

Local universality in biorthogonal Laguerre ensembles

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References

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Laguerre (unitary) ensemble

- X : an $m \times n$ ($m \leq n$) matrix whose entries are independent with a complex Gaussian distribution (**Ginibre matrix**)



Laguerre (unitary) ensemble

- ▶ X : an $m \times n$ ($m \leq n$) matrix whose entries are independent with a complex Gaussian distribution (**Ginibre matrix**)
- ▶ Eigenvalue distribution of XX^* (**Wishart matrix**):

$$\frac{1}{C_{m,n}} \prod_{l=1}^m x_l^\alpha e^{-x_l} \prod_{1 \leq j < k \leq m} (x_j - x_k)^2 = \det [K_m^\alpha(x_j, x_k)]_{j,k=1,\dots,m}$$

where $\alpha = n - m$

- ▶ This is a **determinantal point process** with correlation kernel $K_n^\alpha(x, y)$ expressed in terms of Laguerre polynomials



Large n limits of the correlation kernel

- ▶ Macroscopic limit:

$$\lim_{n \rightarrow \infty} K_n^\alpha(nx, nx) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}}, \quad 0 < x < 4$$

Marchenko-Pastur density



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Marchenko-Pastur density

- ▶ Microscopic limit: principle of universality

- ▷ sine kernel in the bulk and Airy kernel at the soft-edge
- ▷ Bessel kernel at the hard-edge

$$\lim_{n \rightarrow \infty} K_n^\alpha\left(\frac{x}{4n}, \frac{y}{4n}\right)/4n = K^{\text{Bes}, \alpha}(x, y)$$

where

$$K^{\text{Bes}, \alpha}(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)}$$



Biorthogonal Laguerre ensembles

- ▶ Definition: n particles $x_1 < \dots < x_n$ distributed over the positive real axis with jpdf of the form

$$\frac{1}{Z_n} \prod_{1 \leq i < j \leq n} (x_j - x_i) \prod_{1 \leq i < j \leq n} (x_j^\theta - x_i^\theta) \prod_{j=1}^n x_j^\alpha e^{-x_j},$$

where

- ▷ Z_n : the normalization constant
- ▷ $\alpha > -1, \theta > 0$: two parameters



Background

- ▶ First introduced by Muttalib: provide more effective description of disordered conductors in the metallic regime than the classical random matrix theory

[Muttalib, '95]



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- ▶ A more concrete physical example (with $\theta = 2$) for disordered bosons
[Lueck-Sommers-Zirnbauer, '06]



Background

- ▶ First introduced by Muttalib: provide more effective description of disordered conductors in the metallic regime than the classical random matrix theory
[Muttalib, '95]
- ▶ A more concrete physical example (with $\theta = 2$) for disordered bosons
[Lueck-Sommers-Zirnbauer, '06]
- ▶ Realizations as PDF for squared singular values of certain triangular random matrices
[Cheliotis, '14; Forrester-Wang, '15]



Correlation kernels

- ▶ Determinantal forms of the jpdf:

$$\frac{1}{n!} \det \left(K_n^{(\alpha, \theta)}(x_i, x_j) \right)_{i,j=1}^n$$



Correlation kernels

- ▶ Determinantal forms of the jpdf:

$$\frac{1}{n!} \det \left(K_n^{(\alpha, \theta)}(x_i, x_j) \right)_{i,j=1}^n$$

- ▶ Representation of $K_n^{(\alpha, \theta)}(x, y)$:

$$K_n^{(\alpha, \theta)}(x, y) = \sum_{j=0}^{n-1} p_j^{(\alpha, \theta)}(x) q_j^{(\alpha, \theta)}(y^\theta) x^\alpha e^{-x}$$

where

- ▷ $p_j^{(\alpha, \theta)}(x) = \varkappa_j x^j + \dots, \quad q_k^{(\alpha, \theta)}(x) = x^k + \dots, \quad \varkappa_j > 0$
- ▷ $\int_0^\infty p_j^{(\alpha, \theta)}(x) q_k^{(\alpha, \theta)}(x^\theta) x^\alpha e^{-x} dx = \delta_{j,k}, \quad j, k = 0, 1, 2, \dots$



Large n limit of $K_n^{(\alpha, \theta)}(x, y)$ 

Large n limit of $K_n^{(\alpha, \theta)}(x, y)$

- ▶ Hard edge scaling limits:

$$\begin{aligned} & \lim_{n \rightarrow \infty} K_n^{(\alpha, \theta)} \left(\frac{x}{n^{1/\theta}}, \frac{y}{n^{1/\theta}} \right) / n^{1/\theta} \\ &= \sum_{k,l=0}^{\infty} \frac{(-1)^k x^{\alpha+k}}{k! \Gamma \left(\frac{\alpha+1+k}{\theta} \right)} \frac{(-1)^l y^{\theta l}}{l! \Gamma(\alpha+1+\theta l)} \frac{\theta}{\alpha+1+k+\theta l} \\ &= \theta x^\alpha \int_0^1 J_{\frac{\alpha+1}{\theta}, \frac{1}{\theta}}(ux) J_{\alpha+1, \theta}((uy)^\theta) u^\alpha du, \end{aligned}$$

where

- ▷ $J_{a,b}(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j! \Gamma(a+bj)}$ is Wright's generalized Bessel function
[Borodin, '99]



About hard edge scaling limits

- ▶ Also appear in biorthogonal Jacobi and biorthogonal Hermite ensembles

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- ▶ $\theta = 1$: classical Bessel kernel



About hard edge scaling limits

- ▶ Also appear in biorthogonal Jacobi and biorthogonal Hermite ensembles
[Borodin, '99]
- ▶ $\theta = 1$: classical Bessel kernel
- ▶ $\theta = M \in \mathbb{N}$ or $1/\theta = M$: related to certain Meijer G-kernels encountered in the products of M Ginibre matrices
[Kuijlaars-LZ, '14], [Kuijlaars-Stivigny, '14]



Macroscopic behavior

- ▶ For a quite general weight $e^{-nV(x)}$:
 - ▷ An equilibrium problem in the one-cut case with and without hard edge for $\theta > 1 \longrightarrow$ [Claeys-Romano, '14]
 - ▷ A vector equilibrium problem for rational θ : \longrightarrow [Kuijlaars, '16]



Macroscopic behavior

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 - ▷ An equilibrium problem in the one-cut case with and without hard edge for $\theta > 1 \rightarrow$ [Claeys-Romano, '14]
 - ▷ A vector equilibrium problem for rational θ : \rightarrow [Kuijlaars, '16]
- ▶ Behavior of the equilibrium measure for Laguerre weight
 - ▷ Density function

$$f_\theta(x) = \frac{\theta}{2\pi xi} (I_+(x) - I_-(x)), \quad x \in (0, (1 + \theta)^{1+1/\theta}),$$

where $I_{\pm}(x)$ (with $\text{Im}(I_+(x)) > 0$) satisfy

$$J(z) = \theta(z + 1) \left(\frac{z + 1}{z} \right)^{1/\theta} = x$$



Macroscopic behavior

- ▶ Behavior of the equilibrium measure for Laguerre weight
 - ▷ The density blows up with a rate $x^{-1/(1+\theta)}$ near the origin (hard edge), while vanishes as a square root near $(1+\theta)^{1+1/\theta}$ (soft edge)



Macroscopic behavior

- ▶ Behavior of the equilibrium measure for Laguerre weight
 - ▷ The density blows up with a rate $x^{-1/(1+\theta)}$ near the origin (hard edge), while vanishes as a square root near $(1+\theta)^{1+1/\theta}$ (soft edge)
- ▶ More explicit description: connection to Fuss-Catalan distribution
 - ▷ Change of variables: $x_i \rightarrow \theta x_i^{1/\theta} \rightarrow$ distributed over $[0, (1+\theta)^{1+\theta}/\theta^\theta]$
 - ▷ The density function blows up with a rate $x^{-\theta/(1+\theta)}$ near the origin, and vanishes as a square root near $(1+\theta)^{1+\theta}/\theta^\theta$
 - ▷ The limiting mean distribution is recognized as the **Fuss-Catalan distribution**

[Forrester-Liu, '15]



Macroscopic behavior

► Fuss-Catalan distribution:

- ▷ k -th moment: $\frac{1}{(1+\theta)k+1} \binom{(1+\theta)k+k}{k}$, $k = 0, 1, 2, \dots$
- ▷ Limiting mean distribution for square singular values of products of M Ginibre random matrices ($\theta = M$)



Macroscopic behavior

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► The density functions ρ : with parametrization

$$x = \frac{(\sin((1+\theta)\varphi))^{1+\theta}}{\sin \varphi (\sin(\theta\varphi))^{\theta}}, \quad 0 < \varphi < \frac{\pi}{1+\theta},$$

one has

$$\rho(\varphi) = \frac{1}{\pi x} \frac{\sin((1+\theta)\varphi)}{\sin(\theta\varphi)} \sin \varphi = \frac{1}{\pi} \frac{(\sin \varphi)^2 (\sin(\theta\varphi))^{\theta-1}}{(\sin((1+\theta)\varphi))^{\theta}}$$

[Biane, '98; Haagerup-Möller, '13; Neuschel, '14]



Double integral formula for $K_n^{(\alpha,\theta)}(x,y)$

Theorem (LZ, J. Stat. Phys., '15)

We have the follow double integral formula for $K_n^{(\alpha,\theta)}(x,y)$:

$$\frac{\theta}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} ds \oint_{\Sigma} dt \frac{\Gamma(s+1)\Gamma(\alpha+1+\theta s)}{\Gamma(t+1)\Gamma(\alpha+1+\theta t)} \frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} \frac{x^{-\theta s-1}y^{\theta t}}{s-t},$$

where

$$c = \frac{\max\{0, 1 - \frac{\alpha+1}{\theta}\} - 1}{2} < 0,$$

and Σ is a closed contour going around $0, 1, \dots, n-1$ in the positive direction and $\operatorname{Re} t > c$ for $t \in \Sigma$



New representations of hard edge scaling limits

Corollary (LZ, J. Stat. Phys., '15)

With $\alpha \geq -1$, $\theta \geq 1$ being fixed, we have

$$\lim_{n \rightarrow \infty} K_n^{(\alpha, \theta)} \left(\frac{x}{n^{1/\theta}}, \frac{y}{n^{1/\theta}} \right) / n^{1/\theta} = K^{(\alpha, \theta)}(x, y),$$

where $K^{(\alpha, \theta)}(x, y)$ is given by

$$\frac{\theta}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} ds \oint_{\Sigma} dt \frac{\Gamma(s+1)\Gamma(\alpha+1+\theta s)}{\Gamma(t+1)\Gamma(\alpha+1+\theta t)} \frac{\sin \pi s}{\sin \pi t} \frac{x^{-\theta s-1} y^{\theta t}}{s-t}$$

and where Σ encircles the positive real axis positively and $\operatorname{Re} t > c$ for $t \in \Sigma$



Remarks:

- ▶ An equivalent double integral formula for $K_n^{(\alpha,\theta)}(x,y)$ and the same integral representations of $K^{(\alpha,\theta)}(x,y)$ were also obtained independently in

[Forrester-Wang, '15]



Bulk and soft edge universality

- ▶ Two classical universality classes

- ▷ sine kernel:

$$K_{\sin}(x, y) := \frac{\sin \pi(x - y)}{\pi(x - y)}$$

- ▷ Airy kernel:

$$K_{\text{Ai}}(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}$$

- ▶ Parametrization of the spectral argument:

$$x = \frac{(\sin((1 + \theta)\varphi))^{1+\theta}}{\sin \varphi (\sin(\theta\varphi))^{\theta}}, \quad 0 < \varphi < \frac{\pi}{1 + \theta}$$



Bulk universality

Theorem (LZ, J. Stat. Phys., '15)

For $x_0 \in (0, (1 + \theta)^{1+\theta}/\theta^\theta)$ with parametrization shown before, we have, with α, θ being fixed,

$$\lim_{n \rightarrow \infty} \frac{e^{-\pi\eta \cot \varphi}}{e^{-\pi\xi \cot \varphi}} \frac{1}{\rho(\varphi) x_0^{1-\frac{1}{\theta}}} K_n^{(\alpha, \theta)} \left(n\theta \left(x_0 + \frac{\xi}{n\rho(\varphi)} \right)^{\frac{1}{\theta}}, n\theta \left(x_0 + \frac{\eta}{n\rho(\varphi)} \right)^{\frac{1}{\theta}} \right) \\ = K_{\sin}(\xi, \eta),$$

uniformly for ξ and η in any compact subset of \mathbb{R} , where $\rho(\varphi)$ is the density function of Fuss-Catalan distribution



Soft edge universality

Theorem (LZ, J. Stat. Phys., '15)

We have

$$\lim_{n \rightarrow \infty} f(\xi, \eta) K_n^{(\alpha, \theta)} \left(n\theta \left(x_* + \frac{c_* \xi}{n^{\frac{2}{3}}} \right)^{\frac{1}{\theta}}, n\theta \left(x_* + \frac{c_* \eta}{n^{\frac{2}{3}}} \right)^{\frac{1}{\theta}} \right) = K_{\text{Ai}}(\xi, \eta)$$

uniformly valid for ξ and η in any compact subset of \mathbb{R} , where

$$x_* = \frac{(1+\theta)^{1+\theta}}{\theta^\theta}, \quad c_* = \frac{(1+\theta)^{\frac{2}{3}+\theta}}{2^{\frac{1}{3}}\theta^{\theta-1}} \quad \text{and}$$

$$f(\xi, \eta) = \frac{e^{-2^{-\frac{1}{3}}(1+\theta)^{\frac{2}{3}}\eta n^{\frac{1}{3}}}}{e^{-2^{-\frac{1}{3}}(1+\theta)^{\frac{2}{3}}\xi n^{\frac{1}{3}}}} \frac{(1+\theta)^{\frac{2}{3}+\frac{1}{\theta}}}{2^{\frac{1}{3}}} n^{\frac{1}{3}}$$

Local universality in biorthogonal Laguerre ensembles



Ideas of the proof

- ▶ Double contour integral representation of the correlation kernel

- ▷ Integral representation of $q_k^{(\alpha, \theta)}(x)$:

$$\frac{\Gamma(\alpha + 1 + k\theta)k!}{2\pi i} \oint_{\Sigma} \frac{\Gamma(t - k)x^t}{\Gamma(t + 1)\Gamma(\alpha + 1 + \theta t)} dt,$$

where Σ is a closed contour encircling $0, 1, \dots, k$ positively

- ▷ Integral representation of $x^\alpha e^{-x} p_k^{(\alpha, \theta)}(x)$:

$$\frac{1}{2\pi i \Gamma(\alpha + 1 + k\theta)k!} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{s}{\theta} + 1 - \frac{1}{\theta}\right)}{\Gamma\left(\frac{s}{\theta} + 1 - \frac{1}{\theta} - k\right)} \Gamma(\alpha + s)x^{-s} ds,$$

where $c > \max\{-\alpha, 1 - \theta\}$ and $x > 0$

- ▷ Integral representation of $K_n^{(\alpha, \theta)}(x, y)$: a telescoping sum



Ideas of the proof

- ▶ Hard edge scaling limit: straightforward calculation



Ideas of the proof

- ▶ Hard edge scaling limit: straightforward calculation
- ▶ Bulk and soft edge universality: a steepest descent analysis of the contour integral
 - ▷ two saddle points → bulk universality
 - ▷ coalescing of these two saddle points → soft edge universality



Integrable representation of $K^{(\alpha, \theta)}$



Integrable representation of $K^{(\alpha, \theta)}$

Theorem (LZ, '16)

Let $\theta = \frac{m}{n} \in \mathbb{Q}$. If $\alpha > m - 1 - \frac{m}{n} \geq -1$, we have

$$K^{(\alpha, \theta)}(x, y) = m^m n^{n-1} x^{m-1} \frac{\mathcal{B}\left(x^{\alpha+1-m} J_{\frac{\alpha+1}{\theta}, \frac{1}{\theta}}(x), J_{\alpha+1, \theta}(y^\theta)\right)}{x^m - y^m},$$

where $\mathcal{B}(\cdot, \cdot)$ is a bilinear operator defined by

$$(-1)^{n+1} \sum_{j=0}^{m+n-1} (-1)^j (\Delta_x)^j f(x) \sum_{i=0}^{m+n-1-j} \frac{b_{i+j}}{m^{i+j}} (\Delta_y)^i g(y)$$

Integrable representation of $K^{(\alpha, \theta)}$

Theorem (Cont.)

with $\Delta_x = x \frac{d}{dx}$ and $\Delta_y = y \frac{d}{dy}$. The constants b_i are determined by

$$\prod_{i=1}^{m+n-1} (x - \nu_i) = \sum_{i=0}^{m+n-1} b_i x^i,$$

with

$$\nu_i = \begin{cases} \frac{i}{n}, & i = 1, \dots, n-1, \\ 1 - \frac{\alpha}{m} - \frac{i-n+1}{m}, & i = n, n+1, \dots, m+n-1. \end{cases}$$



Integrable representation of $K^{(\alpha, \theta)}$

Theorem (Cont.)

Equivalently, one has

$$K^{(\alpha, \frac{m}{n})}(mn^{\frac{n}{m}}x^{\frac{1}{m}}, mn^{\frac{n}{m}}y^{\frac{1}{m}}) = \frac{x^{1-\frac{1}{m}} \tilde{\mathcal{B}}\left(G_{0,m+n}^{m,0}\left(\begin{array}{c}- \\ -\nu_{m+n-1}, \dots, -\nu_1, \nu_0\end{array} \middle| x\right), G_{0,m+n}^{n,0}\left(\begin{array}{c}- \\ \nu_0, \nu_1, \dots, \nu_{m+n-1}\end{array} \middle| y\right)\right)}{n^{\frac{n}{m}}(x-y)},$$

where $G_{0,m+n}^{n,0}$ is the Meijer G-function and $\tilde{\mathcal{B}}(\cdot, \cdot)$ is a bilinear operator defined by

$$(-1)^{n+1} \sum_{j=0}^{m+n-1} (-1)^j (\Delta_x)^j f(x) \sum_{i=0}^{m+n-1-j} b_{i+j} (\Delta_y)^i g(y)$$



Fredholm determinant

- ▶ $\mathcal{K}_{\alpha,\theta}$: the integral operator with kernel $K^{(\alpha,\theta)}(x,y)\chi_J(y)$ acting on $L^2((0,\infty))$
 - ▷ $\det(I - \mathcal{K}_{\alpha,\theta})$: gap probability on the interval J



Fredholm determinant

- ▶ $\mathcal{K}_{\alpha,\theta}$: the integral operator with kernel $K^{(\alpha,\theta)}(x,y)\chi_J(y)$ acting on $L^2((0,\infty))$
 - ▷ $\det(I - \mathcal{K}_{\alpha,\theta})$: gap probability on the interval J
- ▶ We take

$$J = \bigcup_{j=1}^{\ell} (mn^{\frac{n}{m}} a_{2j-1}^{\frac{1}{m}}, mn^{\frac{n}{m}} a_{2j}^{\frac{1}{m}}), \quad 0 \leq a_1 < a_2 < \dots < a_{2\ell}$$



Fredholm determinant

- ▶ A change of variable shows that

$$\det \left(I - \mathcal{K}_{\alpha, \frac{m}{n}} \right) = \det \left(I - \tilde{\mathcal{K}}_{\alpha, \frac{m}{n}} \right),$$

where $\tilde{\mathcal{K}}_{\alpha, \frac{m}{n}}$ is an integral operator with kernel

$$\begin{aligned} & \tilde{K}^{(\alpha, \frac{m}{n})}(x, y) \chi_{\tilde{J}}(y) \\ &= \frac{\tilde{\mathcal{B}} \left(G_{0, m+n}^{m,0} \left(\begin{array}{c} - \\ -\nu_{m+n-1}, \dots, -\nu_1, \nu_0 \end{array} \middle| x \right), G_{0, m+n}^{n,0} \left(\begin{array}{c} - \\ \nu_0, \nu_1, \dots, \nu_{m+n-1} \end{array} \middle| y \right) \right)}{x - y} \chi_{\tilde{J}}(y) \\ &= \frac{\sum_{i=0}^{m+n-1} \phi_i(x) \psi_i(y)}{x - y} \chi_{\tilde{J}}(y), \end{aligned}$$

and $\tilde{J} = \bigcup_{j=1}^{\ell} (a_{2j-1}, a_{2j})$



The system of partial differential equations

- For $j = 0, 1, \dots, m+n-1$ and $k = 1, \dots, 2\ell$, we set

$$x_{j,k} := (I - \tilde{\mathcal{K}}_{\alpha, \frac{m}{n}})^{-1} \phi_j(a_k), \quad y_{j,k} = (I - \tilde{\mathcal{K}}'_{\alpha, \frac{m}{n}})^{-1} \psi_j(a_k),$$

$$u_j := (-1)^n \int_J \phi_0(x) (I - \tilde{\mathcal{K}}'_{\alpha, \frac{m}{n}})^{-1} \psi_j(x) dx + b_{j-1},$$

$$v_j := (-1)^n \int_J \phi_j(x) (I - \tilde{\mathcal{K}}'_{\alpha, \frac{m}{n}})^{-1} \psi_{m+n-1}(x) dx,$$

where $\tilde{\mathcal{K}}'_{\alpha, \frac{m}{n}}$ is the integral operator with kernel
 $\tilde{\mathcal{K}}^{(\alpha, \frac{m}{n})}(y, x)\chi_J(y)$



The system of partial differential equations

- ▶ All these quantities are actually functions of $a = (a_0, \dots, a_{2\ell})$, and they satisfy a system of PDEs
 - ▷ For $1 \leq k \neq i \leq 2\ell$ and $0 \leq j \leq m + n - 1$, we have

$$\frac{\partial x_{j,k}}{\partial a_i} = (-1)^i \frac{x_{j,i}}{a_k - a_i} \sum_{l=0}^{m+n-1} x_{l,k} y_{l,i},$$

$$\frac{\partial y_{j,k}}{\partial a_i} = (-1)^i \frac{y_{j,i}}{a_i - a_k} \sum_{l=0}^{m+n-1} x_{l,i} y_{l,k}$$

▷ ...

Gap probability on a single interval $(0, s)$

Proposition (LZ, '16)

For $s > 0$, let

$$F_{\alpha, \frac{m}{n}}(s) := \det \left(I - \mathcal{K}_{\alpha, \frac{m}{n}} \Big|_{(0,s)} \right),$$

Then, $F_{\alpha, \frac{m}{n}}(s)$ is given by

$$\begin{aligned} & \exp \left(\frac{(-1)^n}{m^{m-2} n^n} \int_0^s \log \left(\frac{s}{\tau} \right) \tau^{m-1} x_0 \left(\frac{\tau^m}{m^m n^n} \right) y_{m+n-1} \left(\frac{\tau^m}{m^m n^n} \right) d\tau \right) \\ &= \exp \left(m \int_0^s \frac{v_0 \left(\frac{\tau^m}{m^m n^n} \right)}{\tau} d\tau \right), \end{aligned}$$



Gap probability on a single interval $(0, s)$

Proposition (Cont.)

where x_0, y_{m+n-1} and v_0 satisfy

$$s \frac{dx_j(s)}{ds} = -v_j(s)x_0(s) - x_{j+1}(s), \quad 0 \leq j \leq m+n-2,$$

$$s \frac{dx_{m+n-1}(s)}{ds} = ((-1)^{n+1}s - v_{m+n-1}(s))x_0(s) + \sum_{i=0}^{m+n-1} u_i(s)x_i(s),$$

$$s \frac{dy_j(s)}{ds} = y_{j-1}(s) - u_j(s)y_{m+n-1}(s), \quad 1 \leq j \leq m+n-1,$$

$$s \frac{dy_0(s)}{ds} = \sum_{i=0}^{m+n-1} v_i(s)y_i(s) + ((-1)^n s - u_0(s))y_{m+n-1}(s),$$



Gap probability on a single interval $(0, s)$

Proposition (Cont.)

and

$$\frac{du_j(s)}{ds} = (-1)^n x_0(s) y_j(s), \quad 0 \leq j \leq m+n-1,$$

$$\frac{dv_j(s)}{ds} = (-1)^n x_j(s) y_{m+n-1}(s), \quad 0 \leq j \leq m+n-1,$$

+ some initial conditions. In addition, one has

$$\log F_{\alpha, \frac{m}{n}}(s) \sim -\frac{m}{n(\alpha+1)\Gamma(\alpha+2)\Gamma\left(\frac{(\alpha+1)n}{m}\right)} s^{\alpha+1}, \quad s \rightarrow 0$$



Remarks

- ▶ Integrable structure: $J_{a,b}$ satisfies an ordinary differential equation for rational b



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- ▶ The system of PDEs: similar to many classical kernels
[Jimbo-Miwa-Môri-Sato '80; Tracy-Widom '94; Strahov '14]
 - ▷ Introduction of $\tilde{\mathcal{K}}'_{\alpha, \frac{m}{n}}$



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[Jimbo-Miwa-Môri-Sato '80; Tracy-Widom '94; Strahov '14]
 - ▷ Introduction of $\tilde{\mathcal{K}}'_{\alpha, \frac{m}{n}}$
- ▶ Gap probability on $(0, s)$: differential equation for v_0
 - ▷ $\theta = 1$: a particular Painlevé III system → [Tracy-Widom '94]
 - ▷ $\theta = 2$: a 4-th order ODE → [Witte-Forrester '16]
 - ▷ General θ : open



Thanks for your attention!

