Eigenvalue statistics for rank one perturbations of unitary and Hermitian beta-ensembles

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Joint work with Rowan Killip (UCLA)

1. Motivation: open quantum systems
2. Preliminaries
   - Hermitian random matrix ensembles
   - Jacobi matrices and orthogonal polynomials on the real line
   - Dumitriu–Edelman β-ensembles
3. Distribution of resonances
   - as eigenvalues of non-Hermitian perturbations
   - as poles of the resolvent of a Jacobi operator on $\ell^2(\mathbb{Z}_+)$
4. Unitary analogue
   - Unitary random matrix ensembles
   - CMV matrices and orthogonal polynomials on the unit circle
   - Killip–Nenciu β-ensembles
   - Truncations and their eigenvalues
5. Large n limit
   - Asymptotic radial distribution of resonances
   - Universality
   - Idea of the proof

- In quantum chaotic scattering: for a *closed* quantum system:

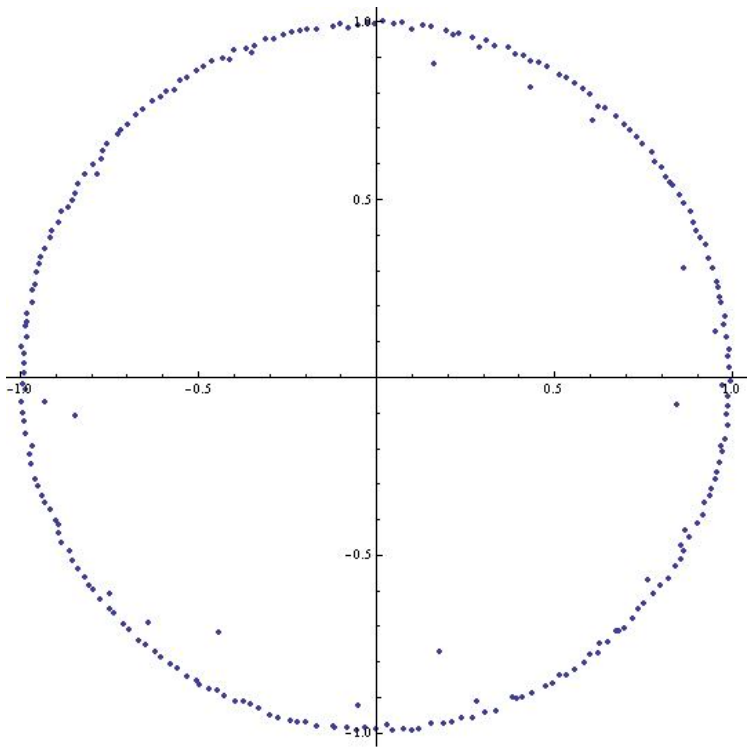
  Hamiltonian = **Hermitian** random matrix (from GOE, GUE, GSE)
  Evolution operator = **unitary** random matrix (from COE, CUE, CSE)
- To model *open* quantum system (interaction with outer world), physicists consider **non-Hermitian** and **non-unitary** perturbations of these ensembles.
- Resonances of an *open* quantum system – long-lived semi-stable states to which discrete energy levels of the closed quantum systems are transformed when system is "opened".

  Mathematically – **eigenvalues** of the perturbed random matrix ensembles.

- Non-unitary perturbations of unitary random matrices.

  Let U be a unitary random matrix (e.g., from one of the circular ensembles or classical groups).
  Delete k rows and corresponding columns.

  Eigenvalues of the truncation:



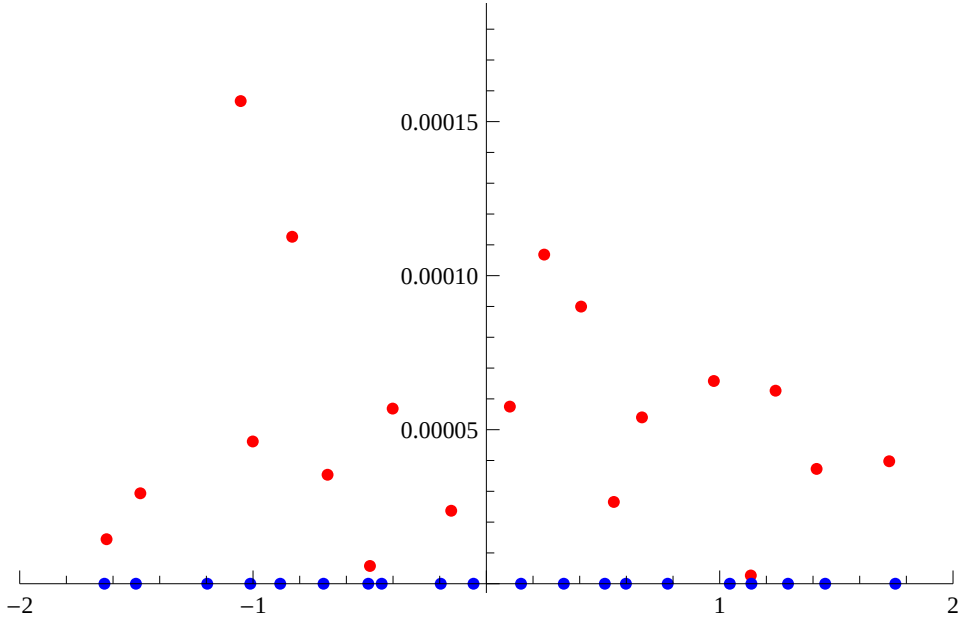
- Non-Hermitian perturbations of Hermitian random matrices

Let H be a Hermitian random matrix (e.g., from one of the Gaussian or Wishart ensembles). Let

$$H_{eff} = H + i\Gamma,$$

where $\Gamma = \Gamma^* > 0$ of rank k.

Eigenvalues:

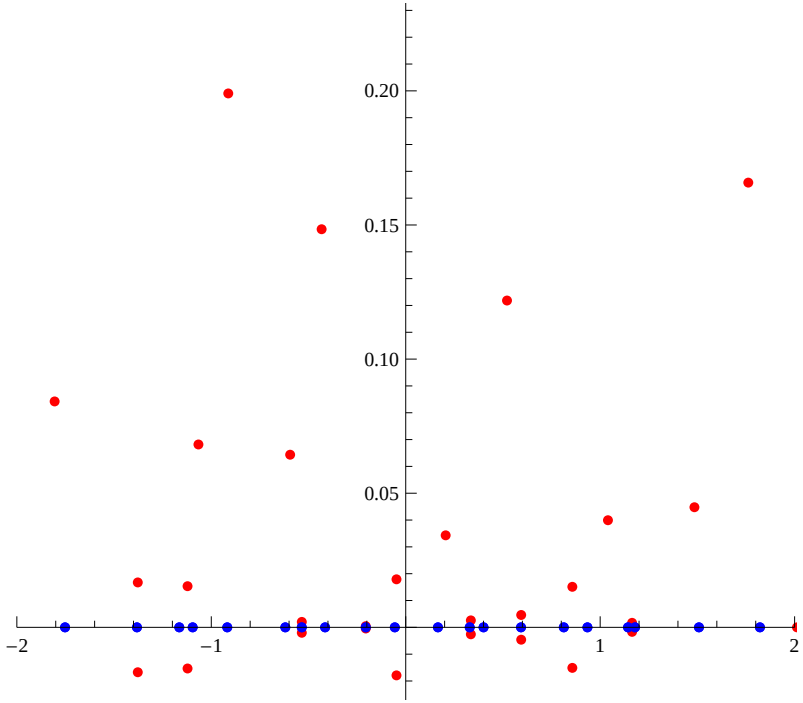


- Coupling of Hermitian random matrix to a discrete Schrödinger operator:

Let $H = (h_{ij})_{i,j=1}^{n}$ be a Hermitian random matrix.
Let

$$\widetilde{H} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} & 0 & 0 & \\ h_{21} & h_{22} & \dots & h_{2n} & 0 & 0 & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ h_{n1} & h_{n2} & \dots & h_{nn} & 1 & 0 & \\ 0 & 0 & \dots & 1 & 0 & 1 & \ddots \\ 0 & 0 & \dots & 0 & 1 & 0 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

Resonances of this operator = poles of the resolvent $(\widetilde{H} - z)^{-1}$ on the second sheet of $\mathbb{C} \setminus [-2, 2]$


0.20
0.15
0.10
0.05
−2
−1
1
2

- Non-Hermitian perturbations of Chiral RMT ensembles:

Let $H = \begin{bmatrix} \mathbf{0}_{m\times m} & X \\ X^* & \mathbf{0}_{n\times n} \end{bmatrix}$, where X is a random Ginibre matrix.

Let

$$\tilde{H} = \begin{bmatrix} i\Gamma_{m\times m} & X \\ X^* & \mathbf{0}_{n\times n} \end{bmatrix},$$

where $\Gamma_{m\times m} = \Gamma^*_{m\times m} \geq \mathbf{0}$ of rank k.

Physical applications — ? (quantum chromodynamics?)

- Multiplicative perturbations of Hermitian RMT ensembles:

  Let H be a Hermitian random matrix (e.g., from one of the Gaussian or Wishart ensembles).

  Let
  $$\tilde{H} = H(I + i\Gamma),$$
  where $\Gamma_{m\times m} = \Gamma^*_{m\times m} \geq \mathbf{0}$ of rank k.

  Physical applications — ?

Classical ensembles of random unitary matrices:

- Group $\mathbb{U}(n)$ of $n \times n$ unitary matrices ($U^* U = \mathbf{1}$).
- Group $\mathbb{O}(n)$ of $n \times n$ orthogonal matrices ($O^T O = \mathbf{1}$).
- Group $\mathbb{USp}(n)$ of $n \times n$ quaternionic unitary matrices (unitary symplectic group) ($Q^* Q = \mathbf{1}$).

- Circular ensembles (Dyson, 1962, as a modification of Wigner Gaussian ensembles):
  - ($\beta = 1$) $COE(n) = \left\{ U^T U, \text{ where } U \in \mathbb{U}(n) \right\}$, symmetric unitary matrices;
  - ($\beta = 2$) $CUE(n) = \mathbb{U}(n)$;
  - ($\beta = 4$) $CSE(n) = \left\{ Q^R Q, \text{ where } Q \in \mathbb{U}(2n) \right\}$, self-dual unitary quaternionic matrices ($U^R = Z^T U^T Z$, where $Z = \operatorname{diag}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$).

Circular Dyson's ensembles:

- $(\beta = 1)$ $COE(n) = \left\{U^T U, \text{ where } U \in \mathbb{U}(n)\right\}$
- $(\beta = 2)$ $CUE(n) = \mathbb{U}(n)$;
- $(\beta = 4)$ $CSE(n) = \left\{Q^R Q, \text{ where } Q \in \mathbb{U}(2n)\right\}$

We think of any element of such an ensemble as a matrix of transition probabilities between the various states of a quantum system

$COE(n)$: systems with time-reversal and rotational symmetry
$CUE(n)$: systems without time-reversal symmetry
$CSE(n)$: systems with time-reversal invariance, but without rotational symmetry

Eigenvalues are distributed according to

$$\prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi},$$

Let U be $n \times n$ unitary, and suppose vector $e_1 = [1, 0, \ldots, 0]^T$ is cyclic for U.

Applying the Gram–Schmidt procedure to $e_1, Ue_1, U^{-1}e_1, U^2e_1, U^{-2}e_1, \ldots$ produces a basis of $\mathbb{C}^n$ in which U has CMV form $\mathcal{C}$.

Definition

$$\mathcal{C}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) = \mathcal{L}\mathcal{M},$$

where $\alpha_j \in \bar{\mathbb{D}} := \{z : |z| \leq 1\}$, and

$$\mathcal{L} = \operatorname{diag}(\Xi_0, \Xi_2, \Xi_4, \ldots) \qquad \mathcal{M} = \operatorname{diag}([1], \Xi_1, \Xi_3, \ldots),$$

where

$$\Xi_k = \begin{bmatrix} \bar{\alpha}_k & \sqrt{1 - |\alpha_k|^2} \\ \sqrt{1 - |\alpha_k|^2} & -\alpha_k \end{bmatrix}$$

In other words, $U = V^* \mathcal{C} V$ with some $V^*V = \mathbf{1}$ satisfying $Ve_1 = V^*e_1 = e_1$.

Named after Cantero–Moral–Velázquez ’2001 (also Bunse-Gerstner–Elsner ’1991).

After multiplication, five-diagonal structure:

$$\mathcal{C} = \begin{bmatrix} \bar{\alpha}_0 & \rho_0\bar{\alpha}_1 & \rho_0\rho_1 & 0 & 0 & 0 & 0 & 0 \\ \rho_0 & -\alpha_0\bar{\alpha}_1 & -\alpha_0\rho_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\alpha}_2\rho_1 & -\alpha_1\bar{\alpha}_2 & \bar{\alpha}_3\rho_2 & \rho_2\rho_3 & 0 & 0 & 0 \\ 0 & \rho_1\rho_2 & -\bar{\alpha}_1\rho_2 & -\alpha_2\bar{\alpha}_3 & -\alpha_2\rho_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\alpha_3\bar{\alpha}_4 & \bar{\alpha}_5\rho_4 & \rho_4\rho_5 & 0 \\ 0 & 0 & 0 & \rho_3\rho_4 & -\alpha_3\rho_4 & -\alpha_4\bar{\alpha}_5 & -\alpha_4\rho_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\alpha}_6\rho_5 & -\alpha_5\bar{\alpha}_6 & \bar{\alpha}_7\rho_6 \\ 0 & 0 & 0 & 0 & 0 & \rho_5\rho_6 & -\alpha_5\rho_6 & -\alpha_6\bar{\alpha}_7 \end{bmatrix},$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$

Characteristic polynomials of the top-left corners

$$\Phi_n(\lambda) = \det\left[\lambda - \mathcal{C}_{n\times n}\right]$$

are orthogonal polynomials with respect to the spectral measure of U:

$$d\mu = \sum_{j=1}^{n} w_j \delta_{\lambda_j}$$

These polynomials satisfy Szegő's recurrence:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\, z^n \overline{\Phi_n(1/\bar{z})}$$

We call $\alpha_n \in \mathbb{D}$ the Verblunsky coefficients.

Motivation for reducing U to the CMV form:

- Sparse matrices – good for simulations;
- Spectral measures (eigenvalues and eigenweights) of U and $\mathcal{C}$ are the same;
- Eigenvalues of U = zeros of orthogonal polynomials Φ_n.
- Non-unitary matrices also have CMV-type representations.

Killip–Nenciu '2004 and Killip-K '2015: CMV-fying Dyson's ensembles or the Classical Compact Groups produces statistically independent Verblunsky coefficients α_k

Matrix model for Dyson's ensembles: CMV$(\alpha_0, \dots, \alpha_{n-2}, e^{i\phi})$ with distribution

$$\prod_{j=0}^{n-2} (1 - |\alpha_j|^2)^{\beta(n-1-j)/2-1} d^2\alpha_0 \dots d^2\alpha_{n-2} d\phi \quad (\beta = 1, 2, 4)$$

Remarks:

- α_j's are independent and rotationally invariant;
- The ensemble makes sense for any $0 < \beta < \infty$ ("circular β-ensemble") with the joint eigenvalue distribution

$$\prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^\beta \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi},$$

Orthogonal group $\mathbb{O}(n)$ with Haar measure.

$\mathbb{O}(2k) \setminus \mathbb{SO}(2k)$: $n \times n$ orthogonal matrices of determinant -1.

Matrix model $\mathbb{O}(n)$: $\mathrm{CMV}(\alpha_0, \dots, \alpha_{n-2}, \pm 1)$ with distribution

$$\prod_{j=0}^{n-2} (1 - \alpha_j^2)^{(n-j-3)/2} d\alpha_0, \dots d\alpha_{n-2}$$

Remarks:

- α_j's are real with even distributions;
- α_j's are independent;
- β-dependent family of CMV ensembles exists ("real orthogonal β-ensemble").

Matrix model for $\mathbb{USp}(n)$: $\mathrm{CMV}(\alpha_0, \ldots, \alpha_{n-2}, q)$ with distribution

$$\prod_{j=0}^{n-2} (1 - |\alpha_j|^2)^{2n-2j-3} d^4\alpha_0, \ldots d^4\alpha_{n-2} d^3 q$$

Notes:

- α_j and q are independent quaternions.
- $d^3 q$ is the uniform measure on unit quaternionic sphere $|q| = 1$.

For H from any of the unitary RMT ensembles: delete one row and corresponding column (rank one perturbation; "one open channel"). Distributions of eigenvalues?

Motivation:

- From the point of view of physics:
  - Scattering resonances of open quantum systems with one open channel;
  - Log-gas interpretation: electrostatic charges in a dielectric medium;
- From the point of view of pure mathematics:
  - Explicitly computable eigenvalue point process for a non-normal ensemble of random matrices;
  - Eigenvalues converge to the zero point process of random Gaussian analytic functions.

From the physics literature the following was known:

Życzkowski–Sommers '2000: eigenvalues of truncations of $CUE(n+1)$ are distributed in $\mathbb{D}^n$ as

$$\frac{1}{\pi^n}\prod_{j<k}|z_k - z_j|^2\, d^2z_1 \dots d^2z_n.$$

Theorem (Killip–K– '2015)

Eigenvalues of truncations of $COE(n+1)$, $CUE(n+1)$, $CSE(n+1)$ are distributed in $\mathbb{D}^n$ as

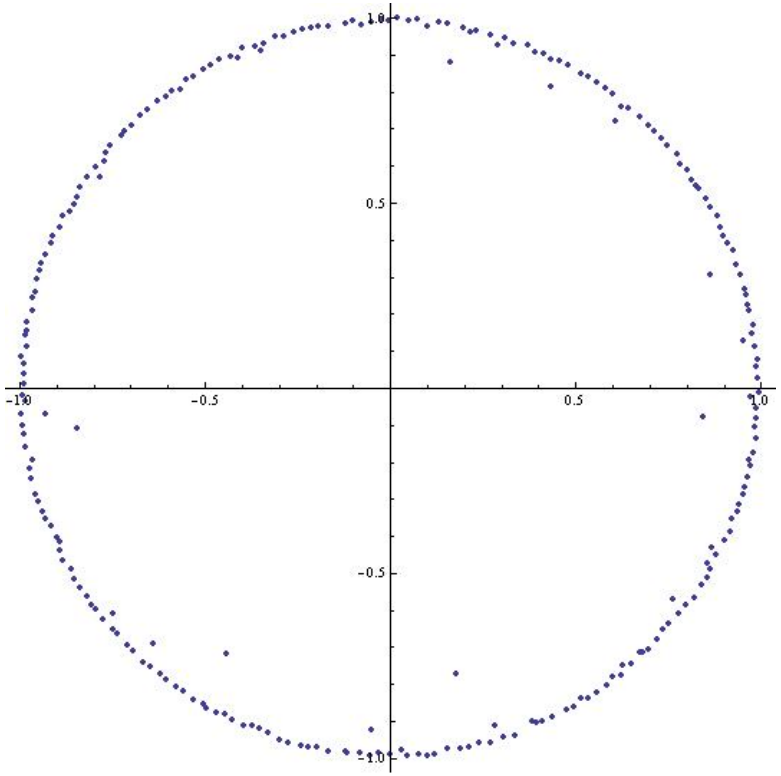
$$\frac{\beta^n}{(2\pi)^n}\prod_{j,k=1}^{n}(1 - z_j\bar{z}_k)^{\frac{\beta}{2}-1}\prod_{j<k}|z_k - z_j|^2\, d^2z_1 \dots d^2z_n.$$

with $\beta = 1, 2,$ and 4, respectively.

Dyson's truncations. Picture

$$\frac{\beta^n}{(2\pi)^n}\prod_{j=1}^{n}\left(|z_j|^{n-1}(1-|z_j|^2)\right)^{\frac{\beta}{2}-1}\prod_{j<k}|z_k-\bar{z}_j^{-1}|^{\beta-2}|z_k-z_j|^2\,d^2z_1\ldots d^2z_n.$$

Typical picture ($n=250$, $\beta=8$):



Khoruzhenko–Sommers–Życzkowski '2010: eigenvalues of truncations of $\mathbb{O}(2n)$, are distributed as

$$\prod_{j=1}^{2n-1} \frac{1}{\sqrt{1-z_j^2}} \prod_{j<k} |z_k - z_j| \, |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_{2n-1}|.$$

Theorem (Killip–K– '2015)

Eigenvalues of truncations of $\mathbb{O}(n)$ ensemble are distributed as

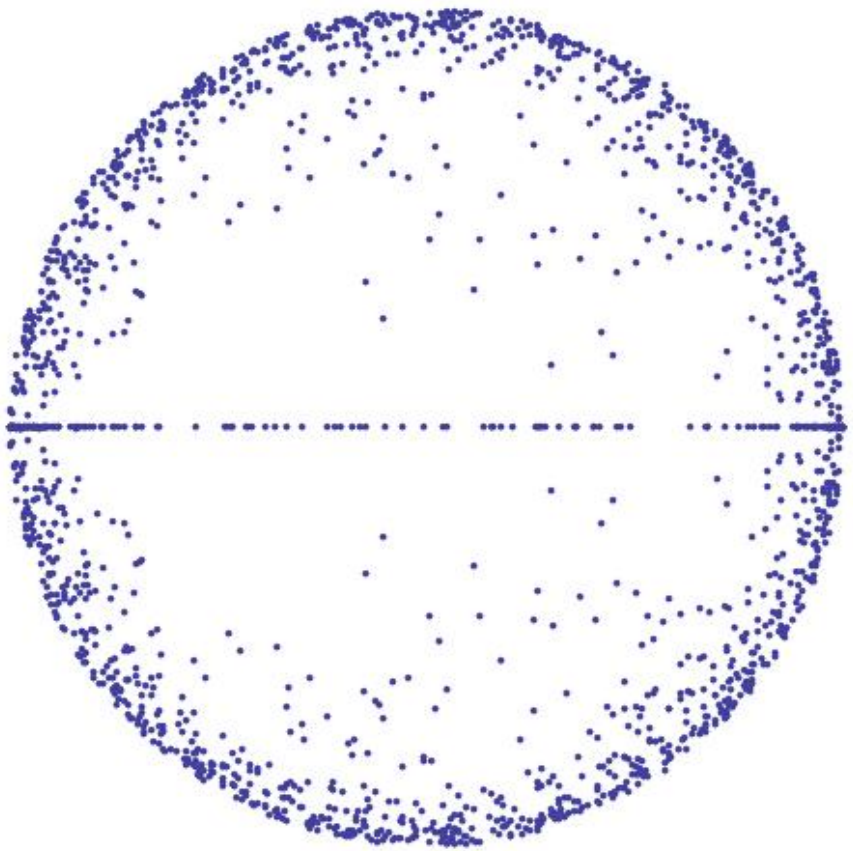
$$\prod_{j,k=1}^{n-1} (1 - z_j \bar{z}_k)^{\frac{\beta}{4} - \frac{1}{2}} \prod_{j=1}^{n-1} (1 - z_j^2)^{-\frac{\beta}{4}} \prod_{j<k} |z_k - z_j| \, |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_{n-1}|,$$

with $\beta = 2$.

Truncations of $\mathbb{O}(n)$. Picture

$$\prod_{j,k=1}^{n-1}(1-z_j\bar{z}_k)^{\frac{\beta}{4}-\frac{1}{2}}\prod_{j=1}^{n-1}(1-z_j^2)^{-\frac{\beta}{4}}\prod_{j<k}|z_k-z_j|\,|dz_1\wedge dz_2\wedge\ldots\wedge dz_{n-1}|,$$

Typical picture ($n=20$, 100 iterations):



$$\prod_{j=1}^{n} \frac{1}{\sqrt{1-z_j^2}} \prod_{j<k} |z_k - z_j| \, |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n|.$$

E.g., suppose $n = 3$: then either $z_1 = x_1, z_2 = x_2, z_3 = x_3$ are all real, in which case the density is

$$\frac{1}{\sqrt{1-x_1^2}} \frac{1}{\sqrt{1-x_2^2}} \frac{1}{\sqrt{1-x_3^2}} \prod_{j<k} (x_k - x_j) \, dx_1 dx_2 dx_3,$$

or $z_1 = x_1 + iy_1, z_2 = x_1 - iy_2, z_3 = x_3$, in which case the density is

$$\frac{1}{|1-(x_1+iy_1)^2|} \frac{1}{\sqrt{1-x_3^2}} |x_1 + iy_1 - x_3|^2 (2y_1) \, (2dx_1 dy_1) dx_3$$

Lemma

Consider a unitary CMV matrix $CMV(\alpha_0, \ldots, \alpha_n)$. Removing one row and one column produces:

- $W^* CMV\left(-\bar{\alpha}_{n-1}\alpha_n, -\bar{\alpha}_{n-2}\alpha_n, \ldots, -\bar{\alpha}_0\alpha_n\right)^T W$, *if n is even,*
- $W^* CMV\left(-\bar{\alpha}_{n-1}\alpha_n, -\bar{\alpha}_{n-2}\alpha_n, \ldots, -\bar{\alpha}_0\alpha_n\right) W$, *if n is odd,*

for some unitary W.

So the following coincide:

- Eigenvalues of truncations of Dyson's ensembles
- Eigenvalues of $\mathrm{CMV}(\alpha_0, \ldots, \alpha_{n-1})$ with distributions

$$\prod_{j=0}^{n-1} (1 - |\alpha_k|^2)^{\frac{\beta(k+1)-2}{2}} d^2\alpha_0 \ldots d^2\alpha_{n-1}$$

- Zeros of random orthogonal polynomials

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n z^n \overline{\Phi_n(1/\bar{z})}$$

with α_k's as above.

Step 1: Let $\Phi_n(z) = z^n + \kappa_1 z^{n-1} + \ldots + \kappa_{n-1} z + \kappa_n$. Compute the Jacobian $\det \frac{\partial(\alpha_0,\ldots,\alpha_{n-1})}{\partial(\kappa_1,\ldots,\kappa_n)}$ is equal to $\prod_{k=0}^{n-1}(1-|\alpha_k|^2)^{-k}$
Proof: induction and recurrence relations.

Step 2: The Jacobian $\det \frac{\partial(\kappa_1,\ldots,\kappa_n)}{\partial(z_1,\ldots,z_n)}$ is equal to $\prod_{j<k}|z_k - z_j|^2$

Thus:

$$\prod_{k=0}^{n-1}(1-|\alpha_k|^2)^{\frac{\beta(k+1)-2}{2}} d\vec{\alpha} = \prod_{k=0}^{n-1}(1-|\alpha_k|^2)^{\frac{\beta(k+1)-2}{2}} \frac{\prod_{j<k}|z_k - z_j|^2}{\prod_{k=0}^{n-1}(1-|\alpha_k|^2)^k} d\vec{z}$$
$$= \prod_{j=0}^{n-1}(1-|\alpha_k|^2)^{\frac{(\beta-2)(k+1)}{2}} \prod_{j<k}|z_k - z_j|^2 d\vec{z}$$

Step 3: $\prod_{k=0}^{n-1}(1-|\alpha_k|^2)^{k+1} = \prod_{j,k=1}^{n}(1 - z_j\bar{z}_k)$ (obtained from the theory of orthogonal polynomials).

Gaussian Hermitian ensembles $G\beta E(n)$ of random matrices:

Let $X = (x_j k)_{j,k=1}^n$ be $n \times n$ matrix with i.i.d. entries with Gaussian $N(0,1)$ distributions (real, complex, or quaternionic). Let $H = X + X^*$.

- ($\beta = 1$) We say that $H \in GOE(n)$ (Gaussian Orthogonal Ensemble) if entries are real;
- ($\beta = 2$) We say that $H \in GUE(n)$ (Gaussian Unitary Ensemble) if entries are complex;
- ($\beta = 4$) We say that $H \in GSE(n)$ (Gaussian Symplectic Ensemble) if entries are quaternionic.

Eigenvalues are distributed in $\mathbb{R}^n$ according to

$$\prod_{j<k} |\lambda_j - \lambda_k|^\beta e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j^2} d\lambda_1 \dots d\lambda_n$$

Laguerre/Wishart Hermitian ensembles $L\beta E(m \times n)$ of random matrices:

Let $X = (x_{jk})_{m\times n}$ be $m \times n$ matrix with i.i.d. entries with Gaussian $N(0,1)$ distributions (real, complex, or quaternionic). Let $H = X^*X$.

- ($\beta = 1$) We say that $H \in LOE(m \times n)$ (Laguerre/Wishart Orthogonal Ensemble) if entries are real;
- ($\beta = 2$) We say that $H \in LUE(m \times n)$ (Laguerre/Wishart Unitary Ensemble) if entries are complex;
- ($\beta = 4$) We say that $H \in LSE(m \times n)$ (Laguerre/Wishart Symplectic Ensemble) if entries are quaternionic.

Eigenvalues are distributed in $\mathbb{R}^n_+$ according to

$$\prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^n \lambda_j^{\beta a/2} e^{-\beta\lambda_j/2} d\lambda_1 \dots d\lambda_n,$$

where $a = m - n + 1 - 2/\beta$.

Chiral $Ch\beta E(m \times n)$ of random matrices:

Let $X = (x_{jk})_{m\times n}$ be $m \times n$ matrix with i.i.d. entries with Gaussian $N(0,1)$ distributions (real, complex, or quaternionic). Let $H = \begin{bmatrix} \mathbf{0}_{m\times m} & X \\ X^* & \mathbf{0}_{n\times n} \end{bmatrix}$.

- ($\beta = 1$) We say that $H \in ChOE(m \times n)$ (Chiral Orthogonal Ensemble) if entries are real;
- ($\beta = 2$) We say that $H \in ChUE(m \times n)$ (Chiral Unitary Ensemble) if entries are complex;
- ($\beta = 4$) We say that $H \in ChSE(m \times n)$ (Chiral Symplectic Ensemble) if entries are quaternionic.

Eigenvalues are $\{\pm\lambda_1, \dots, \pm\lambda_n\}$ (ignoring 0's) with $\lambda_j \in \mathbb{R}_+$ according to

$$\prod_{j<k} |\lambda_j^2 - \lambda_k^2|^\beta \prod_{j=1}^n \lambda_j^{\beta a} e^{-\beta\lambda_j^2/2} d\lambda_1 \dots d\lambda_n.$$

These are just $\pm$ square roots of the eigenvalues of the Laguerre/Wishart ensembles.

Starting from a Hermitian operator/matrix H, applying Gram–Schmidt to $e_1, He_1, H^2e_1, \ldots$, we obtain a basis in which H has Jacobi form

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad a_n > 0, b_n \in \mathbb{R}.$$

In other words, $H = V^*JV$ with some $V^*V = \mathbf{1}$ satisfying $Ve_1 = V^*e_1 = e_1$.

Characteristic polynomials of the top-left corners

$$P_n(\lambda) = \det\left[\lambda - \mathcal{J}_{n\times n}\right]$$

are orthogonal polynomials with respect to the spectral measure of H.

After normalization $p_n(z) = P_n(z)/||P_n(z)||$, sequence $\{p_n\}_{n=0}^{\infty}$ satisfies

$$a_n p_{n-1}(z) + b_{n+1}p_n(z) + a_{n+1}p_{n+1}(z) = zp_n(z), \quad n = 0, 1, \ldots$$

Motivation for reducing H to the Jacobi form:

- Sparse matrices – good for simulations;
- Spectral measures (eigenvalues and eigenweights) of H and $\mathcal{J}$ are the same;
- Eigenvalues of H = zeros of orthogonal polynomials p_n.
- Non-Hermitian matrices also have Jacobi-type representations.

Dumitriu–Edelman '2002: applying Jacobification to $G\beta E(n)$ produces Jacobi matrix with statistically independent Jacobi coefficients!

Matrix model for $G\beta E(n)$: Jacobi matrix

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 & \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad a_n > 0, b_n \in \mathbb{R},$$

with the joint distributions of the coefficients given by

$$\prod_{j=1}^{n-1} a_j^{\beta(n-j)-1} e^{-a_j^2} \prod_{j=1}^{n} e^{-b_j^2/2} da_1 \dots da_{n-1} db_1 \dots db_n.$$

In fact, this ensemble makes sense for any $0 < \beta < \infty$ (Gaussian β-ensembles), rather than just $\beta = 1, 2, 4$ with the joint eigenvalue distribution

$$\prod_{j<k} |\lambda_j - \lambda_k|^\beta e^{-\frac{1}{2}\sum_{j=1}^n \lambda_j^2} d\lambda_1 \dots d\lambda_n$$

Dumitriu–Edelman '2002: applying Jacobification to $L\beta E(m \times n)$ produces Jacobi matrix with the following distribution of entries:

Matrix model for $L\beta E(m \times n)$: Jacobi matrix $\mathcal{J} = B^* B$, where

$$B = \begin{pmatrix} x_m & y_{n-1} & 0 & \\ 0 & x_{m-1} & y_{n-2} & \ddots \\ 0 & 0 & x_{m-2} & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

with the joint distributions of the coefficients given by

$$\prod_{j=0}^{n-1} x_{m-j}^{\beta(m-j)-1} e^{-x_{m-j}^2/2} \prod_{j=1}^{n-1} y_j^{\beta j-1} e^{-y_j^2} dx_{m-n+1} \dots dx_m dy_1 \dots dy_{n-1}.$$

In fact, this ensemble makes sense for any $0 < \beta < \infty$ (Laguerre β-ensembles), rather than just $\beta = 1, 2, 4$ with the joint eigenvalue distribution

$$\prod |\lambda_j - \lambda_k|^\beta \prod_{j=1}^{n} \lambda_j^{\beta a/2} e^{-\beta \lambda_j/2} d\lambda_1 \dots d\lambda_n.$$

Recall $H \in G\beta E(n)$, and we need to compute the eigenvalues of the effective Hamiltonian $H_{eff} = H + i\Gamma$ with $\Gamma = \Gamma^* \geq 0$ with $\mathrm{rank}\Gamma = 1$.

Then:

$$H_{eff} = H + i\Gamma$$

$$(\text{diagonalize } \Gamma = U^*(lI_1)U,$$

$$\text{where } I_1 = \mathrm{diag}(1,0,0,\ldots,0), l = ||\Gamma||_{HS})$$

$$= U^*(UHU^* + ilI_1)U$$

$$(\text{use } UHU^* \in G\beta E(n), \text{ so by Dumitriu–Edelman } UHU^* = V^*\mathcal{J}V)$$

$$= U^*V^*(\mathcal{J} + ilVI_1V^*)VU$$

$$(\text{use } Ve_1 = V^*e_1 = e_1, \text{ so that } VI_1V^* = I_1)$$

$$= U^*V^*(\mathcal{J} + ilI_1)VU$$

Using Dumitriu–Edelman '2002 we obtain matrix form:

$H_{eff} = H + i\Gamma$, where $H \in G\beta E(n)$, $\Gamma \geq 0$, $\mathrm{rank}\Gamma = 1$, can be reduced to the Jacobi form

$$\mathcal{J} = \begin{pmatrix} b_1 + il & a_1 & 0 & & \\ a_1 & b_2 & a_2 & \ddots & \\ 0 & a_2 & b_3 & \ddots & \\ & \ddots & \ddots & \ddots & a_{n-1} \\ & & & a_{n-1} & b_n \end{pmatrix},$$

with $b_j \sim N(0, \sqrt{2/n})$, $a_j \sim \sqrt{2/n}\,\chi(\beta(n-j))$, and $l = ||\Gamma||_{HS}$.

Remark: Works for any $0 < \beta < \infty$.

By Arlinskii–Tsekanovskii '2006, its n eigenvalues are in $\{z : \operatorname{Im} z > 0\}$, and there's 1-to-1 correspondence between all such matrices and all such eigenvalue configurations.

Computing the Jacobian of the transformation we obtain

Theorem (K–, 2015)

If H is taken from the Gaussian β-ensemble, and $||\Gamma||_{HS} \sim \chi^2(\beta n)$, then the eigenvalues of $H_{eff} = H + i\Gamma$ are distributed in $\{z : \operatorname{Im} z > 0\}^n$ as

$$\prod_{j<k} |z_j - z_k|^2 \prod_{j,k} |z_j - \bar{z}_k|^{\beta/2-1} e^{-\frac{1}{2}\sum_{j=1}^n [(\operatorname{Re} z_j)^2 + \operatorname{Im} z_j] - \sum_{j<k} (\operatorname{Im} z_j)(\operatorname{Im} z_k)} d^2 z_1 \dots d^2 z_n$$

Note: this distribution was obtained in the physics literature: Ullah '1969, and Sokolov–Zelevinsky '1989 for $\beta = 1$; and Stöckmann–Šeba '1998 for $\beta = 1, 2$, Fyodorov–Khoruzhenko '1999.

Similar approach can be applied to perturbations of Laguerre/Wishart ensembles:

Theorem (K–, 2015)

If H is taken from the Laguerre (Wishart) β-ensemble $L\beta E(m \times n)$ with $m \geq n$, and $||\Gamma||_{HS} \sim \chi^2(\beta n)$, then the eigenvalues of $H_{eff} = H + i\Gamma$ are distributed in

$$\left\{ (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \in \mathbb{C}^n : \sum_{j=1}^{n} \theta_j < \tfrac{\pi}{2}, \quad 0 < \theta_j < \tfrac{\pi}{2} \right\}$$

according to

$$\prod_{j<k} |z_j - z_k|^2 \prod_{j,k} |z_j - \bar{z}_k|^{\beta/2-1} (\mathrm{Re} \prod_{j=1}^{n} z_j)^{\frac{\beta a}{2}} e^{-\frac{1}{2}\sum_{j=1}^{n}(\mathrm{Re}\, z_j + \mathrm{Im}\, z_j)} d^2 z_1 \ldots d^2 z_n$$

Similar approach can be applied to perturbations of Laguerre/Wishart ensembles:

Theorem (K–, 2015)

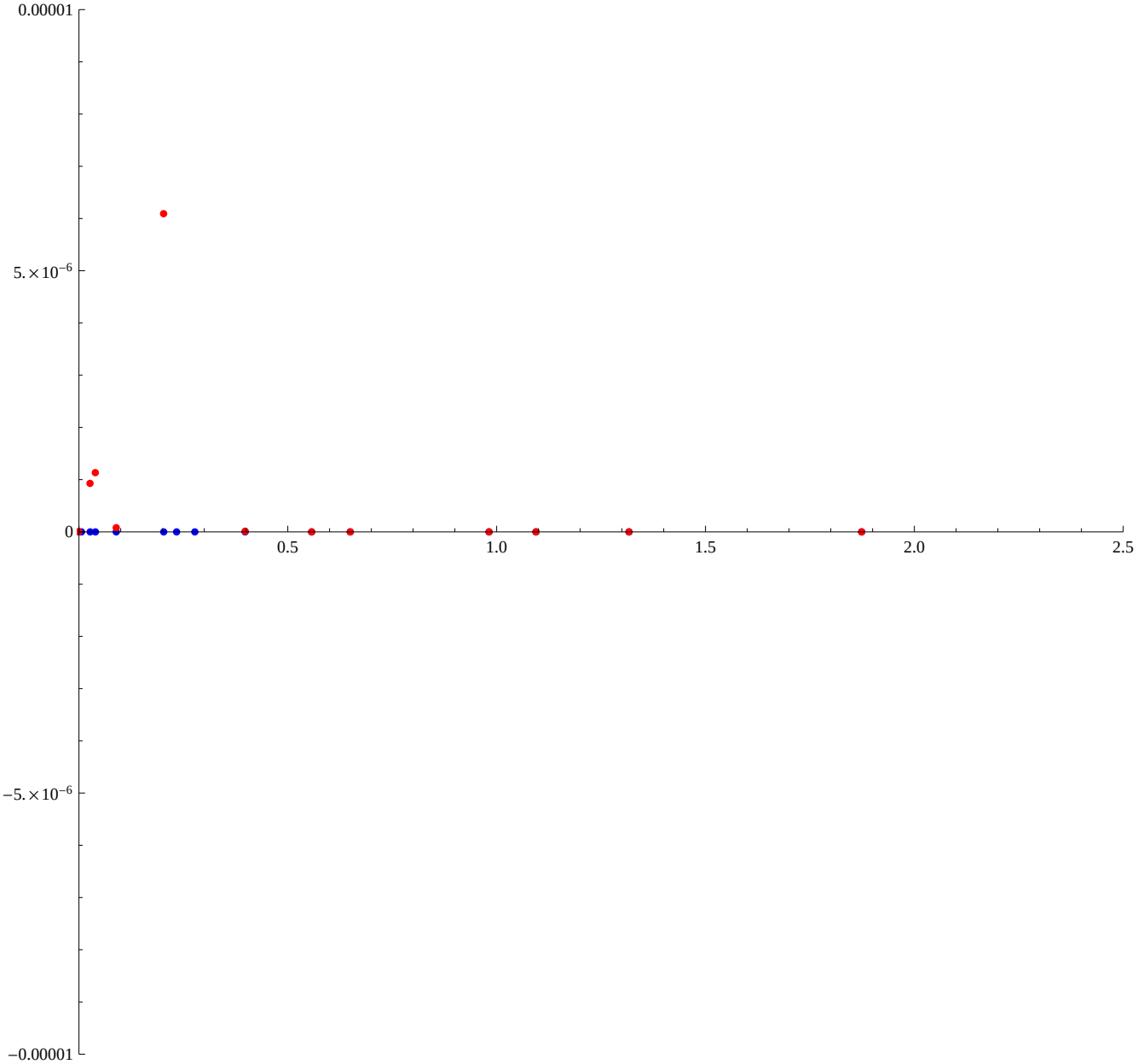
If H is taken from the Laguerre (Wishart) β-ensemble $L\beta E(m \times n)$ with $m < n$, and $||\Gamma||_{HS} \sim \chi^2(\beta n)$, then the eigenvalues of $H_{eff} = H + i\Gamma$ are $\{z_1, \dots, z_{m+1}, 0, \dots, 0\}$ distributed according to

$$\left\{ (r_1 e^{i\theta_1}, \dots, r_{m+1} e^{i\theta_{m+1}}) \in \mathbb{C}^{m+1} : \sum_{j=1}^{m+1} \theta_j = \tfrac{\pi}{2}, \quad 0 < \theta_j < \tfrac{\pi}{2} \right\}$$

according to

$$\prod_{j<k} |z_j - z_k|^2 \prod_{j,k} |z_j - \bar{z}_k|^{\beta/2-1} \prod_{j=1}^{m+1} |z_j|^{\frac{\beta(n-m-1)}{2}} e^{-\frac{1}{2}\sum_{j=1}^{m+1}(\operatorname{Re} z_j + \operatorname{Im} z_j)} d\vec{r}\, d\vec{\theta}$$

Picture



- The method works for the perturbation of the Chiral ensembles:

  $$\tilde{H} = \begin{bmatrix} i\Gamma_{m\times m} & X \\ X^* & \mathbf{0}_{n\times n} \end{bmatrix}, \text{ with } \Gamma_{m\times m} \geq 0, \operatorname{rank}\Gamma = 1.$$

  Again, we get a Jacobi form for any β, and then compute the joint eigenvalue density: $2n$ eigenvalues in $\{z : \operatorname{Im} z > 0\}$ symmetric with respect to $i\mathbb{R}$.
- The method works for the multiplicative perturbations:

  $$\tilde{H} = H(I + i\Gamma), \text{ with } \Gamma \geq 0, \operatorname{rank}\Gamma = 1.$$

  Again, we get a Jacobi form for any β, and then compute the joint eigenvalue density: n eigenvalues in first and third quadrants of $\mathbb{C}$.

- The method works for the operator approach:

$$\widetilde{H} = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1n} & 0 & 0 & \\ h_{21} & h_{22} & \dots & h_{2n} & 0 & 0 & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ h_{n1} & h_{n2} & \dots & h_{nn} & 1 & 0 & \\ 0 & 0 & \dots & 1 & 0 & 1 & \ddots \\ 0 & 0 & \dots & 0 & 1 & 0 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

Using Dumitriu–Edelman '2002 result:

Operator can be reduced to the Jacobi form

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & & & 0 & 0 & 0 & \\ a_1 & b_2 & \ddots & & 0 & 0 & 0 & \\ & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & a_{n-1} & & & \\ 0 & 0 & & a_{n-1} & b_n & 1 & 0 & \\ 0 & 0 & & & 1 & 0 & 1 & \ddots \\ 0 & 0 & & & 0 & 1 & 0 & \ddots \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

with $b_j \sim N(0, \sqrt{2/n})$ and $a_j \sim \sqrt{2/n}\,\chi(\beta j)$.

Theorem

If H is taken from Gaussian (Hermite) β-ensemble, then the resonances of $\widetilde{H}$ (moved from $\mathbb{C} \setminus (-2,2)$ to the unit disk $\mathbb{D}$ via $z + z^{-1} \mapsto z$) are distributed as

$$\prod_{j<k} |z_j - z_k| \prod_{j<k} (1 - z_j \bar{z}_k)^{\beta/2-1} \exp\left(-\tfrac{n}{4} \sum_{j=1}^{2n-1} z_j^2 \right) |dz_1 \wedge \ldots \wedge dz_{2n-1}|$$

Remark: compare with the eigenvalue density

$$\prod_{j<k} |\lambda_j - \lambda_k|^{\beta} \exp\left(-\tfrac{n}{4} \sum_{j=1}^{n} \lambda_j^2 \right) d\lambda_1 \ldots d\lambda_n.$$

Resonances are zeros of the so-called Jost function u_{2n-1}, which is a polynomial satisfying the (slightly modified) Geronimo–Case recursion relations

$$\begin{pmatrix} u_{2k+2}(z) \\ c_{2k+2}(z) \end{pmatrix} = \begin{pmatrix} z & -(a_{k+1}^2 - 1) \\ z & 1 \end{pmatrix} \begin{pmatrix} u_{2k+1}(z) \\ c_{2k+1}(z) \end{pmatrix}$$

and

$$\begin{pmatrix} u_{2k+1}(z) \\ c_{2k+1}(z) \end{pmatrix} = \begin{pmatrix} z & -b_{k+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{2k}(z) \\ c_{2k}(z) \end{pmatrix}$$

where $c_{2k}(z) = c_{2k+1}(z) = z^k P_k(z + z^{-1})$; $u_0 = c_0 = 1$.

Remark: zeros $\{z_j\}$ of u_{2n-1} are the resonances of $\widetilde{H}$ and zeros $\{\lambda_j\}$ of c_{2n-1} are the eigenvalues of H.

Let $z_j, j = 1, \ldots, n$ be the n eigenvalues of truncated $C\beta E(n+1)$.

Theorem (Killip–K. '201?)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{(1-|z_j|)n} = \nu \quad \textit{weakly a.s.}$$

where ν is an absolutely-continuous probability measure on $(0, \infty)$.

- So 100% of zeros are of distance $O(\frac{1}{n})$ from the boundary of the circle.

Remark: eigenvalues of truncated Dyson's ensembles = zeros of OPUC with α_k distributed as $(1-|\alpha_k|^2)^{\frac{\beta(k+1)-2}{2}} d^2\alpha_k$ ($\Rightarrow \mathrm{Var}(\alpha_k) = \frac{2}{\beta}\frac{1}{k}$).

Consider OPUC

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n z^n \overline{\Phi_n(1/\bar{z})}, \quad \text{where}$$

(i) α_k are independent and rotationally invariant;
(ii) $\mathbb{E}\{|\alpha_k|^2\} = \frac{2}{\beta k} + O(\frac{1}{k^2})$;
(iii) $\mathbb{E}\{\frac{|\alpha_k|^6}{1-|\alpha_k|^2}\} = O(\frac{1}{k^3})$

What is asymptotic distribution of zeros?

Equivalently: CMV matrix $\mathcal{C}(\alpha_0, \dots, \alpha_{n-1})$ with random coefficients.
Asymptotic distribution of eigenvalues of such contractions operators?

Theorem (Killip–K. '201?)

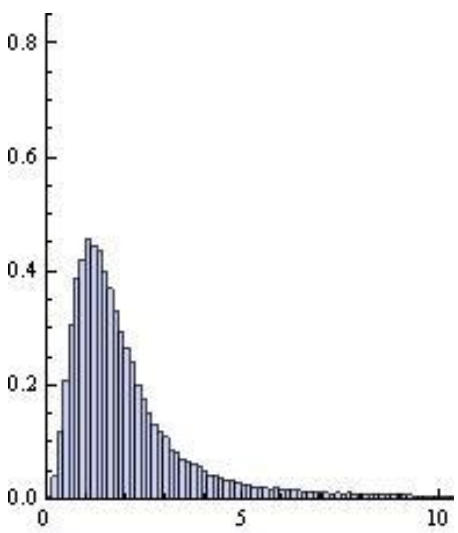
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{(1-|z_j|)n} = \nu \quad \textit{weakly a.s.}$$

where ν is an absolutely-continuous probability measure on $(0, \infty)$.

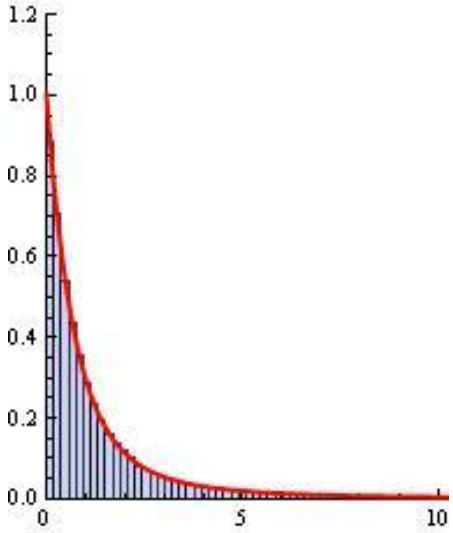
- ν is non-random, depends on β, but does not depend on the distributions of α_k

Numerical approximation of $\frac{d\nu(x)}{dx}$:

$\beta = 8$

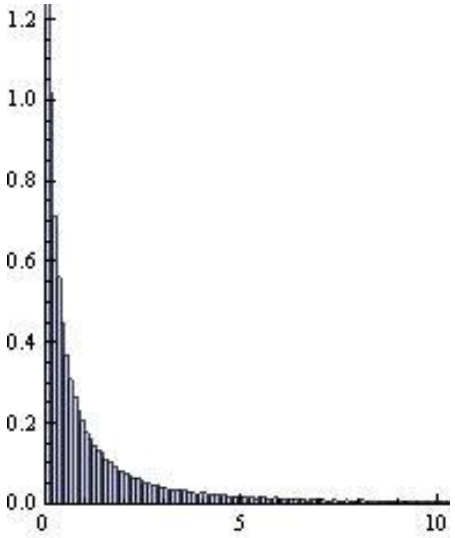


$\beta = 2$



$$\frac{1 - e^{-2x} - 2xe^{-2x}}{2x^2}$$

$\beta = 1$



Instead of studying zeros of $\Phi_n(z)$, it'll be easier to deal with zeros of

$$B_n(z) = \frac{\Phi_n(z)}{z^n \overline{\Phi_n(1/\bar{z})}}$$

Note: B_n is a Blaschke product.

To get the radial distribution of zeros, use Jensen's formula:

$$\int_0^{2\pi} \log |B_n(re^{i\theta})| \frac{d\theta}{2\pi} = \sum_{j=1}^{n} \max\{\log |z_j^{(n)}|, \log(r)\}$$

Using recurrence for Φ_n, we get recurrence

$$B_{n+1}(z) = z\frac{B_n(z) - \bar{\alpha}_n}{1 - \alpha_n B_n(z)}. \tag{9.1}$$

Turns out that $|B_n(e^{-y/n})|^2$ converges to a solution of a certain SDE.

Define

$$M_n(0) = 1$$
$$M_n\left(\tfrac{k+1}{n}\right) = \left|B_k\left(e^{-y/n}\right)\right|^2$$

and for $t \in \left(\frac{k}{n}, \frac{k+1}{n}\right)$, $M_n(t)$ is linear. Note: we need the law of $\lim_{n\to\infty} M_n(1)$

Taking $z = e^{-y/n}$ in (9.1), and expanding into Taylor series one gets:

$$M_n(\tfrac{k+1}{n}) = M_n(\tfrac{k}{n}) - \frac{2y}{n}M_n(\tfrac{k}{n}) + \frac{(1 - M_n(\frac{k}{n}))^2}{\beta k} - 2(1 - M_n(\tfrac{k}{n}))\sqrt{M_n(\tfrac{k}{n})}\,\mathrm{Re}(\alpha_k)$$
$$- 2(1 - M_n(\tfrac{k}{n}))M_n(\tfrac{k}{n})\,\mathrm{Re}(\alpha_k^2) + \varepsilon_{n,k},$$

where $\mathbb{E}\{|\varepsilon_{n,k}|\} \lesssim \frac{1}{k^{3/2}+n^{3/2}}$ and $\mathbb{E}\{|\varepsilon_{n,k}|^2\} \lesssim \frac{1}{k^2+n^2}$

Sequence of processes $M_n(t)$ converge (as measures on $C([0,\infty),\mathbb{R})$ with the topology of uniform convergence on compact sets) to the strong solution of SDE

$$dM_t = \left[-2yM_t + \tfrac{2}{t\beta}(1-M_t)^2\right]dt - \tfrac{2}{\sqrt{t\beta}}\sqrt{M_t}(1-M_t)dW_t$$

$$\mathbb{E}\{1-M_t\} \le 2yt$$

(morally $M_0 = \delta_1$)

Thank you for your attention

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