}$

$U_k(z) \rightarrow$ Tchebyshov polynomials

Partial list of available solutions

$$g(n) = \frac{2}{n+1} \binom{3n}{n}; \text{ Prodinger}$$

$$= \binom{2n}{n} \binom{2(n+1)}{n+1}$$

$$= \text{Catalan}(n) \text{Catalan}(n+1)$$

$$= \binom{4n}{2n}$$

$$= \frac{2(4n+1)!}{(3n+2)!(n+1)!}$$

$$= [\text{Catalan}(n)]^2 \binom{2n}{n}$$

$$= [\text{Catalan}(n)]^p, \quad p=2, 3, \dots$$

for which

$$W_p(x) = \left(\frac{1}{4\sqrt{\pi}}\right)^p G_{p,p}^{p,0}\left(\frac{x}{4^p} \mid \begin{array}{l} \{1\}_{1 \leq k \leq p} \\ \{-\frac{1}{2}\}_{1 \leq k \leq p} \end{array}\right)$$

⋮