Local eigenvalue statistics for random regular graphs

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Bielefeld, 11.12.2015

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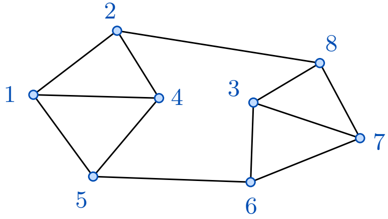
Random regular graphs

Consider (undirected) graphs on $\{1, \dots, N\}$.

d-regular graph: each vertex has d neighbours.

Random d-regular graph (RRG): uniform probability measure on the set of simple d-regular graphs.



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Adjacency matrix $A_{ij} := \mathbf{1}(i \sim j)$ and Laplacian $L := dI - A$.

Trivial eigenvalues d and 0 with eigenvector $\mathbf{e} = N^{-1/2}(1, 1, \dots, 1)^*$.

Key questions

(1) Expansion and spectral gap. Ramanujan graphs are optimal expanders, with a spectral gap at least $d - 2\sqrt{d-1}$. How many regular graphs are Ramanujan, or almost Ramanujan?

(Alon-Boppana, Broder-Shamir, Kahn-Szemerédi, Friedman, Puder, Bordenave)

(2) Quantum chaos. Random regular graphs are a good paradigm of quantum chaos (Kottos-Smilansky, Sarnak, Anantharaman). Two hallmarks of quantum chaos (Bohigas-Giannoni-Schmit, Rudnick-Sarnak):

- Universal random matrix statistics (GOE) of eigenvalues.
- Quantum unique ergodicity ($\approx$ asymptotic flatness) of eigenvectors.

This talk: question (2).

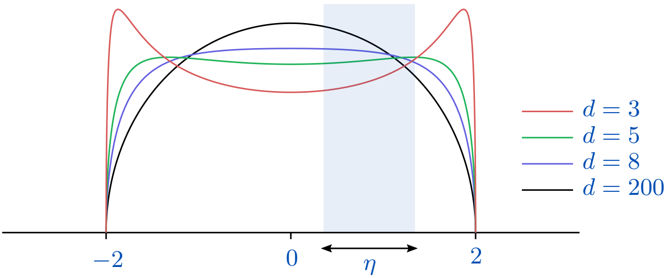
The key tool to address question (2) is a local law for A.

Part 1: local law

Kesten-McKay law

Empirical eigenvalue distribution of $(d-1)^{-1/2}A$ converges as $N \to \infty$ to law with density

$$\left(1 + \frac{1}{d-1} - \frac{x^2}{d}\right)^{-1} \frac{\sqrt{[4-x^2]_+}}{2\pi} \,.$$



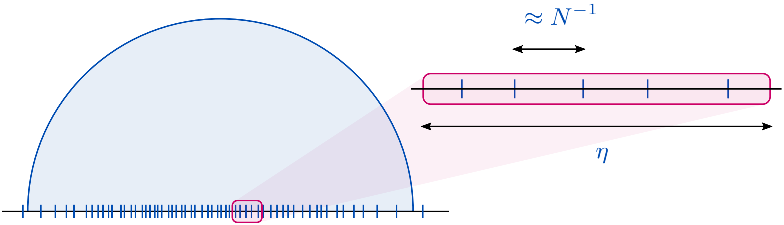
Convergence on macroscopic scale $\eta \sim 1$ (global law).

- Kesten [1957]: d-regular tree
- McKay [1981]: d-regular graph (locally a d-regular tree)

Local laws ($\eta \ll 1$):

	d	scale η
Dumitriu-Pal [2009]	$(\log N)^C$	$(\log N)^{-1}$
Tran-Vu-Wang [2010]	$\gg 1$	$d^{-1/10}$
Anatharaman-Le Masson [2013]	fixed	$(\log N)^{-c}$
Geisinger [2014]	fixed	$(\log N)^{-1}$

Control eigenvalue distribution on small scales, $N^{-1} \ll \eta \ll 1$.



Consider centred matrix

$$H := \frac{1}{\sqrt{d-1}}(A - d\mathbf{e}\mathbf{e}^*)\,.$$

The nontrivial eigenvalues $\lambda_1, \ldots, \lambda_{N-1}$ of H and $(d-1)^{-1/2}A$ coincide (restriction to space $\mathbf{e}^\perp$).

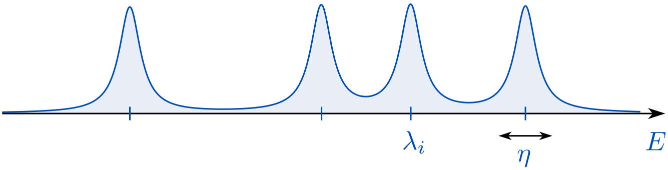
Green function

Local law is best stated using the Green function

$$G(z) \; := \; (H - zI)^{-1}\,, \qquad z \in \mathbb{C}_+\,.$$

Writing $z = E + \mathrm{i}\eta$, we have

$$\operatorname{Im} \frac{1}{N} \operatorname{Tr} G(z) \;=\; \frac{1}{N} \sum_{i=1}^{N} \frac{\eta}{(\lambda_i - E)^2 + \eta^2}$$



Observation: $\eta = \operatorname{Im} z$ is the spectral resolution.

- Global law: control of $G(z)$ for $\eta \approx 1$.
- Local law: control of $G(z)$ for $\eta \gg N^{-1}$.

Local semicircle law

For $d \gg 1$, expect that $\frac{1}{N}\operatorname{Tr} G(z)$ is close to the Stieltjes transform of the semicircle law,

$$m(z) := \int_{-2}^{2} \frac{\sqrt{4-x^2}}{2\pi} \frac{\mathrm{d}x}{x-z}.$$

A local semicircle law entails control of

$$\frac{1}{N}\operatorname{Tr} G(z) - m(z) \qquad \text{or, much better, of} \qquad G_{ij}(z) - \delta_{ij} m(z)$$

for $\eta \gg N^{-1}$.

Local semicircle laws are known for many matrix models with independent entries.

- Erdős-Schlein-Yau [2008]: Wigner matrices
- Erdős-Yau-Yin [2010]: Generalized Wigner matrices, optimal error bounds
- Erdős-K-Yau-Yin [2012]: Sparse matrices, Erdős-Rényi graph

Methods extend to other matrix models with independent entries.
Independence is essential.

Theorem (Bauerschmid-K-Yau [2015]). Define

$$D \;:=\; d \wedge \frac{N^2}{d^3}\,.$$

Then for all

$$D \;\geqslant\; (\log N)^4 \qquad \text{and} \qquad \eta \;\geqslant\; \frac{(\log N)^4}{N}$$

we have

$$G_{ij}(z) - \delta_{ij} m(z) \;=\; O\left((\log N)^2 \sqrt{\frac{1}{N\eta} + \frac{1}{D}}\right)$$

with very high probability. (Recall $\eta := \operatorname{Im} z$.)

Part 2: application to local eigenvalue and eigenvector distribution

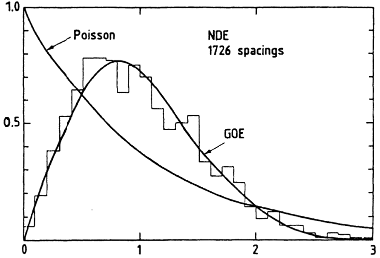
Let Δ be the Laplacian of a bounded domain or a compact Riemannian manifold M, with eigenvalues and eigenvectors $\Delta v_i = \lambda_i v_i$ ($\lambda_1 \leqslant \lambda_2 \leqslant \cdots$).

Let $\mu_N := \frac{1}{N}\sum_{i=1}^{N} \delta_{i^{1-2/d}(\lambda_{i+1}-\lambda_i)}$ denote the (rescaled) empirical eigenvalue spacings between the N first eigenvalues.

Conjecture 1 (Bohigas-Giannoni-Schmit [1984], Berry-Tabor [1977]).

If the geodesic flow on M is ergodic, then μ_N converges to the GOE distribution of random matrix theory.

If the geodesic flow on M is integrable, then μ_N converges to the exponential distribution.



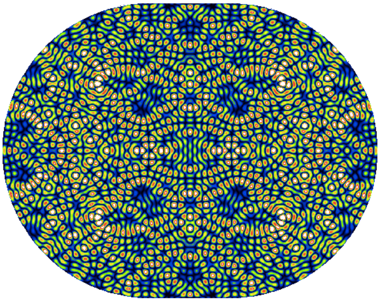
Conjecture 2 (Rudnick-Sarnak [1994]). If the geodesic flow on M is ergodic, then $|v_i(x)|^2$ becomes equidistributed as $i \to \infty$: for any fixed open $A \subset M$ we have

$$\lim_{i \to \infty} \int_A |v_i(x)|^2 \, \mathrm{d}x \;=\; \int_A \mathrm{d}x \,.$$

Quantum unique ergodicity (QUE).

Averaged version of QUE: Shnirelman [1974], Colin de Verdière [1985], Zelditch [1987].

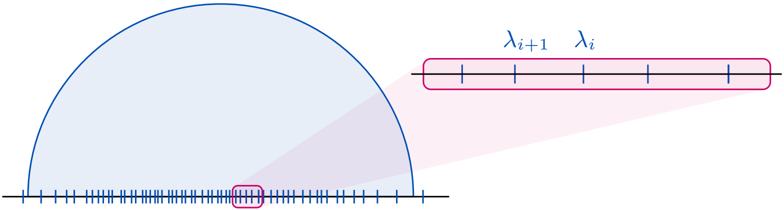
Special cases of QUE: Lindenstrauss [2006], Anantharaman [2008], Holowinski-Soundararajan [2010].



Conjectures 1 and 2 are a manifestation of the general universality conjecture for disordered quantum systems: RMT eigenvalue statistics $\iff$ QUE.

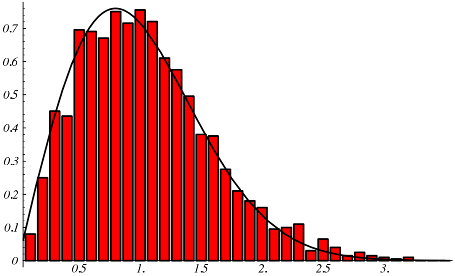
On conjecture 1 for random regular graphs

Asymptotic distribution of individual eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{N-1}$ of $L|_{\mathbf{e}^\perp}$.



Numerical evidence for GOE level spacing distribution: $d = 3$, $N = 1000$.

Jakobson-Miller-Rivin-Rudnick [1999]
Oren-Smilansky [2010]



So far, random matrix statistics have been proved for two classes of models:

- Invariant ensembles. (Pastur-Shcherbina, Bleher-Its, Deift-Kriecherbauer-McLaughlin-Venakides-Zhou, Bourgade-Erdős-Yau, Shcherbina, Bekerman-Figalli-Guionnet)
- Ensembles with independent entries. (Johansson, Erdős-K-Schlein-Yau-Yin, Tao-Vu)

RRG is in neither class.

Theorem (Bauerschmidt-Huang-K-Yau [2015]). For $d \in [N^{\varepsilon}, N^{2/3-\varepsilon}]$ with $\varepsilon > 0$ fixed, the local eigenvalue statistics of the uniform random d-regular graph in the bulk coincide with those of the GOE.

On conjecture 2 for random regular graphs

QUE holds with very high probability on scales $\gg N^{-1}$.

Theorem (Bauerschmidt-K-Yau [2015]). Suppose that $d \wedge \frac{N^2}{d^3} \geqslant (\log N)^4$. If $\sum_{k=1}^N a(k) = 0$ then for any normalized eigenvector $(v(k))_{k=1}^N$ we have

$$\sum_k a(k) v(k)^2 \;=\; O\biggl(\frac{(\log N)^4}{N} \biggl(\sum_k a(k)^2\biggr)^{1/2}\biggr)$$

with very high probability.

In particular,

$$\sum_{k \in V} v(k)^2 \;=\; \sum_{k \in V} \frac{1}{N} + O\biggl(\frac{(\log N)^4 \sqrt{|V|}}{N}\biggr)$$

with very high probability: the measure $k \mapsto v(k)^2$ is close to the uniform measure $k \mapsto N^{-1}$ when integrated over sets of (uniform) measure $\geqslant (\log N)^8/N$.

Theorem (Anantharaman-Le Masson [2013]). For fixed d, an averaged version of QUE holds with high probability on sets of measure $\asymp 1$.

Part 3: some ideas of proof of local law

Previous approach for independent entries

Write

$$H = \left[\begin{array}{c|ccc} H_{11} & H_{12} & \cdots & H_{1N} \\ \hline H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{array}\right]$$

Condition on bottom-right block. The first row/column is independent.

Use Schur complement formula and concentration for first row/column

$$G_{ii} = \frac{1}{H_{ii} - z - \sum_{k,l \neq i} H_{ik} G_{kl}^{(i)} H_{li}} \approx \frac{1}{-z - \frac{1}{N}\sum_k G_{kk}},$$

Averaging over i yields a self-consistent equation for $s := \frac{1}{N}\sum_i G_{ii}$:

$$s^2 + zs + 1 = o(1).$$

For the random regular graph, the red entries are deterministic given the green entries. A new approach is required.

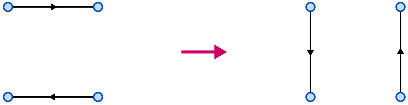
Switchings

Need a resampling of neighbourhood $\partial 1$ that

- leaves the RRG invariant,
- mixes $\partial 1$ sufficiently well,
- has good concentration properties.

Solution: carefully chosen simultaneous double switchings.

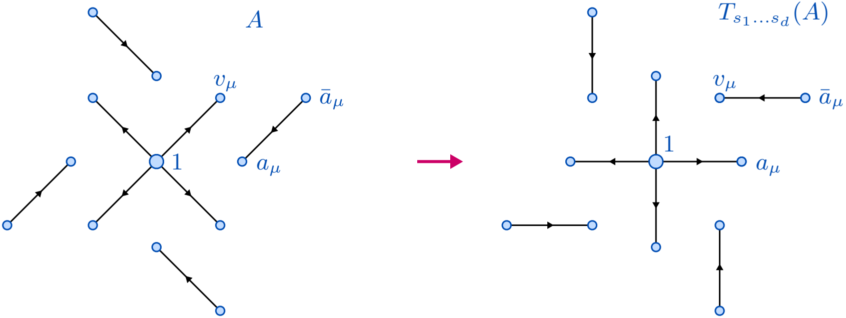
For simplicity, let us just look at simple switchings.



- Order neighbours of 1: $\partial 1 = \{v_1, \ldots, v_d\}$.
- Choose oriented edges $s_1, \ldots, s_d$ at random, where $s_\mu = (\bar{a}_\mu, a_\mu)$.
- Switch the oriented edges $(1, v_\mu)$ and s_μ for all $\mu = 1, \ldots, d$:

$$A \;\mapsto\; T_{s_1 \ldots s_d}(A) \;\overset{\mathrm{d}}{=}\; A\,.$$

- The neighbourhood $\partial 1$ in $T_{s_1 \ldots s_d}(A)$ is given with high probability by $\{a_1, \ldots, a_d\}$.



Work on augmented probability space consisting of $(A, s_1, \ldots, s_d)$, and set H to be $(d-1)^{-1/2}\big(T_{s_1\ldots s_d}(A) - d\mathbf{e}\mathbf{e}^*\big)$ in Green function $G = (H-z)^{-1}$.

Start from trivial identity $I + zG = HG$:

$$\begin{aligned}
1 + z\mathbb{E}_{s_1\ldots s_d} G_{11} &= \mathbb{E}_{s_1\ldots s_d} \sum_j H_{1j} G_{j1} \\
&= \mathbb{E}_{s_1\ldots s_d} \frac{1}{N} \sum_j \frac{1}{\sqrt{d-1}} \sum_{\mu=1}^{d} (G_{a_\mu 1} - G_{j1}) \\
&= \mathbb{E}_{s_1\ldots s_d} \mathbb{E}_{\tilde{s}_1\ldots \tilde{s}_d} \frac{1}{\sqrt{d-1}} \sum_{\mu=1}^{d} (G_{a_\mu 1} - G_{\tilde{a}_\mu 1}) \\
&= \mathbb{E}_{s_1\ldots s_d} \mathbb{E}_{\tilde{s}_1\ldots \tilde{s}_d} \frac{1}{\sqrt{d-1}} \sum_{\mu=1}^{d} (\tilde{G}^{\mu}_{\tilde{a}_\mu 1} - G_{\tilde{a}_\mu 1}) \\
&= -\mathbb{E}_{s_1\ldots s_d} \mathbb{E}_{\tilde{s}_1\ldots \tilde{s}_d} G_{\tilde{a}_\mu \tilde{a}_\mu} G_{11} + o(1)\,,
\end{aligned}$$

where $\tilde{G}^{\mu}$ is obtained from G by replacing a_μ with $\tilde{a}_\mu$ in H.

Use concentration for $\mathbb{E}_{s_1\ldots s_d}$, and average, to get $1 + zs + s^2 = o(1)$.