Distinguishing generic quantum states and symmetrized Marchenko–Pastur distribution

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in collaboration with

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Motivation

A short visit in a **quantum shop**

Suppose you need a quantum state ρ,

you go to a quantum shop, pay for it and

...

you get a state σ instead !

How good the quantum shop is doing ?

Is the state σ we bought at least ϵ–close

to the state ρ we have ordered??

Close with respect to which metric?

If the desired state is pure, $\rho = |\psi\rangle\langle\psi|$

the situation is simple:

You need to maximize the overlap (**fidelity**),
i.e. the expectation value: $F = \langle\psi|\sigma|\psi\rangle$,

What should one do, if the ordered state ρ is mixed?

How to measure the distance
between density operators ρ and σ?

The set Ω_N of mixed states of size N

definition

$$\Omega_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

Distances in the set of quantum states

a) Hilbert–Schmidt distance, $D_{\mathrm{HS}}(\rho,\sigma) := [\mathrm{Tr}(\rho-\sigma)^2]^{1/2}$

b) trace distance, $D_{\mathrm{tr}}(\rho,\sigma) := \frac{1}{2}\mathrm{Tr}|\rho-\sigma|$

c) Bures distance, $D_{\mathrm{B}}(\rho,\sigma) := \left(2[1-\sqrt{F(\rho,\sigma)}]\right)^{1/2}$,

where **fidelity** between two states reads (Uhlmann '76, Jozsa '94),

$$F(\rho,\sigma) := [\mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|]^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)^2 .$$

Generic mixed states

How they appear in quantum physics ?

Reduction of typical (=random) pure states

1) Consider an ensemble of **random pure states** $|\psi\rangle$ of a **composite system** $\mathcal{H}_A \otimes \mathcal{H}_B$ distributed according to a given measure μ.

2) Perform partial trace over a chosen subsystem B to get a **random mixed state**

$$\rho \ := \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

Properties of 'typical' pure states in $\mathcal{H}_N$

One quantum state fixed, one random...

Fix an arbitrary state $|\psi_1\rangle$. Generate randomly the other state $|\psi_2\rangle$.

- What is the average angle χ between these states ?
- What is the distribution $P(\chi)$ of the angle $\chi := \arccos |\langle\psi_1|\psi_2\rangle|$?

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Measure concentration phenomenon

'Fat hiper-equator' of the sphere S^N in $\mathbb{R}^{N+1}$...
It is a consequence of the Jacobian factor for expressing the volume element of the N– sphere. Let $z = \cos\vartheta_1$, so that

$$J \sim (sin\,\vartheta_1)^{N-1} J_2(\vartheta_2, \ldots, \vartheta_N)$$

Hence the typical angle χ is 'close' to $\pi/2$ and two 'typical random states' are orthogonal and the distribution $P(\chi)$ is **'close' to** $\delta(\chi - \pi/2)$.
How close?

Quantitative description of **Measure Concentration**

Levy's Lemma (on higher dimensional spheres)

Let $f : S^N \to \mathbb{R}$ be a **Lipschitz function**,
with the constant η and the mean value $\langle f \rangle = \int_{S^N} f(x) d\mu(x)$.
Pick a point $x \in S^N$ **at random from the sphere**. For large N it is then **unlikely** to get a value of f much different then the average:

$$P\Big(|f(x) - \langle f \rangle| > \alpha\Big) \ \leq \ 2 \exp\Big(-\frac{(N+1)\alpha^2}{9\pi^3\eta^2}\Big)$$

Simple application: the distance from the 'equator'

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Take $f(x_1, ... x_{N+1}) = x_1$. Then **Levy's Lemma** says that the probability of finding a random point of S^N outside a band along the **equator of width** 2α converges **exponentially** to zero as $2 \exp[-C(N+1)\alpha^2]$.

As $N >> 1$ then **every equator** of S^N is **'FAT'**.

Random states and Marchenko-Pastur distribution (1967)

Consider a random state σ of size N obtained by partial trace over K dimensional environment, $\sigma = \mathrm{Tr}_K[U\,|\psi_N, \phi_K\rangle\langle\psi_N, \phi_K|\,U^\dagger]$.

Then its asymptotic **level density** is $P_c(x) = \frac{1}{2\pi x}\sqrt{(x - x_-)\,(x_+ - x)}$, where $x = N\lambda$, rectangularity $c = N/K$ and support $x_\pm = \left(1 \pm \sqrt{c}\right)^2$.

For equal subsystems $c = 1$ this expression reduces to the standard

Marchenko – Pastur

distribution

$P_1(x) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}}, \quad x \in [0, 4]$,

equivalent to setting $x = y^2$ with y distributed according to **Wigner semicircle**

Vladimir Marchenko & Leonid Pastur (2000)

Symmetrized Marchenko–Pastur distribution I

Trace distance between two states. $D_{\mathrm{Tr}}(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1 = \frac{1}{2}\mathrm{Tr}|\rho-\sigma|$ is used to describe their **distinguishability**.

What is the distribution of eigenvalue μ of the **Helstrom matrix**,

$\Gamma = \rho - \sigma$, where both states are random?

It is given by **symmetrized Marchenko–Pastur** distribution,

$SMP_c(x) = MP_c(x) \boxplus MP_c(-x)$, where $x = N\mu$.

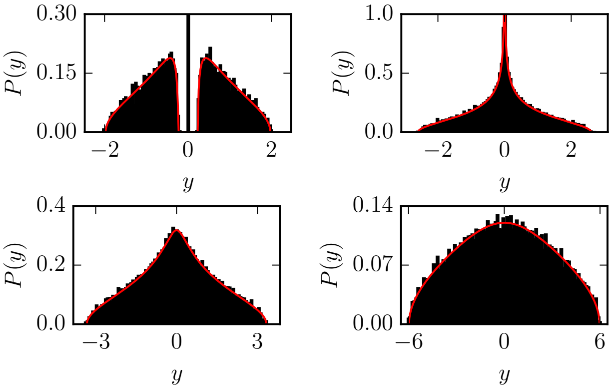
In the case of HS measure, (rectangularity $c = 1$) we obtain the normalized, symmetric MP distribution

$$SMP_1(x) = \frac{-1 - 3x^2 + \left(1 + 3x\left(\sqrt{3 + 33x^2 - 3x^4} + 6x\right)\right)^{2/3}}{2\sqrt{3}\pi x\left(1 + 3x\left(\sqrt{3 + 33x^2 - 3x^4} + 6x\right)\right)^{1/3}}. \tag{1}$$

and analogous analytical formulae for $SMP_c(x)$ with an arbitrary parameter $c = N/K > 0$.

Symmetrized Marchenko–Pastur distribution II

Level density $SMP_c(y)$ of the rescaled eigenvalue $y = \lambda_1 N$ for rectangularity $c = N/K = 0.2, 0.5, 1.0$ and 4.0



The case, $c = 1$ - free commutator of two semicircular distributions, studied by **Nica & Speicher** (1998),

and called **tetilla law**, **Deya & Nourdin** (2012).

In limiting case $c \to \infty$ one obtaines (rescaled) semicircle.

co-author **Zbyszek Puchała** during his research visit in Spain

Average distance between 2 random states

Take two random states σ and ρ acting on $\mathcal{H}_N$,
generated according to the flat (HS) measure ($c = 1$).

For large N their trace distance tends to an integral over the symmetrized MP distribution, which describes the spectrum of the **Helstrom matrix**, $\Gamma = \rho - \sigma$,

$$D_{\rm tr}(\rho,\sigma) \to \frac{1}{2}\int SMP_1(y)|y|dy = \tilde{D} := \frac{1}{4} + \frac{1}{\pi} \approx 0.5683$$

Average distance of a random state ρ to
a) the center $\rho_* = \mathbb{1}/N$ reads

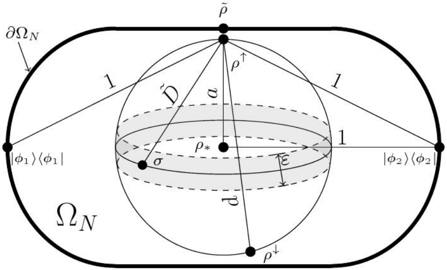
$$D_{\rm tr}(\rho,\rho_*) \xrightarrow[N\to\infty]{a.s.} \frac{1}{2}\int dt|t-1|MP(t) = a = \frac{3\sqrt{3}}{4\pi} \simeq 0.4135.$$

b) the closest pure state, $D_{\rm tr}(\rho, |\phi\rangle\langle\phi|) \xrightarrow[N\to\infty]{a.s.} 1 \;=\; \textbf{diam}(\Omega_N)$

c) the closest boundary state $\tilde{\rho}$, $D_{\rm tr}(\rho,\tilde{\rho}) \xrightarrow[N\to\infty]{a.s.} 0$

The space of quantum states Ω_N for large N

Entire mass of Ω_N is concentrated in a ϵ-vininity of a generic orbit $\rho' = U\rho U^\dagger$, where U is a Haar random unitary and ρ is a **random** state with MP level density. Here $a = 0.413$ and $\tilde{D} = 0.568$, while



the **diameter** d of the orbit is equal to the distance between two diagonal matrices with opposite order of the eigenvalues,

$d = D_{\rm Tr}(p^\uparrow, p^\downarrow) = \int_0^4 dx \, {\rm sign}(x - M) \; x \; MP(x) \simeq 0.7875,$

where M is the **median**, $\int_0^M dx \; MP(x) = 1/2$.

Concentration of measure in high dimensions

Consider two random states of dimension $N \gg 1$
The average value of their trace distance reads

$$\langle D_{\rm tr}(\rho,\sigma)\rangle = \tilde{D} = 1/4 + 1/\pi \ ,$$

but this distribution becomes singular: for $N \to \infty$ one has

$$P\big(D_{\rm tr}(\rho,\sigma)\big) \to \delta(D - \tilde{D})$$

This distance converges *almost surely* to a single value $\tilde{D}$!

How this might be possible ???

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concentration of measure !

What is the expected distance between two random points in a unit ball in $\mathbb{R}^N$?

and in a unit ball in $\mathbb{R}^3$?

Concentration of measure in high dimensions

What is the expected Euclidean distance between two random points in a unit ball in $\mathbb{R}^N$?

The answer is

Concentration of measure in high dimensions

What is the expected Euclidean distance between two random points in a unit ball in $\mathbb{R}^N$?

The answer is

$$D(x, y) \to \sqrt{2} \ !$$

as a) full measure of the ball is concentrated at the **surface**

b) for any point at the sphere another random point will belong to the **equator**, so their Euclidean distance is $D_2(x, y) = \sqrt{1 + 1}$,

while their taxi distance is $D_1(x, y) = 1 + 1 = 2$.

For two random states of large dimension N their Hilbert Schmidt (=Euclidean) distance vanishes as

$$D_{\rm HS}^2(\rho, \sigma) = {\rm Tr}(\rho - \sigma)^2 = {\rm Tr}\rho^2 + {\rm Tr}\sigma^2 - 2{\rm Tr}\rho\sigma \to 0.$$

However, their average **trace distance** is larger and non-trivial,

$$D_{\rm tr}(\rho, \sigma) \to \tilde{D} := \frac{1}{4} + \frac{1}{\pi} \approx 0.568$$

Why do we care about the **trace distance** ?

Distinguishing random states

Helstrom theorem (1967)

Suppose one is given a quantum state $\rho \in \{\rho_1, \rho_2\}$.
Probability P of discriminating between these states is bounded by

$$P \leq \tfrac{1}{2} + \tfrac{1}{2} D_{\rm tr}(\rho_1, \rho_2)$$

For instance, for orthogonal states $D_{\rm tr} = 1$, so that $P = 1$

Distinguishing two generic quantum states

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Theorem. Two random states of large dimension $N \gg 1$ can be distinguished in a single–shot experiment with probability bounded by

$$P \leq \frac{1}{2} + \frac{1}{2}\tilde{D} = \frac{5}{8} + \frac{1}{2\pi} \simeq 0.784155.$$

universal bound for distinguishability in high dimensions

Asymptotic distinguishability results

for two random states σ and ρ acting on $\mathcal{H}_N$,
Related **asymptotic** results ($N \gg 1$) for the average:

a) **relative entropy**: $S(\rho\,||\sigma) = \mathrm{Tr}\rho\log\rho/\sigma$

$$S(\rho\,||\sigma) \to \int dt \int ds(t\log t - t\log s)MP(t)MP(s) = \frac{3}{2}$$

Asymmetric distinguishability by **quantum Sanov** theorem:
Performing n measurements on ρ one obtains results compatible with σ with probability $P \sim \exp(-3n/2)$.

c) **Chernoff information** $Q(\rho,\sigma) \;:=\; \min_{s\in[0,1]} \mathrm{Tr}\rho^s\sigma^{1-s}$.

Chernoff bound for random states:

$$Q(\rho,\sigma) \to Q_* = \langle \mathrm{Tr}\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\rangle \to \left(\int dt\sqrt{t}MP(t)\right)^2 = \left(\frac{8}{3\pi}\right)^2 \approx 0.72.$$

Symmetric distinguishability by **quantum Chernoff** bound:
Performing n measurements on ρ and σ one cannot distinguish them with probability $P \sim \exp(-Q_* n)$.

a) Suppose you need a quantum state ρ,

you go to a quantum shop, pay for it and

you get a state σ instead so that their **fidelity** $F(\rho, \sigma)$ equals (say) 0.8636.

Are these states **close** enough? We know that $0 \leq F \leq 1$.

Assume you do numerical computations (or perform measurements) and get result that the **fidelity** between the **desired state** ρ and the actual state σ is equal to $F_1 = 0.8636$.

Is the fidelity F_1 a 'big number' (**hifi** = high fidelity) or a small one, (**low fidelity**)?

Products of random matrices and average fidelity

Related **asymptotic** results ($N \gg 1$) for the average:

a) **root fidelity** - a benchmark for experimental and theoretical studies involving fidelity:

$\sqrt{F(\rho,\sigma)} = \mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}| \to \sum_i \sqrt{\lambda_i(\rho\sigma)} \to \int dx \sqrt{x} \mathcal{FC}(x) = \frac{3}{4}$,

where

$$\mathcal{FC}(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi} \frac{\left[\sqrt[3]{2}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}\right]}{x^{\frac{2}{3}}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{1}{3}}},$$

denotes **Fuss–Catalan** distribution, $\mathcal{FC} = MP \boxtimes MP$,

which describes level density of a **product** $\rho\sigma$ of two random states.

Related quantities:

b) **Bures distance**

$$D_B(\rho,\sigma) = \sqrt{2(1 - \sqrt{F(\rho,\sigma)})} \to \frac{\sqrt{2}}{2}.$$

c) **Quantum Hellinger distance**

$$D_H(\rho,\sigma) = \sqrt{2 - 2\mathrm{Tr}\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}} \to \sqrt{2 - \frac{128}{9\pi^2}} \approx 0.746$$

Asymptotic average entanglement

Consder a random bipartite **pure state** $|\psi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$, so that level density of the reduced state $\rho = \mathrm{Tr}_N |\psi\rangle\langle\psi|$ is given by MP_1 distribution

Then the density of partially transposed matrix, ρ^{T_A}, converges to the shifted semicircle (**Aubrun 2012**),

$$\lambda(\rho^{\mathrm{T}_A}) \sim \frac{1}{2\pi}\sqrt{4-(x-1)^2}, \quad \text{for } x \in [-1,3]$$

This implies that

a) the **fraction** of negative eigenvalues converges to

$$\int_{-1}^{0} \frac{1}{2\pi}\sqrt{4-(x-1)^2}dx = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \simeq 0.1955$$

b) the **average negativity** tends to

$$\langle \mathcal{N} \rangle_\psi \to \int_{-1}^{3} \frac{|x|-x}{2} \frac{1}{2\pi}\sqrt{4-(x-1)^2}dx \simeq 0.080.$$

Let $G(|\psi\rangle) = N(\det\rho)^{1/N}$ be the G**–concurrence** of a state (**Gour 2005**).

Then the average G**–concurrence** of a random state $|\psi\rangle$ converges to

$$\langle G \rangle_\psi \to_{N\to\infty} \exp\left(\int_0^4 \log t \; MP(t)dt\right) = exp(-1) = 1/e \approx 0.368$$


beeline

Concluding Remarks

- We derived symmetric MP distribution for level density of Helstrom matrix $\rho - \sigma$ for two random states and found their average trace distance $\tilde{D} = \frac{1}{4} + \frac{1}{\pi}$, valid almost surely for any states due to **concentration of measure** effect.
- $\Longrightarrow$ universal **Helstrom distinguishability** bound,
$$P_d \leq 1/2 + \tilde{D}/2 \approx 0.784$$
- Average **fidelity** obtained for $N \to \infty$ reads $\langle F(\rho, \sigma) \rangle = F_* = 9/16$. It describes well results for a dimension N of order ten and provides a **universal benchmark** - a reference value for this quantity
(even if the dimension N is unknown!)
- It the state σ offered by the quantum shop has **fidelity** with respect to the **ordered state** ρ only *slightly larger* than $F_* = 9/16$
better go to another shop !

Bench commemorating discussion between
Stefan Banach and **Otton Nikodym** (**Kraków, summer 1916**)

Opening : Planty Garden, Cracow, Friday, Oct. 14, 2016 at 12.00