Eigenvalue statistics for rank one perturbations of unitary and Hermitian beta-ensembles

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1 Motivation: open quantum systems

2 Preliminaries
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   - Jacobi matrices and orthogonal polynomials on the real line
   - Dumitriu–Edelman $\beta$-ensembles

3 Distribution of resonances
   - as eigenvalues of non-Hermitian perturbations
   - as poles of the resolvent of a Jacobi operator on $\ell^2(\mathbb{Z}_+)$

4 Unitary analogue
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   - CMV matrices and orthogonal polynomials on the unit circle
   - Killip–Nenciu $\beta$-ensembles
   - Truncations and their eigenvalues

5 Large $n$ limit
   - Asymptotic radial distribution of resonances
   - Universality
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In quantum chaotic scattering: for a *closed* quantum system:

- **Hamiltonian** = **Hermitian** random matrix (from GOE, GUE, GSE)
- **Evolution operator** = **unitary** random matrix (from COE, CUE, CSE)

To model *open* quantum system (interaction with outer world), physicists consider **non-Hermitian** and **non-unitary** perturbations of these ensembles.

**Resonances** of an *open* quantum system – long-lived semi-stable states to which discrete energy levels of the closed quantum systems are transformed when system is “opened”.

Mathematically – **eigenvalues** of the perturbed random matrix ensembles.
Non-unitary perturbations of unitary random matrices.

Let $U$ be a unitary random matrix (e.g., from one of the circular ensembles or classical groups).
Delete $k$ rows and corresponding columns.

Eigenvalues of the truncation:
Non-Hermitian perturbations of Hermitian random matrices

Let $H$ be a Hermitian random matrix (e.g., from one of the Gaussian or Wishart ensembles). Let

$$H_{\text{eff}} = H + i\Gamma,$$

where $\Gamma = \Gamma^* > 0$ of rank $k$.

Eigenvalues:
Coupling of Hermitian random matrix to a discrete Schrödinger operator:

Let $H = (h_{ij})_{i,j=1}^n$ be a Hermitian random matrix.

Let

$$\tilde{H} = \begin{bmatrix}
    h_{11} & h_{12} & \ldots & h_{1n} & 0 & 0 \\
    h_{21} & h_{22} & \ldots & h_{2n} & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    h_{n1} & h_{n2} & \ldots & h_{nn} & 1 & 0 \\
    0 & 0 & \ldots & 1 & 0 & 1 \\
    0 & 0 & \ldots & 0 & 1 & 0 \\
\end{bmatrix}$$

Resonances of this operator = poles of the resolvent $(\tilde{H} - z)^{-1}$ on the second sheet of $\mathbb{C} \setminus [-2, 2]$
Mathematical models
Non-Hermitian perturbations of Chiral RMT ensembles:

Let

\[ H = \begin{bmatrix} \mathbf{0}_{m \times m} & X \\ X^* & \mathbf{0}_{n \times n} \end{bmatrix}, \]

where \( X \) is a random Ginibre matrix.

Let

\[ \tilde{H} = \begin{bmatrix} i\Gamma_{m \times m} & X \\ X^* & \mathbf{0}_{n \times n} \end{bmatrix}, \]

where \( \Gamma_{m \times m} = \Gamma^*_{m \times m} \geq \mathbf{0} \) of rank \( k \).

Physical applications — ? (quantum chromodynamics?)
Multiplicative perturbations of Hermitian RMT ensembles:

Let $H$ be a Hermitian random matrix (e.g., from one of the Gaussian or Wishart ensembles).

Let

$$\tilde{H} = H(I + i\Gamma),$$

where $\Gamma_{m\times m} = \Gamma^*_{m\times m} \geq 0$ of rank $k$.

Physical applications — ?
Classical ensembles of random unitary matrices:

- Group $\mathbb{U}(n)$ of $n \times n$ unitary matrices ($U^*U = 1$).
- Group $\mathbb{O}(n)$ of $n \times n$ orthogonal matrices ($O^TO = 1$).
- Group $\mathbb{USp}(n)$ of $n \times n$ quaternionic unitary matrices (unitary symplectic group) ($Q^*Q = 1$).

Circular ensembles (Dyson, 1962, as a modification of Wigner Gaussian ensembles):

- $(\beta = 1)$ COE$(n) = \{ U^T U, \text{ where } U \in \mathbb{U}(n) \}$, symmetric unitary matrices;
- $(\beta = 2)$ CUE$(n) = \mathbb{U}(n)$;
- $(\beta = 4)$ CSE$(n) = \{ Q^R Q, \text{ where } Q \in \mathbb{U}(2n) \}$, self-dual unitary quaternionic matrices ($U^R = Z^T U^T Z$, where $Z = \text{diag} \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$).
Circular Dyson’s ensembles:

- \((\beta = 1)\) \(COE(n) = \{U^T U, \text{ where } U \in \mathbb{U}(n)\}\)
- \((\beta = 2)\) \(CUE(n) = \mathbb{U}(n)\);
- \((\beta = 4)\) \(CSE(n) = \{Q^R Q, \text{ where } Q \in \mathbb{U}(2n)\}\)

We think of any element of such an ensemble as a matrix of transition probabilities between the various states of a quantum system.

- \(COE(n)\): systems with time-reversal and rotational symmetry
- \(CUE(n)\): systems without time-reversal symmetry
- \(CSE(n)\): systems with time-reversal invariance, but without rotational symmetry

Eigenvalues are distributed according to

\[
\prod_{j < k} |e^{i\theta_j} - e^{i\theta_k}|^\beta \frac{d\theta_j}{2\pi} \cdots \frac{d\theta_n}{2\pi},
\]
Let $U$ be $n \times n$ unitary, and suppose vector $e_1 = [1, 0, \ldots, 0]^T$ is cyclic for $U$. Applying the Gram–Schmidt procedure to $e_1, Ue_1, U^{-1}e_1, U^2e_1, U^{-2}e_1, \ldots$ produces a basis of $\mathbb{C}^n$ in which $U$ has CMV form $\mathcal{C}$.

**Definition**

$$\mathcal{C}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) = \mathcal{L}\mathcal{M},$$

where $\alpha_j \in \bar{\mathbb{D}} := \{z : |z| \leq 1\}$, and

$$\mathcal{L} = \text{diag } (\Xi_0, \Xi_2, \Xi_4, \ldots) \quad \mathcal{M} = \text{diag } ([1], \Xi_1, \Xi_3, \ldots),$$

where

$$\Xi_k = \begin{bmatrix} \bar{\alpha}_k & \sqrt{1 - |\alpha_k|^2} \\ \sqrt{1 - |\alpha_k|^2} & -\alpha_k \end{bmatrix}$$

In other words, $U = V^*\mathcal{C}V$ with some $V^*V = 1$ satisfying $Ve_1 = V^*e_1 = e_1$.

After multiplication, five-diagonal structure:

\[
C = \begin{bmatrix}
\bar{\alpha}_0 & \rho_0 \bar{\alpha}_1 & \rho_0 \rho_1 & 0 & 0 & 0 & 0 \\
\rho_0 & -\alpha_0 \bar{\alpha}_1 & -\alpha_0 \rho_1 & 0 & 0 & 0 & 0 \\
0 & \bar{\alpha}_2 \rho_1 & -\alpha_1 \bar{\alpha}_2 & \bar{\alpha}_3 \rho_2 & \rho_2 \rho_3 & 0 & 0 \\
0 & \rho_1 \rho_2 & -\bar{\alpha}_1 \rho_2 & -\alpha_2 \bar{\alpha}_3 & -\alpha_2 \rho_3 & 0 & 0 \\
0 & 0 & 0 & \bar{\alpha}_4 \rho_3 & -\alpha_3 \bar{\alpha}_4 & \bar{\alpha}_5 \rho_4 & \rho_4 \rho_5 \\
0 & 0 & 0 & \rho_3 \rho_4 & -\alpha_3 \rho_4 & -\alpha_4 \bar{\alpha}_5 & -\alpha_4 \rho_5 \\
0 & 0 & 0 & 0 & 0 & \bar{\alpha}_6 \rho_5 & -\alpha_5 \bar{\alpha}_6 \\
0 & 0 & 0 & 0 & 0 & \rho_5 \rho_6 & -\alpha_5 \rho_6 & -\alpha_6 \bar{\alpha}_7
\end{bmatrix},
\]

where \( \rho_k = \sqrt{1 - |\alpha_k|^2} \)
Characteristic polynomials of the top-left corners

$$\Phi_n(\lambda) = \det [\lambda - C_{n \times n}]$$

are orthogonal polynomials with respect to the spectral measure of $U$:

$$d\mu = \sum_{j=1}^{n} w_j \delta_{\lambda_j}$$

These polynomials satisfy Szegő’s recurrence:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n z^n \overline{\Phi_n(1/\bar{z})}$$

We call $\alpha_n \in \mathbb{D}$ the Verblunsky coefficients.
Motivation for reducing $U$ to the CMV form:

- Sparse matrices – good for simulations;
- Spectral measures (eigenvalues and eigenweights) of $U$ and $C$ are the same;
- Eigenvalues of $U = \text{zeros of orthogonal polynomials } \Phi_n$.
- Non-unitary matrices also have CMV-type representations.
Killip–Nenciu ’2004 and Killip-K ’2015: CMV-fying Dyson’s ensembles or the Classical Compact Groups produces statistically independent Verblunsky coefficients $\alpha_k$.

Matrix model for Dyson’s ensembles: $CMV(\alpha_0, \ldots, \alpha_{n-2}, e^{i\phi})$ with distribution

$$\prod_{j=0}^{n-2} (1 - |\alpha_j|^2)^{\beta(n-1-j)/2-1} d^2 \alpha_0 \ldots d^2 \alpha_{n-2} d\phi \quad (\beta = 1, 2, 4)$$

Remarks:

- $\alpha_j$’s are independent and rotationally invariant;
- The ensemble makes sense for any $0 < \beta < \infty$ (“circular $\beta$-ensemble”) with the joint eigenvalue distribution

$$\prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}| \beta \frac{d\theta_1}{2\pi} \ldots \frac{d\theta_n}{2\pi},$$
Orthogonal group $\mathbb{O}(n)$ with Haar measure.

$\mathbb{O}(2k) \setminus \mathbb{SO}(2k)$: $n \times n$ orthogonal matrices of determinant $-1$.

Matrix model $\mathbb{O}(n)$: $\text{CMV}(\alpha_0, \ldots, \alpha_{n-2}, \pm 1)$ with distribution

$$
\prod_{j=0}^{n-2} (1 - \alpha_j^2)^{(n-j-3)/2} \, d\alpha_0, \ldots, d\alpha_{n-2}
$$

Remarks:

- $\alpha_j$’s are real with even distributions;
- $\alpha_j$’s are independent;
- $\beta$-dependent family of CMV ensembles exists (“real orthogonal $\beta$-ensemble”).
Matrix model for $\mathbb{USp}(n)$: $\text{CMV}(\alpha_0, \ldots, \alpha_{n-2}, q)$ with distribution

$$n-2 \prod_{j=0}^{n-2} (1 - |\alpha_j|^2)^{2n-2j-3} d^4 \alpha_0, \ldots d^4 \alpha_{n-2} d^3 q$$

Notes:

- $\alpha_j$ and $q$ are independent quaternions.
- $d^3 q$ is the uniform measure on unit quaternionic sphere $|q| = 1$. 

Rostyslav Kozhan | Rank one perturbations of beta-ensembles
Approach 3. Truncations of unitary random matrices. Motivation

For $H$ from any of the unitary RMT ensembles: delete one row and corresponding column (rank one perturbation; “one open channel”). Distributions of eigenvalues?

Motivation:

- **From the point of view of physics:**
  - Scattering resonances of open quantum systems with one open channel;
  - Log-gas interpretation: electrostatic charges in a dielectric medium;

- **From the point of view of pure mathematics:**
  - Explicitly computable eigenvalue point process for a non-normal ensemble of random matrices;
  - Eigenvalues converge to the zero point process of random Gaussian analytic functions.
From the physics literature the following was known:

Życzkowski–Sommers ’2000: eigenvalues of truncations of $CUE(n + 1)$ are distributed in $\mathbb{D}^n$ as

$$\frac{1}{\pi^n} \prod_{j < k} |z_k - z_j|^2 \, d^2z_1 \ldots d^2z_n.$$ 

Theorem (Killip–K– ’2015)

Eigenvalues of truncations of $COE(n + 1)$, $CUE(n + 1)$, $CSE(n + 1)$ are distributed in $\mathbb{D}^n$ as

$$\frac{\beta^n}{(2\pi)^n} \prod_{j,k=1}^{n} (1 - z_j \bar{z}_k)^{\frac{\beta}{2} - 1} \prod_{j < k} |z_k - z_j|^2 \, d^2z_1 \ldots d^2z_n.$$ 

with $\beta = 1, 2, \text{ and } 4, \text{ respectively.}$
\[
\frac{\beta^n}{(2\pi)^n} \prod_{j=1}^{n} \left( |z_j|^{n-1} \left( 1 - |z_j|^2 \right) \right)^{\frac{\beta}{2} - 1} \prod_{j<k} \left| z_k - \bar{z}_j \right|^{\beta-2} \left| z_k - z_j \right|^2 \, d^2 z_1 \ldots d^2 z_n.
\]

Typical picture ($n = 250, \beta = 8$):
Khoruzhenko–Sommer–Życzkowski ’2010: eigenvalues of truncations of $\mathbb{O}(2n)$, are distributed as

$$\prod_{j=1}^{2n-1} \frac{1}{\sqrt{1 - z_j^2}} \prod_{j<k} |z_k - z_j| |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_{2n-1}|.$$ 

Theorem (Killip–K– ’2015)

Eigenvalues of truncations of $\mathbb{O}(n)$ ensemble are distributed as

$$\prod_{j,k=1}^{n-1} (1 - z_j \bar{z_k})^{\frac{\beta}{4} - \frac{1}{2}} \prod_{j=1}^{n-1} (1 - z_j^2)^{-\frac{\beta}{4}} \prod_{j<k} |z_k - z_j| |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_{n-1}|,$$

with $\beta = 2$. 
\[
\prod_{j,k=1}^{n-1} (1 - z_j z_k)^{\frac{\beta}{4} - \frac{1}{2}} \prod_{j=1}^{n-1} (1 - z_j^2)^{-\frac{\beta}{4}} \prod_{j<k} |z_k - z_j| dz_1 \wedge dz_2 \wedge \ldots \wedge dz_{n-1},
\]

Typical picture (\(n = 20\), 100 iterations):
Truncations of $O(n)$. Wedges

$$\prod_{j=1}^{n} \frac{1}{\sqrt{1 - z_j^2}} \prod_{j<k} |z_k - z_j| |dz_1 \wedge dz_2 \wedge \ldots \wedge dz_n|.$$ 

E.g., suppose $n = 3$: then either $z_1 = x_1, z_2 = x_2, z_3 = x_3$ are all real, in which case the density is

$$\frac{1}{\sqrt{1 - x_1^2}} \frac{1}{\sqrt{1 - x_2^2}} \frac{1}{\sqrt{1 - x_3^2}} \prod_{j<k} (x_k - x_j) \, dx_1 dx_2 dx_3,$$

or $z_1 = x_1 + iy_1, z_2 = x_1 - iy_2, z_3 = x_3$, in which case the density is

$$\frac{1}{|1 - (x_1 + iy_1)^2|} \frac{1}{\sqrt{1 - x_3^2}} |x_1 + iy_1 - x_3|^2 (2y_1) (2dx_1 dy_1) dx_3.$$
Lemma

Consider a unitary CMV matrix $\text{CMV}(\alpha_0, \ldots, \alpha_n)$. Removing one row and one column produces:

- $W^* \text{CMV} \left( -\bar{\alpha}_{n-1} \alpha_n, -\bar{\alpha}_{n-2} \alpha_n, \ldots, -\bar{\alpha}_0 \alpha_n \right)^T W$, if $n$ is even,
- $W^* \text{CMV} \left( -\bar{\alpha}_{n-1} \alpha_n, -\bar{\alpha}_{n-2} \alpha_n, \ldots, -\bar{\alpha}_0 \alpha_n \right) W$, if $n$ is odd,

for some unitary $W$.

So the following coincide:

- Eigenvalues of truncations of Dyson’s ensembles
- Eigenvalues of $\text{CMV}(\alpha_0, \ldots, \alpha_{n-1})$ with distributions

$$\prod_{j=0}^{n-1} \left( 1 - |\alpha_k|^2 \right)^{\frac{\beta(k+1)-2}{2}} \alpha_0 \ldots \alpha_{n-1}^2$$

- Zeros of random orthogonal polynomials

$$\Phi_{n+1}(z) = z \Phi_n(z) - \bar{\alpha}_n z^n \Phi_n \left( \frac{1}{\bar{z}} \right)$$

with $\alpha_k$’s as above.
Step 1: Let $\Phi_n(z) = z^n + \kappa_1 z^{n-1} + \ldots + \kappa_{n-1} z + \kappa_n$. Compute the Jacobian $\det \frac{\partial (\alpha_0, \ldots, \alpha_{n-1})}{\partial (\kappa_1, \ldots, \kappa_n)}$ is equal to $\prod_{k=0}^{n-1} (1 - |\alpha_k|^2)^{-k}$

Proof: induction and recurrence relations.

Step 2: The Jacobian $\det \frac{\partial (\kappa_1, \ldots, \kappa_n)}{\partial (z_1, \ldots, z_n)}$ is equal to $\prod_{j<k} |z_k - z_j|^2$

Thus:

$$\prod_{k=0}^{n-1} (1 - |\alpha_k|^2)^{-\frac{\beta(k+1)-2}{2}} d\vec{\alpha} = \prod_{k=0}^{n-1} (1 - |\alpha_k|^2)^{-\frac{\beta(k+1)-2}{2}} \frac{\prod_{j<k} |z_k - z_j|^2}{\prod_{k=0}^{n-1} (1 - |\alpha_k|^2)^k} d\vec{z} = \prod_{j=0}^{n-1} (1 - |\alpha_k|^2)^{-\frac{(\beta-2)(k+1)}{2}} \prod_{j<k} |z_k - z_j|^2 d\vec{z}$$

Step 3: $\prod_{k=0}^{n-1} (1 - |\alpha_k|^2)^{k+1} = \prod_{j,k=1}^{n} (1 - z_j \bar{z_k})$ (obtained from the theory of orthogonal polynomials).
Gaussian Hermitian ensembles $G\beta E(n)$ of random matrices:

Let $X = (x_{jk})_{j,k=1}^n$ be $n \times n$ matrix with i.i.d. entries with Gaussian $N(0, 1)$ distributions (real, complex, or quaternionic). Let $H = X + X^*$. 

- $(\beta = 1)$ We say that $H \in GOE(n)$ (Gaussian Orthogonal Ensemble) if entries are real;
- $(\beta = 2)$ We say that $H \in GUE(n)$ (Gaussian Unitary Ensemble) if entries are complex;
- $(\beta = 4)$ We say that $H \in GSE(n)$ (Gaussian Symplectic Ensemble) if entries are quaternionic.

Eigenvalues are distributed in $\mathbb{R}^n$ according to

$$\prod_{j<k} |\lambda_j - \lambda_k|^{-\beta} e^{-\frac{1}{2} \sum_{j=1}^n \lambda_j^2} d\lambda_1 \ldots d\lambda_n$$
Laguerre/Wishart Hermitian ensembles $L\beta E(m \times n)$ of random matrices:

Let $X = (x_{jk})_{m \times n}$ be $m \times n$ matrix with i.i.d. entries with Gaussian $N(0, 1)$ distributions (real, complex, or quaternionic). Let $H = X^* X$.

- $(\beta = 1)$ We say that $H \in LOE(m \times n)$ (Laguerre/Wishart Orthogonal Ensemble) if entries are real;
- $(\beta = 2)$ We say that $H \in LUE(m \times n)$ (Laguerre/Wishart Unitary Ensemble) if entries are complex;
- $(\beta = 4)$ We say that $H \in LSE(m \times n)$ (Laguerre/Wishart Symplectic Ensemble) if entries are quaternionic.

Eigenvalues are distributed in $\mathbb{R}_+^n$ according to

$$\prod_{j < k} |\lambda_j - \lambda_k|^{\beta} \prod_{j = 1}^{n} \lambda_j^{\beta a/2} e^{-\beta \lambda_j / 2} d\lambda_1 \ldots d\lambda_n,$$

where $a = m - n + 1 - 2/\beta$. 
Chiral $Ch\beta E(m \times n)$ of random matrices:

Let $X = (x_{jk})_{m \times n}$ be $m \times n$ matrix with i.i.d. entries with Gaussian $N(0, 1)$ distributions (real, complex, or quaternionic). Let $H = \begin{bmatrix} 0_{m \times m} & X \\ X^* & 0_{n \times n} \end{bmatrix}$.

- ($\beta = 1$) We say that $H \in ChOE(m \times n)$ (Chiral Orthogonal Ensemble) if entries are real;
- ($\beta = 2$) We say that $H \in ChUE(m \times n)$ (Chiral Unitary Ensemble) if entries are complex;
- ($\beta = 4$) We say that $H \in ChSE(m \times n)$ (Chiral Symplectic Ensemble) if entries are quaternionic.

Eigenvalues are $\{\pm \lambda_1, \ldots, \pm \lambda_n\}$ (ignoring 0’s) with $\lambda_j \in \mathbb{R}_+$ according to

$$
\prod_{j<k} |\lambda_j^2 - \lambda_k^2|^{\beta} \prod_{j=1}^{n} \lambda_j^{\beta a} e^{-\beta \lambda_j^2 / 2} d\lambda_1 \ldots d\lambda_n.
$$

These are just $\pm$ square roots of the eigenvalues of the Laguerre/Wishart ensembles.
Starting from a Hermitian operator/matrix $H$, applying Gram–Schmidt to $e_1, He_1, H^2e_1, \ldots$, we obtain a basis in which $H$ has **Jacobi form**

$$\mathcal{J} = \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 \\ 0 & a_2 & b_3 \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad a_n > 0, b_n \in \mathbb{R}.$$  

In other words, $H = V^*JV$ with some $V^*V = 1$ satisfying $Ve_1 = V^*e_1 = e_1$.

Characteristic polynomials of the top-left corners

$$P_n(\lambda) = \det [\lambda - \mathcal{J}_{n\times n}]$$

are orthogonal polynomials with respect to the spectral measure of $H$.

After normalization $p_n(z) = P_n(z)/||P_n(z)||$, sequence $\{p_n\}_{n=0}^{\infty}$ satisfies

$$a_n p_{n-1}(z) + b_{n+1} p_n(z) + a_{n+1} p_{n+1}(z) = z p_n(z), \quad n = 0, 1, \ldots$$
Motivation for reducing $H$ to the Jacobi form:

- Sparse matrices – good for simulations;
- Spectral measures (eigenvalues and eigenweights) of $H$ and $J$ are the same;
- Eigenvalues of $H = \text{zeros of orthogonal polynomials } p_n$.
- Non-Hermitian matrices also have Jacobi-type representations.
Dumitriu–Edelman ’2002: applying Jacobification to $G\beta E(n)$ produces Jacobi matrix with statistically independent Jacobi coefficients!

Matrix model for $G\beta E(n)$: Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad a_n > 0, b_n \in \mathbb{R},$$

with the joint distributions of the coefficients given by

$$\prod_{j=1}^{n-1} a_j^{\beta(n-j)-1} e^{-a_j^2} \prod_{j=1}^{n} e^{-b_j^2/2} da_1 \ldots da_{n-1} db_1 \ldots db_n.$$

In fact, this ensemble makes sense for any $0 < \beta < \infty$ (Gaussian $\beta$-ensembles), rather than just $\beta = 1, 2, 4$ with the joint eigenvalue distribution

$$\prod_{i < k} |\lambda_j - \lambda_k|^\beta e^{-\frac{1}{\beta} \sum_{j=1}^{n} \lambda_j^2} d\lambda_1 \ldots d\lambda_n$$
Dumitriu–Edelman ’2002: applying Jacobification to $L\beta E(m \times n)$ produces Jacobi matrix with the following distribution of entries:

Matrix model for $L\beta E(m \times n)$: Jacobi matrix $\mathcal{J} = B^* B$, where

$$B = \begin{pmatrix}
  x_m & y_{n-1} & 0 \\
  0 & x_{m-1} & y_{n-2} \\
  0 & 0 & x_{m-2} \\
  \vdots & \vdots & \vdots \\

e \end{pmatrix}$$

with the joint distributions of the coefficients given by

$$\prod_{j=0}^{n-1} x_j^{(m-j)-1} e^{-x_j^2/2} \prod_{j=1}^{n-1} y_j^{(m-j)-1} e^{-y_j^2} dx_{m-n+1} \ldots dx_m dy_{1} \ldots dy_{n-1}.$$ 

In fact, this ensemble makes sense for any $0 < \beta < \infty$ (Laguerre $\beta$-ensembles), rather than just $\beta = 1, 2, 4$ with the joint eigenvalue distribution

$$\prod |\lambda_j - \lambda_k|^\beta \prod_{j}^{n} \lambda_j^{\beta a/2} e^{-\beta \lambda_j/2} d\lambda_1 \ldots d\lambda_n,$$
Recall $H \in G\beta E(n)$, and we need to compute the eigenvalues of the effective Hamiltonian $H_{\text{eff}} = H + i\Gamma$ with $\Gamma = \Gamma^* \geq 0$ with rank\(\Gamma = 1\).

Then:

$$H_{\text{eff}} = H + i\Gamma$$

(diagonalize $\Gamma = U^*(lI_1)U$,

where $I_1 = \text{diag}(1, 0, 0, \ldots, 0), l = ||\Gamma||_{HS}$

$$= U^*(UHU^* + ilI_1)U$$

(use $UHU^* \in G\beta E(n)$, so by Dumitriu–Edelman $UHU^* = V^*JV$)

$$= U^*V^*(J + ilVI_1V^*)VU$$

(use $Ve_1 = V^*e_1 = e_1$, so that $VI_1V^* = I_1$)

$$= U^*V^*(J + ilI_1)VU$$
Using Dumitriu–Edelman ’2002 we obtain matrix form:

\[ H_{\text{eff}} = H + i\Gamma, \text{ where } H \in G\beta E(n), \Gamma \geq 0, \text{ rank } \Gamma = 1, \text{ can be reduced to the Jacobi form} \]

\[
\mathcal{J} = \begin{pmatrix}
 b_1 + il & a_1 & 0 \\
 a_1 & b_2 & a_2 & \ddots \\
 0 & a_2 & b_3 & \ddots \\
 & \ddots & \ddots & \ddots & a_{n-1} \\
 & & a_{n-1} & b_n
\end{pmatrix},
\]

with \( b_j \sim N(0, \sqrt{2/n}) \), \( a_j \sim \sqrt{2/n} \chi(\beta(n - j)) \), and \( l = ||\Gamma||_{HS} \).

Remark: Works for any \( 0 < \beta < \infty \).
By Arlinskii–Tsekanovskii ’2006, its $n$ eigenvalues are in $\{z : \Im z > 0\}$, and there’s 1-to-1 correspondence between all such matrices and all such eigenvalue configurations.

Computing the Jacobian of the transformation we obtain

**Theorem (K–, 2015)**

*If $H$ is taken from the Gaussian $\beta$-ensemble, and $||\Gamma||_{HS} \sim \chi^2(\beta n)$, then the eigenvalues of $H_{\text{eff}} = H + i\Gamma$ are distributed in $\{z : \Im z > 0\}^n$ as*

$$
\prod_{j<k} |z_j - z_k|^2 \prod_{j,k} |z_j - \overline{z}_k|^\beta/2 - 1 e^{-\frac{1}{2} \sum_{j=1}^{n} ((\Re z_j)^2 + \Im z_j)} - \sum_{j<k} (\Im z_j)(\Im z_k) d^2z_1 \ldots d^2z_n
$$

Note: this distribution was obtained in the physics literature: Ullah ’1969, and Sokolov–Zelevinsky ’1989 for $\beta = 1$; and Stöckmann–Šeba ’1998 for $\beta = 1, 2$, Fyodorov–Khoruzhenko ’1999.
Similar approach can be applied to perturbations of Laguerre/Wishart ensembles:

**Theorem (K–, 2015)**

If $H$ is taken from the Laguerre (Wishart) $\beta$-ensemble $L_\beta E(m \times n)$ with $m \geq n$, and $||\Gamma||_{HS} \sim \chi^2(\beta n)$, then the eigenvalues of $H_{\text{eff}} = H + i\Gamma$ are distributed in

\[
\left\{ (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n}) \in \mathbb{C}^n : \sum_{j=1}^{n} \theta_j < \frac{\pi}{2}, \quad 0 < \theta_j < \frac{\pi}{2} \right\}
\]

according to

\[
\prod_{j<k} |z_j - z_k|^2 \prod_{j,k} |z_j - \bar{z}_k|^\beta/2 - 1 \left( \text{Re} \prod_{j=1}^{n} z_j \right)^{\beta a/2} e^{-\frac{1}{2} \sum_{j=1}^{n} (\text{Re} z_j + \text{Im} z_j)} d^2 z_1 \ldots d^2 z_n
\]
Similar approach can be applied to perturbations of Laguerre/Wishart ensembles:

**Theorem (K−, 2015)**

*If* $H$ *is taken from the Laguerre (Wishart) $\beta$-ensemble* $L\beta E(m \times n)$ *with* $m < n$, *and* $||\Gamma||_{HS} \sim \chi^2(\beta n)$, *then the eigenvalues of* $H_{\text{eff}} = H + i\Gamma$ *are* $\{z_1, \ldots, z_{m+1}, 0, \ldots, 0\}$ *distributed according to*

$$\left\{(r_1e^{i\theta_1}, \ldots, r_{m+1}e^{i\theta_{m+1}}) \in \mathbb{C}^{m+1} : \sum_{j=1}^{m+1} \theta_j = \frac{\pi}{2}, \ 0 < \theta_j < \frac{\pi}{2}\right\}$$

*according to*

$$\prod_{j<k} |z_j - z_k|^2 \prod_{j,k} |z_j - \bar{z}_k|^{\beta/2-1} \prod_{j=1}^{m+1} |z_j|^{\frac{\beta(n-m-1)}{2}} e^{-\frac{1}{2} \sum_{j=1}^{m+1} (\text{Re} z_j + \text{Im} z_j)} d\vec{r} d\vec{\theta}$$
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**Picture**

![Graph showing rank one perturbations of beta-ensembles](image-url)

Rostyslav Kozhan

Rank one perturbations of beta-ensembles
The method works for the perturbation of the Chiral ensembles:

\[
\tilde{H} = \begin{bmatrix} i\Gamma_{m \times m} & X \\ X^* & 0_{n \times n} \end{bmatrix}, \text{ with } \Gamma_{m \times m} \geq 0, \text{ rank } \Gamma = 1.
\]

Again, we get a Jacobi form for any \( \beta \), and then compute the joint eigenvalue density: 2\( n \) eigenvalues in \( \{z : \text{Im} \, z > 0\} \) symmetric with respect to \( i\mathbb{R} \).

The method works for the multiplicative perturbations:

\[
\tilde{H} = H(I + i\Gamma), \text{ with } \Gamma \geq 0, \text{ rank } \Gamma = 1.
\]

Again, we get a Jacobi form for any \( \beta \), and then compute the joint eigenvalue density: \( n \) eigenvalues in first and third quadrants of \( \mathbb{C} \).
The method works for the operator approach:

\[ \tilde{H} = \begin{bmatrix} h_{11} & h_{12} & \ldots & h_{1n} & 0 & 0 \\ h_{21} & h_{22} & \ldots & h_{2n} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n1} & h_{n2} & \ldots & h_{nn} & 1 & 0 \\ 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 \\
\end{bmatrix} \]
Using Dumitriu–Edelman ’2002 result:

Operator can be reduced to the Jacobi form

\[
\mathcal{J} = \begin{pmatrix}
 b_1 & a_1 & 0 & 0 & 0 \\
 a_1 & b_2 & 0 & 0 & 0 \\
 & \ddots & \ddots & \ddots & \\
 & & a_{n-1} & a_{n-1} & b_n \\
 0 & 0 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 \\
 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

with \( b_j \sim N(0, \sqrt{2/n}) \) and \( a_j \sim \sqrt{2/n} \chi(\beta_j) \).
Theorem

*If* $H$ *is taken from Gaussian (Hermite) $\beta$-ensemble, then the resonances of* $\tilde{H}$ *(moved from* $\mathbb{C} \setminus (-2, 2)$ *to the unit disk* $\mathbb{D}$ *via* $z + z^{-1} \mapsto z$ *) are distributed as*

$$\prod_{j < k} |z_j - z_k| \prod_{j < k} (1 - z_j \bar{z}_k)^{\beta/2-1} \exp \left( -\frac{n}{4} \sum_{j=1}^{2n-1} z_j^2 \right) |dz_1 \wedge \ldots \wedge dz_{2n-1}|$$

*Remark: compare with the eigenvalue density*

$$\prod_{j < k} |\lambda_j - \lambda_k|^{\beta} \exp \left( -\frac{n}{4} \sum_{j=1}^{n} \lambda_j^2 \right) d\lambda_1 \ldots d\lambda_n.$$
Resonances are zeros of the so-called Jost function $u_{2n-1}$, which is a polynomial satisfying the (slightly modified) Geronimo–Case recursion relations

$$
\begin{pmatrix}
u_{2k+2}(z) \\
c_{2k+2}(z)
\end{pmatrix} = \begin{pmatrix} z & -(a_{k+1}^2 - 1) \\ z & 1 \end{pmatrix} \begin{pmatrix} u_{2k+1}(z) \\
c_{2k+1}(z)
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
u_{2k+1}(z) \\
c_{2k+1}(z)
\end{pmatrix} = \begin{pmatrix} z & -b_{k+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_{2k}(z) \\
c_{2k}(z)
\end{pmatrix}
$$

where $c_{2k}(z) = c_{2k+1}(z) = z^k P_k(z + z^{-1})$; $u_0 = c_0 = 1$.

Remark: zeros $\{z_j\}$ of $u_{2n-1}$ are the resonances of $\tilde{H}$ and zeros $\{\lambda_j\}$ of $c_{2n-1}$ are the eigenvalues of $H$. 
Let $z_j, j = 1, \ldots, n$ be the $n$ eigenvalues of truncated $C\beta E(n + 1)$.

**Theorem (Killip–K. ’201?)**

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{1 - |z_j|} = \nu \quad \text{weakly a.s.}$$

where $\nu$ is an absolutely-continuous probability measure on $(0, \infty)$.

- So 100% of zeros are of distance $O\left(\frac{1}{n}\right)$ from the boundary of the circle.

Remark: eigenvalues of truncated Dyson’s ensembles = zeros of OPUC with $\alpha_k$ distributed as $(1 - |\alpha_k|^2) \frac{\beta(k+1)-2}{2} d^2 \alpha_k \quad (\Rightarrow \text{Var}(\alpha_k) = \frac{2}{\beta} \frac{1}{k})$. 


Consider OPUC

\[ \Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n z^n \Phi_n(1/\bar{z}), \] where

(i) \( \alpha_k \) are independent and rotationally invariant;
(ii) \( \mathbb{E}\{|\alpha_k|^2\} = \frac{2}{\beta_k} + O\left(\frac{1}{k^2}\right) \);
(iii) \( \mathbb{E}\{\frac{|\alpha_k|^6}{1-|\alpha_k|^2}\} = O\left(\frac{1}{k^3}\right) \)

What is asymptotic distribution of zeros?

Equivalently: CMV matrix \( C(\alpha_0, \ldots, \alpha_{n-1}) \) with random coefficients. Asymptotic distribution of eigenvalues of such contractions operators?

**Theorem (Killip–K. ’201?)**

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \delta(1-|z_j|) \]  

\[ n = \nu \text{ weakly a.s.} \]

where \( \nu \) is an absolutely-continuous probability measure on \((0, \infty)\).

- \( \nu \) is non-random, depends on \( \beta \), but does not depend on the distributions of \( \alpha_k \).
Numerical approximation of $\frac{d\nu(x)}{dx}$:

\[ \beta = 8 \]

\[ \beta = 2 \]

\[ \beta = 1 \]

\[ \frac{1 - e^{-2x} - 2xe^{-2x}}{2x^2} \]
Instead of studying zeros of $\Phi_n(z)$, it’ll be easier to deal with zeros of

$$B_n(z) = \frac{\Phi_n(z)}{z^n \Phi_n(1/\bar{z})}$$

Note: $B_n$ is a Blaschke product.

To get the radial distribution of zeros, use Jensen’s formula:

$$\int_0^{2\pi} \log |B_n(re^{i\theta})| \frac{d\theta}{2\pi} = \sum_{j=1}^{n} \max\{\log |z_j^{(n)}|, \log(r)\}$$
Using recurrence for $\Phi_n$, we get recurrence

$$B_{n+1}(z) = z \frac{B_n(z) - \bar{\alpha}_n}{1 - \alpha_n B_n(z)}.$$  \hspace{1cm} (9.1)

Turns out that $|B_n(e^{-y/n})|^2$ converges to a solution of a certain SDE.

Define

$$M_n(0) = 1$$

$$M_n \left( \frac{k+1}{n} \right) = \left| B_k \left( e^{-y/n} \right) \right|^2$$

and for $t \in \left( \frac{k}{n}, \frac{k+1}{n} \right)$, $M_n(t)$ is linear. Note: we need the law of

$$\lim_{n \to \infty} M_n(1)$$

Taking $z = e^{-y/n}$ in (9.1), and expanding into Taylor series one gets:

$$M_n \left( \frac{k+1}{n} \right) = M_n \left( \frac{k}{n} \right) - \frac{2y}{n} M_n \left( \frac{k}{n} \right) + \frac{(1 - M_n \left( \frac{k}{n} \right))^2}{\beta k} - 2(1 - M_n \left( \frac{k}{n} \right)) \sqrt{M_n \left( \frac{k}{n} \right)} \text{Re}(\alpha_k)$$

$$- 2(1 - M_n \left( \frac{k}{n} \right)) M_n \left( \frac{k}{n} \right) \text{Re}(\alpha_k^2) + \varepsilon_{n,k},$$

where $\mathbb{E}\{|\varepsilon_{n,k}|\} \lesssim \frac{1}{k^{3/2} + n^{3/2}}$ and $\mathbb{E}\{|\varepsilon_{n,k}|^2\} \lesssim \frac{1}{k^2 + n^2}$. 

Rostyslav Kozhan
Sequence of processes $M_n(t)$ converge (as measures on $C([0, \infty), \mathbb{R})$ with the topology of uniform convergence on compact sets) to the strong solution of SDE

$$
dM_t = \left[-2yM_t + \frac{2}{t\beta} (1 - M_t)^2\right] dt - \frac{2}{\sqrt{t\beta}} \sqrt{M_t}(1 - M_t)dW_t
$$

$$
\mathbb{E}\{1 - M_t\} \leq 2yt
$$

(morally $M_0 = \delta_1$)
Thank you for your attention