# Persistence of singular behavior in unitary ensembles plus GUE

Arno Kuijlaars (KU Leuven, Belgium)

with

Tom Claeys, Karl Liechty, Dong Wang

XI Brunel-Bielefeld Workshop on RMT

Bielefeld, Germany, 12 December 2015

# 1. Unitary Ensemble plus GUE

# Unitary ensemble (UE)

$M$ **is random** $n \times n$ **Hermitian matrix from UE**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr} V(M)} dM$$

- **Eigenvalues have joint p.d.f.**

$$\frac{1}{\tilde{Z}_n} \Delta_n(x)^2 \prod_{j=1}^{n} e^{-nV(x_j)}, \qquad \Delta_n(x) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

- **Equilibrium measure** $\mu_V$ **is the minimizer of**

$$\iint \log \frac{1}{|s-t|} d\mu(s) d\mu(t) + \int V(t) d\mu(t)$$

- $\mu_V$ **is limiting eigenvalue distribution as** $n \to \infty$

# GUE matrix

$H$ **is** $n \times n$ **(scaled) GUE matrix with distribution**

$$\frac{1}{Z_n} e^{-\frac{n}{2} \operatorname{Tr} H^2} dH$$

- **Eigenvalues of** $\sqrt{\tau} H$ **follow semi-circle law with variance** $\tau$

$$d\lambda_\tau(s) = \frac{1}{2\pi\tau} \sqrt{4\tau - s^2}\, ds, \qquad s \in [-2\sqrt{\tau}, 2\sqrt{\tau}].$$

# Sum of UE and GUE

**We are interested in eigenvalues of**

$$X = M + \sqrt{\tau} H$$

**with** $M$ **from a unitary ensemble,** $H$ **from a (scaled) GUE, independent from** $M$**, and** $\tau > 0$**.**

- **Eigenvalues are determinantal point process**
- **Interpretation as non-intersecting paths**
- **Free probability:**

  $$\mu_V \boxplus \lambda_\tau$$

  **is limiting distribution of eigenvalues as** $n \to \infty$

# Determinantal point process

**Brézin and Hikami (1998), Zinn-Justin (1998), Johansson (2001)**

- **If** $M$ **is fixed with eigenvalues** $a_1, \ldots, a_n$**, then eigenvalues of** $M + \sqrt{\tau} H$ **have joint density**

$$\propto \frac{1}{\Delta_n(a)} \cdot \Delta_n(x) \cdot \det \left[ e^{-\frac{n}{2\tau}(x_k - a_j)^2} \right]_{j,k=1}^{n}$$

- **If** $M$ **is random from** $UE$**, then after averaging over** $a_1, \ldots, a_n$**,**

$$\propto \Delta_n(x) \cdot \det \left[ \int_{-\infty}^{\infty} a^{j-1} e^{-\frac{n\tau}{2}(x_k - a)^2} e^{-nV(a)} da \right]_{j,k=1}^{n}$$

- **This is a polynomial ensemble (special case of DPP)**

# Non-intersecting paths

**Johansson (2001), Bleher and Kuijlaars (2004):**

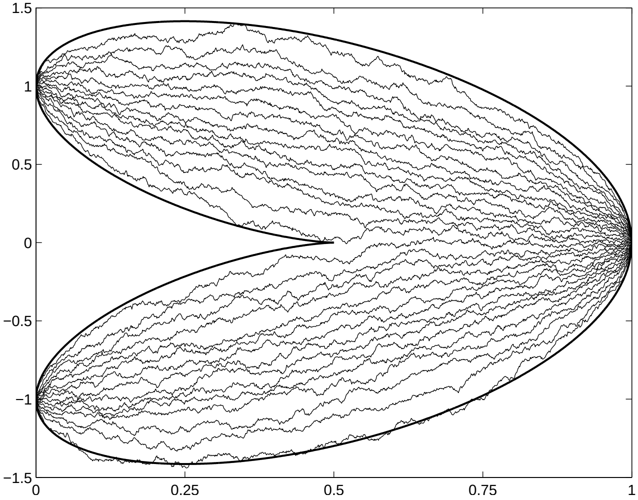
- **Non-intersecting Brownian bridges with starting positions $a_1, \ldots, a_n$ at time $t = 0$ and ending positions $b_1, \ldots, b_n$ at time $t = T$**
- **Joint density for particles at time $t \in (0, T)$:**

$$\propto \frac{1}{\Delta_n(a)\Delta_n(b)} \cdot \det\left[e^{-\frac{n}{2t}(a_j - x_k)^2}\right]_{j,k=1}^{n} \cdot \det\left[e^{-\frac{n}{2(T-t)}(x_j - b_k)^2}\right]_{j,k=1}^{n}$$

- **In limit when all $b_k \to 0$**

$$\propto \frac{1}{\Delta_n(a)} \cdot \det\left[e^{-\frac{n}{2t}(a_j - x_k)^2}\right]_{j,k=1}^{n} \cdot \Delta_n(x) \cdot \prod_{j=1}^{n} e^{-\frac{n}{2(T-t)} x_j^2}$$

# Figure




**Picture if all $a_j \to \pm 1$.**

# Random starting points

- **If $a_j$ are random eigenvalues of matrix from UE ensemble $\frac{1}{Z_n}e^{-nV(M)}dM$ then, after averaging over $a_1, \ldots, a_n$,**

$$\propto \Delta_n(x) \cdot \det \left[ \int_{-\infty}^{\infty} a^{j-1} e^{-\frac{n}{2t}(x_k - a)^2} e^{-nW(a)} da \right]_{j,k=1}^{n} \cdot \prod_{j=1}^{n} e^{-\frac{n}{2(T-t)} x_j^2}$$

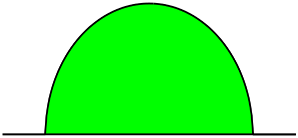
- **For $T \to \infty$, this is exactly the same as eigenvalues of $M + \sqrt{t}H$.**

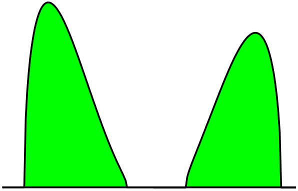
# 2. Singular potential

# Typical behavior of equilibrium measure

**Suppose $V$ is real analytic.**

- **$\mu_V$ is supported on finitely many intervals with density $\psi_V$**
- **Typically: $\psi_V$ vanishes like a square root at each endpoint, and is positive in the interior of each of the intervals.**





# Singular behavior

**Singular interior point**

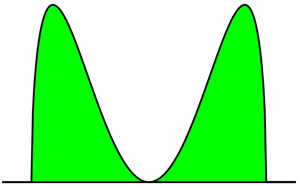
- $\psi_V$ **vanishes at interior point** $x^*$

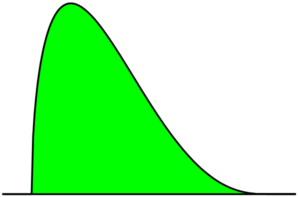
$$\psi_V(x) \sim (x - x^*)^\kappa \qquad \kappa = 2k \text{ for integer } k \geq 1.$$

**Singular edge point**

- $\psi_V$ **vanishes at edge point** $x^*$

$$\psi_V(x) \sim |x - x^*|^\kappa \qquad \kappa = 2k + 1 \text{ for integer } k \geq 1.$$





# Correlation kernels

**Limiting correlation kernels**

- **Sine kernel** at regular interior point
- **Airy kernel** at regular edge point
- **Painlevé kernels** at singular interior and singular edge points

# Singular interior point

- **Suppose interior vanishing**

$$\psi_V(x) \sim (x - x^*)^2$$

- $q$ **is the Hastings-McLeod solution of Painlevé II**

$$q'' = sq^2 + 2q^3, \qquad q(s) = \mathrm{Ai}(s)(1 + o(1)) \quad \textbf{as } s \to +\infty.$$

- **Critical kernel is built out of** **Lax pair solutions** **associated with** $q$

$$K_{crit}(x, y; s) = \frac{-\Phi_1(x; s)\Phi_2(y; s) + \Phi_2(x; s)\Phi_1(y, s)}{2\pi i(x - y)}$$

# Critical kernels

$$K_{crit}(x, y; s) = \frac{-\Phi_1(x; s)\Phi_2(y; s) + \Phi_2(x; s)\Phi_1(y, s)}{2\pi i(x - y)}$$

**where**

$$\frac{\partial}{\partial x}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} = \begin{pmatrix}-4ix^2 - i(s + 2q^2) & 4xq + 2ir \\ 4xq - 2ir & 4ix^2 + i(s + 2q^2)\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}$$

$$\frac{\partial}{\partial s}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix} = \begin{pmatrix}-ix & q \\ q & ix\end{pmatrix}\begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}$$

**Bleher-Its (2003), Claeys-K (2006)**

# 3. Main results

# Propagation of singular density

**Suppose** $X = M + \sqrt{\tau} H$

- **Eigenvalues of** $M$ **have limiting density** $\mu_V$ **where**

$$\frac{d\mu_V(s)}{ds} \sim c_0^{\kappa+1} |s - x^*|^\kappa \qquad \textbf{for interior or edge point } x^*$$

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**Theorem**

**For** $\tau < \tau_{cr} = \frac{2}{-V''(x^*)} > 0$**, there is value**

$$x_\tau^* = x^* + \frac{\tau}{2} V'(x^*)$$

**such that** $\mu_\tau = \mu_V \boxplus \lambda_\tau$ **has density satisfying**

$$\frac{d\mu_\tau(s)}{ds} \sim c_\tau^{\kappa+1} |s - x_\tau^*|^{\kappa}, \qquad c_\tau = \frac{\tau_{cr}}{\tau_{cr-\tau}} c_0$$

# Propagation of singularity

**The singularity at an interior point or edge point propagates in the model $M + \sqrt{\tau} H$.**

- **Interpretation in terms of non-intersecting Brownian paths**
- **Connection with two-matrix model**

**Duits (2014)**

# Propagation of critical kernel

- $M$ **is from unitary ensemble with eigenvalue correlation kernel** $K_n^M(x,y)$ **and scaling limit**

$$\lim_{n\to\infty} \frac{1}{c_0 n^\gamma} K_n^M\left(x^* + \frac{x}{c_0 n^\gamma}, x^* + \frac{y}{c_0 n^\gamma}\right) = \mathcal{K}_{crit,\kappa}(x,y)$$

  **with** $\gamma = (\kappa+1)^{-1}$**.**
- $X = M + \sqrt{\tau} H$ **has correlation kernel** $K_n^X(x,y)$

# Propagation of critical kernel

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  **with** $\gamma = (\kappa+1)^{-1}$**.**

- $X = M + \sqrt{\tau} H$ **has correlation kernel** $K_n^X(x,y)$

**Theorem**

**Under these conditions, plus something extra,**

$$\lim_{n\to\infty} \frac{e^{-H_n(x)+H_n(y)}}{c_\tau n^\gamma} K_n^X \left( x_\tau^* + \frac{x}{c_\tau n^\gamma}, x_\tau^* + \frac{y}{c_\tau n^\gamma} \right) = \mathcal{K}_{crit,\kappa}(x,y)$$

**for certain function** $H_n$

# Extra assumption 1

**Assumption 1**

- **The scaling limit**

$$\lim_{n\to\infty} \frac{1}{c_0 n^\gamma} K_n^M \left( x^* + \frac{x}{c_0 n^\gamma}, x^* + \frac{y}{c_0 n^\gamma} \right) = \mathcal{K}_{crit,\kappa}(x, y)$$

**holds uniformly for** $x$ **and** $y$ **in compact subsets of the complex plane**

# Orthogonal polynomials

- $p_{j,n}$ **is the orthonormal polynomial for** $e^{-nV}$

$$\int_{-\infty}^{\infty} p_{j,n}(x) p_{k,n}(x) e^{-nV(x)} dx = \delta_{j,k}$$

**with leading coefficient** $\kappa_{j,n}$

- **Let**

$$Y(z) = \begin{pmatrix} \kappa_{n,n}^{-1} p_{n,n}(z) & \frac{\kappa_{n,n}^{-1}}{2\pi i} \int_{-\infty}^{\infty} \frac{p_{n,n}(s) e^{-nV(s)}}{s-z} ds \\ -2\pi i \kappa_{n-1,n} p_{n-1,n}(z) & -\kappa_{n-1,n} \int_{-\infty}^{\infty} \frac{p_{n-1,n}(s) e^{-nV(s)}}{s-z} ds \end{pmatrix}$$

**be the solution of the Riemann-Hilbert problem for orthogonal polynomials.**

# Extra assumption 2

**Assumption 2**

$$\left| \left[ Y^{-1}(y) Y(z) \right]_{2,2} \right| \leq C e^{-n \operatorname{Re}(g(y) - g(z))} \times \left( 1 + |y - x^*|^{-1/4} + |z - x^*|^{-1/4} \right)$$

**where** $g(z) = \int \log(z - s) d\mu_V(s)$

# 4. About the proofs

# Double contour integral formula

**For $X = M + \sqrt{\tau} H$,**

$$K_n^X(x,y) = \frac{n}{2\pi i \tau} \int_{x^*-i\infty}^{x^*+i\infty} ds \int_{-\infty}^{\infty} dt \, K_n^M(s,t)$$

$$\times \, e^{\frac{n}{2}(V(s)-V(t))} e^{\frac{n}{2\tau}((s-x)^2-(t-y)^2)}$$

**Claeys-K-Wang (2015)**

- **Proof is basically a steepest descent analysis of this double integral.**

# Local part and rest

**Separate**

$$K_n^X(x,y) = K_{n,loc}^X(x,y) + K_{n,rest}^X(x,y)$$

**with**

$$K_{n,loc}^X(x,y) = \frac{n}{2\pi i\tau}\int_{x^*-iRn^{-\gamma}}^{x^*+iRn^{-\gamma}} ds \int_{x^*-Rn^{-\gamma}}^{x^*+Rn^{-\gamma}} dt\, K_n^M(s,t)$$

$$\times\, e^{\frac{n}{2}(V(s)-V(t))} e^{\frac{n}{2\tau}((s-x)^2-(t-y)^2)}$$

- $R$ **is large, but fixed constant, independent of** $n$**, but it will depend on** $x$ **and** $y$**.**

# Analysis of local part

**After change of variables**

$$\frac{1}{c_\tau n^\gamma} K^X_{n,loc}\left(x^*_\tau + \frac{x}{c_\tau n^\gamma}, x^*_\tau + \frac{y}{c_\tau n^\gamma}\right)$$
$$= \frac{n^{1-2\gamma}}{2\pi i c_0 c_\tau} \int_{x^*-ic_0 R}^{x^*+ic_0 R} ds \int_{x^*-c_0 R}^{x^*+c_0 R} dt\, e^{F_n(s,x)-F_n(t,y)}$$
$$\times \underbrace{\frac{1}{c_0 n^\gamma} K^M_n\left(x^* + \frac{s}{c_0 n^\gamma}, x^* + \frac{t}{c_0 n^\gamma}\right)}_{\to \mathcal{K}_{crit,\kappa}(s,t)}$$

**with**

$$F_n(s,x) = \frac{n}{2} V\left(x^* + \frac{s}{c_0 n^\gamma}\right) + \frac{n}{2\tau}\left(\frac{s}{c_0 n^\gamma} - \frac{\tau}{2} V'(x^*) - \frac{x}{c_\tau n^\gamma}\right)$$

# Taylor expansion of $F_n$

**Expand as** $n \to \infty$

$$F_n(s,x) = \frac{n}{2}V(x^*) + \frac{\tau n}{8}V'(x^*)^2 + \frac{n^{1-\gamma}}{2c_\tau}V'(x^*)x + \frac{n^{1-2\gamma}}{4c_0c_\tau}V''(x^*)x^2$$
$$+ \frac{n^{1-2\gamma}}{2\tau c_0 c_\tau}\left[(s-x)^2 + O(n^{-\gamma})\right]$$

- **No term** $sn^{-\gamma}$ **because of choice for** $x_\tau^*$
- **Complete square** $(s-x)^2$ **because of choice for** $c_\tau$
- **Saddle point equation** $\dfrac{\partial F_n}{\partial s} = 0$ **gives the saddle**

$$x + O(n^{-\gamma})$$

# Analysis of rest

$$K^X_{n,rest}(x,y) = K^X_n(x,y) - K^X_{n,loc}(x,y)$$

**Scaled version**

$$e^{-H_n(x)+H_n(y)} K^X_{n,rest}\left(x^*_\tau + \frac{x}{c_\tau n^\gamma}, x^*_\tau + \frac{y}{c_\tau n^\gamma}\right)$$

**becomes $O(e^{-cn^{1-\gamma}})$ as $n \to \infty$ if $R$ is large enough.**

- **Proof uses deformation of $t$-contour into the complex plane, and depends on assumption 2**

$$\left|\left[Y^{-1}(y)Y(z)\right]_{2,2}\right| \leq Ce^{-n\,\mathrm{Re}(g(y)-g(z))} \times \left(1 + |y - x^*|^{-1/4} + |z - x^*|^{-1/4}\right)$$

**Thank you**