Products of large random matrices: Old laces and new pieces

Maciej A. Nowak

Mark Kac Complex Systems Research Center,
Marian Smoluchowski Institute of Physics,
Jagiellonian University, Kraków, Poland

August 22th, 2016
Random Product Matrices
University of Bielefeld

Supported in part by the grant DEC-2011/02/A/ST1/00119 of National Centre of Science.

Outline and acknowledgements

- **Old laces**
  Ewa Gudowska-Nowak (UJ), Romuald A. Janik (UJ), Jerzy Jurkiewicz (UJ)
  Nucl. Phys. **B670** (2003) 479, New J. Phys. **7** (2005) 54
  Jean-Paul Blaizot (IPhT Saclay), Zdzisław Burda (AGH, Kraków), Jacek Grela (UJ), Wojciech Tarnowski (UJ), Piotr Warchoł (UJ) , Yizhuang Liu and Ismail Zahed (Stony Brook), Pierpaolo Vivo (King's College)
  J-PB,JG,MAN,WT,PW; JSTAT **5** (2016) 54037 and references therein.
- **New pieces**
  Serban Belinschi (Toulouse U.), MAN (UJ), Roland Speicher (Saarland U.), Wojciech Tarnowski (UJ)
  arXiv:1608.04923

Quincunx

Original Galton's quincunx, University College, London

Matricial quincunx?

- Additive (algebraic) random walk: Galton board a.k.a "bean machine"
- Multiplicative (geometric) random walk: take a log of the "bean", since $e^x \cdot e^y = e^{x \cdot y}$
- Matricial additive random walk: make an array of "bean machines" and look at the flow of eigenvalues
- Matricial multiplicative random walk ???

  Lace piece for random product matrices !!!

  [Lace piece - (Shipbuilding) the main piece of timber which supports the beak or head projecting beyond the stem of a ship.]

Matricial Multiplicative Random Walk

- $N = 2$ case [Jackson,Lautrup,Johansen,Nielsen;2002]
  $Y_2(\tau) = \prod_{i=1}^{M}(1 + \sigma\sqrt{\frac{\tau}{M}}X_i)$, where X_i represent two by two complex Gaussian matrices
- $N = \infty$ case [Gudowska-Nowak,Janik,Jurkiewicz,MAN;2003]
  $Y_\infty(\tau) = \prod_{i=1}^{M}(1 + \sigma\sqrt{\frac{\tau}{M}}X_i)$, where X_i represent mutually free Ginibre matrices

Matricial Multiplicative Random Walk: Tools

- Complex spectrum - generalized Green's functions
- Strings of matrices $X_1 .... X_M$ - linearization trick

Non-hermitian case - large N - electrostatic analogy

Analytic methods break down, since spectra are complex $\rho(z) = \frac{1}{N}\left\langle \sum_i \delta^{(2)}(z-\lambda_i)\right\rangle$.

- Electrostatic potential [Girko;1984],[Brown;1986],[Sommers et al.;1988]
  $\phi(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\ln[|z-X|^2+\epsilon^2]\right\rangle$
- Green's function (electric field)
  $g = \partial_z\phi = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\frac{\bar{z}-X^\dagger}{|z-X|^2+\epsilon^2}\right\rangle$
- Gauss law $\rho(z,\tau) = \frac{1}{\pi}\partial_{\bar{z}} g|_{\epsilon=0} = \frac{1}{\pi}\frac{\partial^2\phi}{\partial z\partial\bar{z}}|_{\epsilon=0}$
  Proof: $\delta^{(2)}(z) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon^2}{(|z|^2+\epsilon^2)^2}$

"Generalized Green's functions" [Janik, MAN, Papp, Zahed; 1997]

- $\phi(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\ln[|z-X|^2+\epsilon^2]\right\rangle$
  $= \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\ln D_N\right\rangle$ where
  $D_N(z,\bar{z},\epsilon) = \det(Z\otimes \mathbf{1}_N - \mathcal{X})$ with
  $$Z = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$$
- $\mathcal{G}(z,\bar{z}) = \frac{1}{N}\left\langle \mathrm{btr}\frac{1}{(Z-\mathcal{X})}\right\rangle = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix}$
  $$\mathrm{btr}\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{2N\times 2N} \equiv \begin{pmatrix} \mathrm{tr}A & \mathrm{tr}B \\ \mathrm{tr}C & \mathrm{tr}D \end{pmatrix}_{2\times 2}$$
- $\mathcal{G}_{11} = g(z,\bar{z})$ yields spectral function
- $\mathcal{G}_{1\bar{1}}\cdot\mathcal{G}_{\bar{1}1}$ yields elements of a certain eigenvector correlator [Janik, Noerenberg, MAN, Papp, Zahed;1999].

Linearization trick [Gudowska-Nowak, Janik, Jurkiewicz, MAN; 2003]

- Exact relation between the eigenvalues of a products of N by N matrices $A = X_1 X_2 .. X_M$ and the eigenvalues of a block matrix

$$\mathcal{X} = \begin{pmatrix} 0 & X_1 & 0 & \cdots & 0 \\ 0 & 0 & X_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & X_{M-1} \\ X_M & 0 & 0 \cdots & 0 & 0 \end{pmatrix}$$

If $\{\lambda_i\}$ are eigenvalues of A, then the eigenvalues of block matrix are $\{\lambda_i^{1/M} e^{2\pi i k/M}\}$, where $k = 0, 1, \ldots, M-1$.

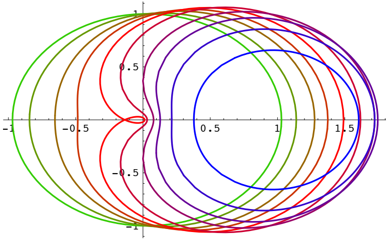
Combining both tricks

- We define auxiliary $2MN$ by $2MN$ Green's function of the form
$$\mathcal{G}(u) = \left\langle \left(\begin{array}{cc} \mathcal{U} & -\sqrt{\tau/M}\mathcal{X} \\ -\sqrt{\tau/M}\mathcal{X}^\dagger & \mathcal{U}^\dagger \end{array} \right)^{-1} \right\rangle$$
where
$$\mathcal{U} = \left(\begin{array}{ccccc} u & -1 & 0 & \cdots & 0 \\ 0 & u & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u & -1 \\ -1 & 0 & 0\cdots & 0 & u \end{array} \right)$$
where $u^M = z$.
- Spectral density follows from
$\rho(x,y) = \frac{1}{\pi}\frac{1}{M}\frac{u\bar{u}}{z\bar{z}}\partial_{\bar{u}}\left[\frac{1}{N}\mathrm{Tr}_{BM}\mathcal{G}(u)\right]_{11}$

Features of the solution

Using diagrammatics and (heavily) properties of the cyclic matrices we solved in 2003 the problem for any M, including $M = \infty$. Highlights of the solution are:

- Topological phase transition for finite τ.
- Hole never vanishes, leading to singularity at the origin in the large τ limit
- Links to Burgers equation



Verifications and extensions of the results

- Topological phase transition confirmed [Lohmeyer, Neuberger, Wettig; 2008], link to free Segal-Bargmann transformation [Biane; 1997]
- The large τ limit - radial symmetry, relation between SV for product of Ginibres and product of Wishart ensembles, singularity from Haagerup-Larsen (single ring theorem), finite N unfolding (G-Meijer kernels) [see Random Product Matrices participation list]
- Links to Burgers equations (spectral and eigenvalue shock waves in RMM, $1/N$ as a viscosity)[Kraków & friends group]
- Is anything left undone in the realm of random product matrices?

Loophole in the standard arguments

- For non-hermitian matrices X, we have left and right eigenvectors $X = \sum_k \lambda_k |R_k><L_k|$ where $X|R_k>= \lambda_k |R_k>$ and $<L_k|X = \lambda_k <L_k|$
- $<L_j|R_k>= \delta_{jk}$, but $<L_i|L_j>\neq 0$ and $<R_i|R_j>\neq 0$.
- $D_N = \det(Z - \mathcal{X}) = \det[A^{-1}(Z-\mathcal{X})A] =$ $\det \begin{pmatrix} z\mathbf{1}_N - \Lambda & -i\epsilon <L|L> \\ i\epsilon <R|R> & \bar{z}\mathbf{1}_N - \bar{\Lambda} \end{pmatrix}$
- Spectrum (Λ) entangled with diagonal part of the overlap of eigenvectors $O_{ij} \equiv <L_i|L_j><R_j|R_i>$.
- Naive limit $\epsilon \to 0$ kills the entanglement leading to incomplete description of the non-hermitian RM

Cure: Hidden variable

We promote $i\epsilon$ to full, complex-valued dynamical variable.
Then, "orthogonal direction" w unravels the eigenvector correlator $O(z) = \frac{1}{N^2} \left\langle \sum_k O_{kk} \delta^{(2)}(z - \lambda_k) \right\rangle$, where $O_{ij} =< L_i | L_j >< R_j | R_i >$ and $|L_i >$ ($|R_i >$) are left (right) eigenvectors of X. [Savin, Sokolov; 1997], [Chalker,Mehlig; 1997]
Replacing $Z = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix}$ by **quaternion** $Q = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$ provides algebraic generalization of free random variables calculus for nonhermitian RMM: $\mathcal{R}[\mathcal{G}(Q)] + \mathcal{G}^{-1}(Q) = Q$
[Janik,MAN,Papp, Zahed; 1997], [Feinberg, Zee; 1997],[Jarosz, MAN; 2006], [Belinschi, Sniady, Speicher; 2015].

"Quaternionization trick"

- $\phi(z,w) \equiv \lim_{N\to\infty} \left\langle \frac{1}{N}\mathrm{tr} \ln[|z-X|^2 + w\bar{w}]\right\rangle$
- $\mathcal{G}(z,w) = \frac{1}{N}\left\langle \mathrm{btr}\frac{1}{(Q-\mathcal{X})}\right\rangle = \begin{pmatrix} \partial_z\phi & \partial_w\phi \\ -\partial_{\bar{w}}\phi & \partial_{\bar{z}}\phi \end{pmatrix}$
- On top of the vector "electric field" $G(z,w) = \partial_z\phi$ we have second vector, "velocity" field $V(z,w) = \partial_w\phi$.
- $\rho(z) = \frac{1}{\pi}\partial_{\bar{z}}G(z,0) = \frac{1}{\pi}\partial_{z\bar{z}}\phi(z,0)$ gives spectral function
- $O(z) = \frac{1}{\pi}|V(z,0)|^2$ yields eigenvector correlator
- $\partial_w G(z,w) = \partial_z V(z,w)$ so both vector fields are intertwined.

Simplified proof [Belinschi, MAN, Speicher, Tarnowski; 2016]

- $\det(Q - \mathcal{X}) = \det \begin{pmatrix} z\mathbf{1}_N - \Lambda & -\bar{w} < L|L > \\ w < R|R > & \bar{z}\mathbf{1}_N - \bar{\Lambda} \end{pmatrix}$
  $\det(A + F) = \det A + \det F + S_1 + S_2 + \ldots + S_{2N-1}$, where

$$S_k = \sum_{1 \le i_1 < \ldots < i_k \le 2N} \frac{\det A}{a_{i_1} \ldots a_{i_k}} \det F_{i_1, \ldots, i_k},$$

  a_i denotes the i-th element on the diagonal of A, and $F_{i_1,\ldots,i_k}$ is a $k \times k$ submatrix the element of which lie at the intersections of $i_1, \ldots, i_k$-th rows and columns.

- In the $|w| \to 0$ limit, the dominant term in the expansion is

$$S_2 = \sum_{k,l=1}^{N} \frac{\det A}{(z - \lambda_k)(\bar{z} - \bar{\lambda}_l)} \det \begin{pmatrix} 0 & -\bar{w} < L_k|L_l > \\ w < R_l|R_k > & 0 \end{pmatrix}$$

Proof cont.

- $S_2 = \frac{\det A}{|z-\lambda_i|^2}|w|^2 O_{ii} + |w|^2 \mathcal{O}(|z-\lambda_i|)$
- Since $\partial_w \log\det(Q-\mathcal{X}) = \frac{\bar{w}}{\frac{|z-\lambda_i|^2}{O_{ii}}+|w|^2}$, therefore (for any N)

$$\pi O(z,\bar{z}) = \lim_{|w|\to 0} < \frac{1}{N^2}\partial_w \log\det(Q-\mathcal{X})\partial_{\bar{w}} \log\det(Q-\mathcal{X}) >$$

- Large N:

$$\pi O(z,\bar{z}) = \lim_{|w|\to 0} \partial_w \Phi \partial_{\bar{w}} \Phi = -\mathcal{G}_{1\bar{1}}\mathcal{G}_{\bar{1}1}\big|_{|w|=0}.$$

Why bother about $O(z, \bar{z})$?

- Sensitivity to perturbation of non-normal operators $X(\epsilon) = X + \epsilon P$.
- $|\lambda_i(\epsilon) - \lambda_i(0)| = |\epsilon < L_i|P|R_i > | \leq |\epsilon|\sqrt{< L_i|L_i >< R_i|R_i >}||P||$,
- The bound is reached if the perturbation is of rank one $P = |L_i >< R_i|$ (Wilkinson matrix)
- $\sqrt{O_{ii}}$ is known in the numerical analysis community as the eigenvalue condition number
- In RMT $c(z, \bar{z}) = \mathbb{E}P\left(O_{ii}N^{-1}|\lambda_i = z\right) = \int \frac{O_{ii}}{N} \frac{p(O_{ii},\lambda_i=z)}{p(\lambda_i=z)} dO_{ii} = \frac{1}{\rho(z,\bar{z})} \int \frac{O_{ii}}{N} \delta^{(2)}(z - \lambda_i) p(X) dX = \frac{O(z,\bar{z})}{\rho(z,\bar{z})}$ is known as a Petermann factor (excess noise factor), reflecting the non-orthogonality of the cavity modes in open chaotic scattering [Beenakker, Berry, Fyodorov, Savin, Sokolov, Chalker-Mehlig]
- Crucial element in free and Ornstein-Uhlenbeck Ginibre processes [Blaizot, Burda, Grela, MAN, Tarnowski, Warchoł; 2015-2016]

Single ring (Haagerup-Larsen theorem)

- R-diagonal operator reads $X = PU$, where U is Haar and P positive and P, U are mutually free. Then
- **H-L th.**: For R-diagonal operators, cumulative radial spectral distribution $F(r)$ follows simply from $S_{p^2}(F(r) - 1) = \frac{1}{r^2}$ and the spectrum $\rho(r) = \frac{1}{2\pi r} \frac{dF}{dr}$ is spanned between two circles with known radii r_{inner} (from $F(r) = 0$) and r_{outer} (from $F(r) = 1$).
- R-diagonal operators are non-normal. Does it mean that there is an "eigenvector" part in Haagerup-Larsen theorem?
- **H-L th.**+: [Belinschi, MAN, Speicher, Tarnowski; 2016] Eigenvector correlator reads $O(r) = \frac{1}{\pi r^2} F(r)(1 - F(r))$
- Proof borrows heavily from formalism of [Belinschi, Speicher, Śniady; 2015].

Example: Product of n free Ginibres

$O_{Y,n}(r) = \frac{1-r^{2/n}}{\pi r^{2-2/n}}\theta(1-r)$, For completeness we note that $\rho_{Y,n}(r) = \frac{1}{\pi n} r^{\frac{2}{n}-2}\theta(1-r)$. Interestingly, even for the case $n=1$ the result for $O_{Y,1}(r) = \pi^{-1}(1-r^2)\theta(1-r)$ was obtained first time 33 years after the spectral density result $\rho_{Y,1} = \pi^{-1}\theta(1-r)$ derived in the seminal paper by Ginibre. Figure confronts our prediction with the numerical calculations.

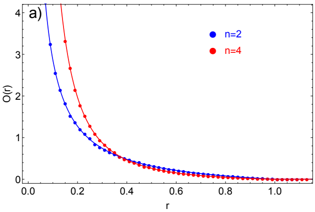


Figure: A numerical simulation (dots) of the eigenvector correlator for the product of $n = 2, 4$ complex Ginibre matrices of the size 1000 by 1000, averaged over the sample of 2000 matrices. The solid lines represent the analytic prediction.

Example: Product of n truncated unitary matrices

$Y = X_1 X_2 ... X_n$, where X_i are truncated unitary matrices, i.e. Haar matrices of the size $(N+L) \times (N+L)$, in which L columns and rows are removed. In the limit where both L and N tend to infinity in such a way that $\kappa = L/N$ is fixed, $F_Y = \kappa \frac{r^{2/n}}{1-r^{2/n}}$ for $r < (1+\kappa)^{-n/2}$ and 1 otherwise, hence $O_{Y,n,\kappa}(r) = \frac{\kappa}{\pi} \frac{1-r^{2/n}(1+\kappa)}{r^{2(n-1)/n}(1-r^{2/n})^2} \theta\left(\frac{1}{(1+\kappa)^{n/2}} - r\right)$. Figure confronts our prediction with the numerical calculations.

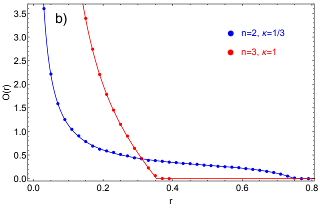


Figure: $n = 2, 3$ truncated unitary matrices of the size $L = 1000$, $N = \kappa L$ done on the samples of 2000 matrices. The solid lines represent the analytic prediction.

Example: Ratios of Ginibre's

Let us consider k Ginibre ensembles X_i and the same number of inverse Ginibre ensembles $\tilde{X}_i^{-1}$. Defining Y as the product $Y = X_1 \dots X_k \tilde{X}_1^{-1} \dots \tilde{X}_k^{-1}$, we get a rather unexpected result: $O_{Y,k}(r) = \frac{1}{\pi r^2} \frac{r^{\frac{2}{k}}}{(1+r^{2/k})^2} = k\rho_{Y,k}(r)$. Figure confronts our result for the eigenvector correlator with numerical simulations.

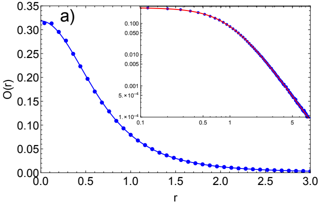


Figure: The eigenvector correlator calculated by a numerical diagonalization of 4000 matrices that are ratio of two Ginibres (also known as the spherical ensemble) of size $N = 1000$ presented on linear and double logarithmic (inset) scales.

Example: Finite N Ginibre

Combining results of [Chalker, Mehlig;1998] and [Walters, Starr; 2016] $c(z,\bar{z}) = 1 - |z|^2 + \frac{1}{N}\frac{e^{-N|z|^2}(N|z|^2)^N}{\Gamma(N,N|z|^2)}$

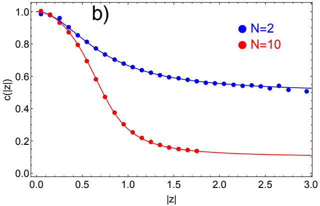


Figure: Mean eigenvalue condition number of the Ginibre ensemble. The complex plane was divided into the hollowed cylinders of radii r and $r + \Delta r$, eigenvalues and their condition numbers were assigned to cylinders, according to the modulus of the eigenvalue. The dots denote the average eigenvalue condition number within each cylinder, the lines present formula above formula. Numerical distribution was obtained by the diagonalization of 10^6 matrices of size $N = 2$ and $4 \cdot 10^5$ matrices of size $N = 10$.

Conclusions and open problems

- Paradigm shift: This is the time of eigenvectors.
- Several open problems: Finite N formalism, arbitrary β, role of the off-diagonal O_{ik}, hierarchy of "Ward identities" from bi-unitarity invariance, combinatorics, beyond radial symmetry, microscopic properties....
- Applications in the theory of stability