A model of coupled positive matrices; universality results and conjectures

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Abstract Universality results in the statistical behaviour of spectra of random matrices typically consider the fluctuations near a macroscopic point of the spectrum. In an ensemble of positive definite matrices, there are two essentially different distinguished points; the largest eigenvalues and the smallest. Since they are positive definite the study of the smallest eigenvalue is the study of the statistics near the origin of the spectral axis.
The simplest instance is the Laguerre ensemble; in this case the fluctuations are described by a determinant random point field defined in terms of the celebrated Bessel kernel. This behaviour is also proven in the literature to be "universal".
We consider a model of several coupled matrices of Laguerre type (with a specific interaction) and address the corresponding study of the origin of the spectrum. We find a natural generalization of the Bessel random point field to a multi-specie analog that involves special functions (Meijer-G). It is then natural to formulate a universality conjecture. Time permitting I will also discuss results concerning the expectation of ratios of characteristic polynomials

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The Cauchy chain of positive matrices

Definition 1 (The p-chain-Cauchy Matrix-Model)

Let $\mathcal{M}_{n,+}^{(p)}$ be the set of p–tuples of *positive semidefinite* Hermitean matrices with the following class of measures

$$\mathrm{d}\mu(\vec{M}) = \mathcal{Z}\,\frac{\prod_{j=1}^{p} \det(M_j)^{a_j} \mathrm{e}^{-N\mathrm{Tr}\,U_j(M_j)}\mathrm{d}M_j}{\prod_{j=1}^{p-1} \det(M_j + M_{j+1})^n}$$

The scaling parameter N is taken proportional to the size when considering the limit of large sizes $n \to \infty$.

Theorem 2

The eigenvalues $\mathfrak{X}_j = (x_{j,1}, \ldots, x_{j,n})$ $j = 1, 2, \ldots, p$ *form a* **(multi-specie) determinantal point field**

The Cauchy matrix chain (B.Gekhtman-Szmigielski 09 [3]) can be reduced to spectral variables (Mehta-Shubin 94)

$$P(X_1, \ldots, X_p) = \frac{\Delta(X_1)\Delta(X_p)}{\mathcal{Z}_n} \prod_{m=1}^{p}\prod_{j=1}^{n} e^{-U_m(x_{mj})}\mathrm{d}x_{mj} \prod_{\ell=1}^{p-1} \det\left[\frac{1}{x_{\ell j} + x_{\ell+1,k}}\right]_{j,k=1}^{n}$$

$$X_\ell = (x_{\ell,1}, \ldots, x_{\ell,n}) = \text{ Spectrum of } M_\ell$$

A general angular integral

The reduction to the eigenvalue formula above relies upon the following

Theorem 3 (`Gross--Richards '89, Harnad--Orlov, '06`)

$$\int_{U(n)} \det(\mathbf{1} - z\, AUBU^\dagger)^{-r} \mathrm{d}U \propto \frac{\det\left[(1 - za_i b_j)^{n-r-1}\right]_{1\le i,j\le n}}{\Delta(A)\Delta(B)}$$

Set $A = X^{-1}, B = Y,\ r = n,\ z = -1$, multiply by $\det(X)^{-n}$

Corollary 4

$$\int_{U(n)} \frac{\mathrm{d}U}{\det\left(X + UYU^\dagger\right)^n} \propto \frac{\det\left[\frac{1}{x_i + y_j}\right]_{i,j}}{\Delta(X)\Delta(Y)}$$

The correlation functions are expressed as a determinant (Eynard-Mehta '98 for the Itzykson–Zuber case (but essentially general), B-Bothner '14)

$$\mathcal{R}^{(\ell_1,\dots,\ell_p)}(\{x_{1j}\}_1^{\ell_1},\dots,\{x_{pj}\}_1^{\ell_p}) = \left[\prod_{j=1}^{p}\frac{n!}{(n-\ell_j)!}\right]\frac{1}{\mathcal{Z}_n}$$

$$\times\int_{\mathbb{R}_+^{n-\ell_1}}\cdots\int_{\mathbb{R}_+^{n-\ell_p}} P(\{x_{1j}\}_1^n,\dots,\{x_{pj}\}_1^n)\prod_{j=1}^{p}\prod_{m_j=\ell_j+1}^{n}\mathrm{d}x_{jm_j}$$

$$= \det\left[\begin{array}{ccc} \boxed{\begin{array}{c}\mathbb{K}_{11}(x_{1r},x_{1s})\\ 1\le r\le\ell_1, 1\le s\le\ell_1\end{array}} & \cdots & \boxed{\begin{array}{c}\mathbb{K}_{1p}(x_{1r},x_{ps})\\ 1\le r\le\ell_1, 1\le s\le\ell_p\end{array}} \\ \vdots & \ddots & \vdots \\ \boxed{\begin{array}{c}\mathbb{K}_{p1}(x_{pr},x_{1s})\\ 1\le r\le\ell_p, 1\le s\le\ell_1\end{array}} & \cdots & \boxed{\begin{array}{c}\mathbb{K}_{pp}(x_{pr},x_{ps})\\ 1\le r\le\ell_p, 1\le s\le\ell_p\end{array}} \end{array}\right]_{(\sum_1^p \ell_i)\times(\sum_1^p \ell_i)},$$

with correlation kernels

$$\mathbb{K}_{j\ell}(x,y) = e^{-\frac{1}{2}U_j(x)-\frac{1}{2}U_\ell(y)}\mathbb{M}_{j\ell}(x,y),\quad \mathbb{M}_{p1}(x,y) = \sum_{\ell=0}^{n-1}\phi_\ell(x)\psi_\ell(y)\frac{1}{h_\ell}$$

These can be expressed in terms of iterated integrals of certain *biorthogonal polynomials* $\{\phi_\ell,\psi_\ell\}$;

Definition 5

The monic (Cauchy) biorthogonal polynomials $\{\psi_n(x), \phi_n(x)\}_{n\geq 0}$ are defined by the requirements

$$\int_0^\infty \int_0^\infty \psi_n(x)\phi_m(y)\eta_p(x,y)\,\mathrm{d}x\mathrm{d}y = h_n\delta_{nm} \tag{1}$$

$$\psi_n(x) = x^n + \mathcal{O}\left(x^{n-1}\right), \qquad x\to\infty; \qquad \phi_n(x) = x^n + \mathcal{O}\left(x^{n-1}\right), \qquad x\to\infty.$$

with

$$\eta_p(x,y) \quad = \quad \int_{\mathbb{R}_+^{p-2}} \frac{w_1(x)}{x+\xi_1}\left(\frac{\prod_{j=2}^{p-1} w_j(\xi_j)}{\prod_{j=1}^{p-3}(\xi_j+\xi_{j+1})}\right)\frac{w_p(y)}{\xi_{p-2}+y}\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{p-2}.$$

The pair $\{\psi_n(x), \phi_n(x)\}, n\geq 1$ is given in terms of determinants:

$$I_{j\ell} = \int_0^\infty \int_0^\infty x^j y^\ell \eta_p(x,y)\,\mathrm{d}x\mathrm{d}y, \qquad j,\ell\geq 0 \tag{2}$$

In terms of (2), the biorthogonal polynomials can be written as

$$\psi_n(x) = \frac{1}{\Delta_n}\det\left[I_{j\ell}\,|\,x^j\right]_{j,\ell=0}^{n,n-1}, \qquad \phi_n(y) = \frac{1}{\Delta_n}\det\begin{bmatrix} I_{j\ell} \\ y^\ell \end{bmatrix}_{j,\ell=0}^{n-1,n}; \qquad \Delta_n = \det[I_{j\ell}]_{j,\ell=0}^{n-1}.$$

Example: the Cauchy-Laguerre 2–chain

$$\mathrm{d}\mu(M_1, M_2) = \frac{1}{\mathcal{Z}_n} \frac{\det(M_1)^{a_1} \det(M_2)^{a_2} \mathrm{e}^{-n\mathrm{Tr}(M_1+M_2)}}{\det(M_1 + M_2)^n}$$

Serendipity allows to find explicitly the correlation functions for any size n so that the limit as $n \to \infty$ becomes a simple inspection [3]

$$R^{(k,\ell)}(x_1, \ldots, x_k; y_1, \ldots, y_\ell) = \det \left[\begin{array}{c|c} \underset{1\le i,j\le k}{\mathbb{K}_{01}^{(n)}(x_i, x_j)} & \underset{1\le i\le k;1\le j\le \ell}{\mathbb{K}_{00}^{(n)}(x_i, y_j)} \\ \hline \underset{1\le i\le \ell;1\le j\le k}{\mathbb{K}_{11}^{(n)}(y_i, x_j)} & \underset{1\le i,j\le \ell}{\mathbb{K}_{10}^{(n)}(y_i, y_j)} \end{array} \right]$$

Theorem 6 (B.-Gekhtman-Szmigielski '14)

$$\mathbb{K}_{00}^{(n)}(x,y) = \int_0^1 H_{a_1,n}(tx)H_{a_2,n}(ty)\mathrm{d}t \qquad \mathbb{K}_{01}^{(n)}(x,x') = \int_0^1 H_{a_1,n}(tx)\widetilde{H}_{a_1,n}(tx')\mathrm{d}t$$

$$\mathbb{K}_{10}^{(n)}(y',y) = \int_0^1 \widetilde{H}_{a_1,n}(ty')H_{a_2,n}(ty)\mathrm{d}t \qquad \mathbb{K}_{11}^{(n)}(y,x) = \int_0^1 \widetilde{H}_{a_1,n}(ty)\widetilde{H}_{a_2,n}(tx)\mathrm{d}t - \frac{1}{x+y}$$

$$H_{c,n}(z) := \int_\gamma \frac{\mathrm{d}u}{2i\pi} \frac{z^{-u}}{\Gamma(1-u)} \frac{\Gamma(u+c)\Gamma(n+a_1+a_2-c+1-u)}{\Gamma(c+n+u)\Gamma(a_1+a_2-c-u+1)}$$

$$\widetilde{H}_{c,n}(w) := \int_\gamma \frac{\mathrm{d}u}{2i\pi} \frac{\Gamma(u)\Gamma(u+c)\Gamma\left(n+a_1+a_2-c+1-u\right)}{\Gamma(a_1+a_2-c+1-u)\Gamma\left(c+n+u\right)} w^{-u}$$

Note

The kernels have the same (multiplicative) "convolution" form as the Bessel Kernel.

Scaling behaviour near the origin of the spectrum

It is then straightforward (Stirling formula) to obtain scaling limit $x = \frac{\xi}{n^2}$; $\qquad y = \frac{\zeta}{n^2}$.

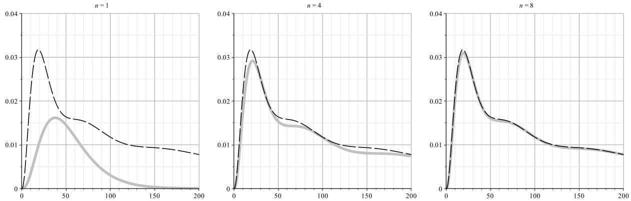
Definition 7 (Meijer-G random point field and universal class)

In the scaling limit the correlations of the eigenvalues of M_1, M_2 are determined by the two–level random point field with kernels.

$$\mathcal{G}_{00}(\xi,\zeta) = \int_0^1 H_a(t\xi)H_b(t\zeta)\mathrm{d}t \qquad \mathcal{G}_{01}(\xi,\xi') = \int_0^1 H_a(t\xi)\widetilde{H}_b(t\xi')\mathrm{d}t$$

$$\mathcal{G}_{10}(\zeta',\zeta) = \int_0^1 \widetilde{H}_a(t\zeta')H_b(t\zeta)\mathrm{d}t \qquad \mathcal{G}_{11}(\zeta,\xi) = \int_0^1 \widetilde{H}_a(t\zeta)\widetilde{H}_b(t\xi)\mathrm{d}t - \frac{1}{\xi+\zeta}$$

$$H_c(\zeta) := \int_\gamma \frac{\mathrm{d}u}{2i\pi} \frac{\Gamma(u+c)}{\Gamma(1-u)\Gamma(\alpha-c+1-u)}\zeta^{-u}\,, \qquad \widetilde{H}_c(\zeta) := \int_\gamma \frac{\mathrm{d}u}{2i\pi} \frac{\Gamma(u)\Gamma(u+c)}{\Gamma(\alpha-c+1-u)}\zeta^{-u}.$$



Meijer-G random point field for p-chain

The integrals (Mellin-Barnes) that appear in the formulæ above are all Meijer-G functions. These are generalizations of hypergeometric functions (which they contain as a subclass) (see review of Szmigielski-Beals: "Meijer G–Functions: A Gentle Introduction").

Definition 8

Let $a_j, b_j \in \mathbb{C}$ and $0 \leq m \leq q, 0 \leq n \leq p$ be integers. The Meijer-G function is defined through the Mellin-Barnes integral formula

$$G^{m,n}_{p,q}\left({a_1,\dots,a_p \atop b_1,\dots,b_q} \,\middle|\, \zeta \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{\ell=1}^{m} \Gamma(b_\ell + s)}{\prod_{\ell=m}^{q-1} \Gamma(1 - b_{\ell+1} - s)} \frac{\prod_{\ell=1}^{n} \Gamma(1 - a_\ell - s)}{\prod_{\ell=n}^{p-1} \Gamma(a_{\ell+1} + s)} \zeta^{-s} \, \mathrm{d}s$$

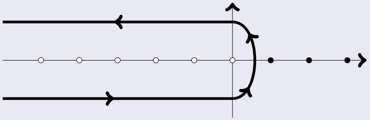


Figure: A choice for L corresponding to $a_j = b_j = 0$.

These special functions have appeared recently in the statistical analysis of singular values of products of Ginibre random matrices (AB 12 [1], AKW 13 [2], KZ 13 [7], CKW 15 [8]).

Meijer-G functions appear pervasively because they form a family stable under convolution

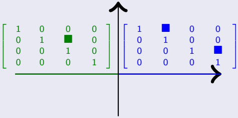
The p–chain

No explicit formulas for finite size n, $p \geq 3$; however

Theorem 9 (Riemann-Hilbert characterization for $\{\psi_k, \phi_k\}_{k\geq 0}$)

Determine a $(p+1)\times(p+1)$ *function* $\Gamma(z) = \Gamma_n(z)$ *with jump on* $\mathbb{R}$ ($\mu_j(z) = z^{a_j} \mathrm{e}^{-U_j(z)}$)*;*

$$\Gamma_+(z) = \Gamma_-(z) \begin{pmatrix} 1 & \mu_1(z)\chi_+ & 0 & & \\ 0 & 1 & \mu_2(-z)\chi_- & 0 & \\ & & 1 & \dots & \\ & & & & 1 \end{pmatrix}$$



$$\Gamma(z) = \left(\mathbf{1} + \mathcal{O}\left(z^{-1}\right)\right) \operatorname{diag}\left[z^n, 1, \dots, 1, z^{-n}\right], \quad z \to \infty.$$

Moreover the correlation kernels $\mathbb{K}_{j\ell}$ *are given by (B-Bothner '15 [6])*

$$\mathbb{K}_{j\ell}(x,y) = e^{-\frac{1}{2}U_j(x) - \frac{1}{2}U_\ell(y)} \mathbb{M}_{j\ell}(x,y),$$

$$\mathbb{M}_{j\ell}(x,y) = \frac{(-)^{\ell-1}}{(-2\pi i)^{j-\ell+1}} \left[\frac{\Gamma^{-1}(w;n)\Gamma(z;n)}{w-z}\right]_{j+1,\ell} \Bigg|_{\substack{w = x(-)^{j+1} \\ z = y(-)^{\ell-1}}}$$

The Biorthogonal polynomials are recovered:

$$\psi_n(z) = (\Gamma_n(z))_{11} \ , \qquad \phi_n(z) = (-1)^{n(p+1)} \left(\Gamma_n^{-1}((-)^{p+1}z)\right)_{p+1,p+1}$$

The proof is by induction on $n \geq 0$ using recursion relations;

$$\begin{aligned} \Gamma_{n+1}(z) &= R_n(z)\Gamma_n(z) \\ R_n(z) &= zE_{11} + B_n \end{aligned}$$

B_n explicit. Also For any $n \in \mathbb{Z}_{\geq 0}$

$$R_n^{-1}(w)R_n(z) = I - (z-w)\frac{E_{p+1,1}}{[Y_{1n}]_{1,p+1}}, \qquad z, w \in \mathbb{C}. \tag{3}$$

Conjecture 1 (BB 14 [6])

For any $p = 2, 3, \dots$, there exists $c_0 = c_0(p)$ and $\{\eta_j\}_1^p$ which depend on $\{a_j\}_1^p$ such that

$$\lim_{n\to\infty} \frac{c_0}{n^{p+1}} n^{\eta_\ell - \eta_j} \mathbb{K}_{j\ell}\left(\frac{c_0}{n^{p+1}}\xi, \frac{c_0}{n^{p+1}}\eta\right) \propto \mathcal{G}_{j\ell}^{(p)}\left(\xi, \eta; \{a_j\}_1^p\right)$$

uniformly for ξ, η chosen from compact subsets of $(0, \infty)$. Here the limiting correlation kernels equal

$$\mathcal{G}_{j\ell}^{(p)}\left(\xi, \eta; \{a_j\}_1^p\right) = \int_L \int_{\widehat{L}} \frac{\prod_{s=0}^{\ell-1}\Gamma(u - a_{1s})}{\prod_{s=\ell}^{p}\Gamma(1 + a_{1s} - u)} \frac{\prod_{s=j}^{p}\Gamma(a_{1s} - v)}{\prod_{s=0}^{j-1}\Gamma(1 - a_{1s} + v)} \frac{\xi^v \eta^{-u}}{1 - u + v} \frac{\mathrm{d}v\,\mathrm{d}u}{(2\pi i)^2}$$
$$+ \sum_{s\in\mathcal{P}\cup\{0\}} \operatorname*{res}_{v=s} \frac{\prod_{s=0}^{\ell-1}\Gamma(1 + v - a_{1s})}{\prod_{s=\ell}^{p}\Gamma(a_{1s} - v)} \frac{\prod_{s=j}^{p}\Gamma(a_{1s} - v)}{\prod_{s=0}^{j-1}\Gamma(1 + v - a_{1s})} \frac{\xi^v \eta^{-v}}{(-)^j\xi - (-)^\ell\eta}$$

with $\mathcal{P} = \{a_{1\ell} := \sum_{j=1}^{\ell} a_j,\ 1 \le \ell \le p\}$.

Remark

The blue part is present only for $j < \ell$. In the statistical analysis of singular values of products of Ginibre random matrices (AB 12 [1], AKW 13 [2], KZ 13 [7]) only the correlation kernel of one product was considered; thus we can only compare it to one (the $(1, 1)$ specifically) of the kernels.

Theorem 10 (BB 14 [6])

The conjecture holds for $p = 2, 3$ and potentials $U_j(x) = Nx - a_j \ln x$, $(N = n)$

Idea of Proof. •: use the formula for kernels

$$\mathbb{K}_{j\ell}(x, y) = e^{-\frac{1}{2}U_j(x) - \frac{1}{2}U_\ell(y)} \mathbb{M}_{j\ell}(x, y),$$

$$\mathbb{M}_{j\ell}(x, y) = \frac{(-)^{\ell-1}}{(-2\pi i)^{j-\ell+1}} \left[\frac{\Gamma^{-1}(w; n)\Gamma(z; n)}{w - z} \right]_{j+1,\ell} \Bigg|_{\substack{w = x(-)^{j+1} \\ z = y(-)^{\ell-1}}}$$

• In of $|z| < \epsilon$ introduce $\zeta \simeq n^{p+1} z$ and solve a "model RHP" in the ζ plane; the boundary

$$|z| = \epsilon \ \mapsto |\zeta| = n^{p+1}\epsilon \to \infty$$

Different asymptotics at $\zeta = \infty$ (.... next slide...)

Theorem 11 (Solution of the model problem)

$$g_j^{(\pm)}(\zeta) = \frac{c_j}{2\pi i} \int_L \frac{\prod_{\ell=1}^{j} \Gamma(s + a_{\ell,j-1})}{\prod_{\ell=j}^{p} \Gamma(1 + a_{j\ell} - s)} e^{\pm i\pi s \sigma_j} \zeta^{-s} \,\mathrm{d}s, \quad 1 \le j \le p+1 \quad \zeta \in \mathbb{C}\backslash(-\infty, 0].$$

$$\sigma_j = (j+1) \mod 2 \qquad c_j = (2\pi i)^{p+1-j} \sqrt{\frac{p+1}{(2\pi)^p}},$$

$$\mathbb{G}^{(\pm)}(\zeta) = \left[\left(\Delta_\zeta - a_{1,k-1}\right)^{j-1} g_k^{(\pm)}(\zeta) \right]_{j,k=1}^{p+1}, \quad \zeta \in \mathbb{C}\backslash(-\infty, 0], \qquad \Delta_\zeta := \zeta \frac{\mathrm{d}}{\mathrm{d}\zeta}.$$

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- Estimate remainder of approximation ($\star\star$ *Hard part* $\star\star$). Needs detailed knowledge of the solution of a vector equilibrium problem. In general the densities have a singularity $|x|^{-\frac{p}{p+1}}$.

This is why partly conjecture!

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• Write nice formulæ for $\frac{\left(\mathbb{G}^{(\pm)}(\zeta)\right)^{-1}\mathbb{G}^{(\pm)}(\eta)}{\zeta-\eta}$. (Bilinear Concomitant formulæ).

Ginibre random matrices: (Akemann-Burda '12, Akemann-Kieburg-Wei '13, Kuijlaars-Zhang '13, Strahov '15).

Chain separation

We can make a **consistency check** about these formulæ; if we send $a_q \to \infty$ with $1 \leq q \leq p$, the chain should (and does) "separate".

Theorem 12 (Chain separation)

Let $1 \leq q \leq p$ and consider the kernels $\mathcal{G}^{(p)}_{j\ell}(\zeta, \eta; \{a_1, \ldots, a_q\})$. In the limit as $\Lambda = a_q \to \infty$ we have the following behaviors;

$$\Lambda^{p-q+1}\left[\mathcal{G}^{(p)}_{j\ell}(\Lambda^{p-q+1}\zeta, \Lambda^{p-q+1}\eta; \vec{a}\right]_{j,\ell=1}^{p} = \left[\begin{array}{cc} \boxed{\begin{array}{c}\mathcal{G}^{(q-1)}_{j\ell}(\xi, \eta; \vec{a}) \\ 1\leq j,\ell\leq q-1\end{array}} & \mathcal{O}(1) \\ \mathcal{O}\left(\Lambda^{-1}\right) & \mathcal{O}\left(\lambda^{-1}\right)\end{array}\right],$$

$$\Lambda^{q}\left[\mathcal{G}^{(p)}_{j\ell}(\Lambda^{q}\zeta, \Lambda^{q}\eta; \vec{a})\right]_{j,\ell=1}^{p} = \left[\begin{array}{cc} \mathcal{O}(\Lambda^{-1}) & \mathcal{O}(1) \\ \mathcal{O}(\Lambda^{-1}) & \boxed{\begin{array}{c}\left(\frac{\xi}{\eta}\right)^{a_{1q}}\mathcal{G}^{(p-q)}_{j\ell}(\xi, \eta; \{a_{q+1}, \ldots, a_p\}) \\ 1\leq j,\ell\leq p-q\end{array}}\end{array}\right].$$

That is, the p-chain random point field split into two independent multi-level random point fields corresponding to two subchains of lengths $q-1$, $p-q$ with scaling at the indicated rates. *($\rightsquigarrow$ continued on next slide)*

Theorem 13 (Chain separation, cont'd)

In the case that $p - q = q - 1$ (i.e. p is odd and $p = 2q - 1$) so that the two subchains scale at the same rate, we have

$$\Lambda^q \left[\mathcal{G}^{(p)}_{j\ell}(\Lambda^q\zeta, \Lambda^q\eta; \vec{a})\right]_{j,\ell=1}^p = \left[\begin{array}{ccc} \boxed{\begin{array}{c}\mathcal{G}^{(q-1)}_{j\ell}(\xi,\eta;\{a_k\}_{k=1}^{q-1}) \\ 1\le j,\ell\le q-1\end{array}} & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\Lambda^{-1}) & \mathcal{O}(\Lambda^{-1}) & \mathcal{O}(1) \\ \mathcal{O}(\Lambda^{-1}) & \mathcal{O}(\Lambda^{-1}) & \boxed{\begin{array}{c}(\frac{\xi}{\eta})^{a_{1q}}\mathcal{G}^{(p-q)}_{j\ell}(\xi,\eta;\{a_k\}_{k=q+1}^{p}) \\ 1\le j,\ell\le p-q\end{array}} \end{array}\right],$$

and hence they still are independent subchains because the correlation functions factorize to leading order.

$$\mathrm{d}\mu(\vec{M}) = \mathcal{Z}\,\frac{\prod_{j=1}^p \det(M_j)^{a_j}\,\mathrm{e}^{-N\mathrm{Tr}(M_j)}\,\mathrm{d}M_j}{\prod_{j=1}^{p-1}\det(M_j + M_{j+1})^n}$$

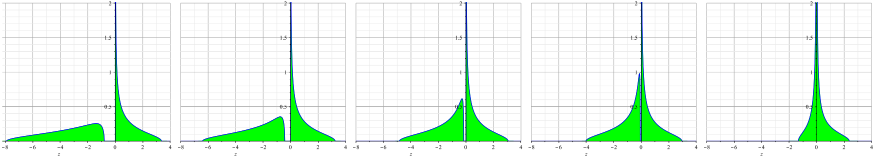


Figure: The limiting densities of eigenvalues for the $p = 3$ separated chain for $a_2 = n\beta$ and different values of β and a_1 is independent of n. Note that the support of the density of the matrix M_2 is separated from the origin. By comparison we also show the densities for $\beta = 0$ (connected chain). The profile of the density on the negative axis is the asymptotic macroscopic density of the eigenvalues of $-M_2$, while on the positive axis the profile corresponds to the densities of M_1, M_3 (they are identical).

Another use of the Riemann-Hilbert formulation: Expectations of rational characteristic functions I

We want to consider also the expectation of **rational characteristic functions** (product/ratios of charpolys); same was done by **[Baik-Deift-Strahov, Fyodorov-Strahov ('03)]** for the one-matrix model.

$$\mathcal{C}^{(\vec{s},\vec{r})}(\{\vec{z}_\nu, \vec{w}_\nu\}_{\nu=1}^{p}) = \left\langle \prod_{\nu=1}^{p} \frac{\prod_{j=1}^{s_\nu} \det\left(z_{\nu,j}\mathbf{1} - M_\nu\right)}{\prod_{j=1}^{r_\nu} \det\left(w_{\nu,j}\mathbf{1} - M_\nu\right)} \right\rangle =$$

$$= \frac{1}{Z_{p,n}} \int_{(\mathcal{H}_n^+)^p} \prod_{\nu=1}^{p} \frac{\prod_{j=1}^{s_\nu} \det\left(z_{\nu,j}\mathbf{1} - M_\nu\right)}{\prod_{j=1}^{r_\nu} \det\left(w_{\nu,j}\mathbf{1} - M_\nu\right)} \frac{\prod_{\ell=1}^{p} (\det M_\ell)^{a_\ell} \mathrm{e}^{-U_\ell(M_\ell)} \mathrm{d}M_\ell}{\prod_{j=1}^{p-1} \det(M_j + M_{j+1})^n}$$

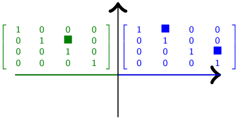
They can all be written solely in terms of the solution of the RHP $\Gamma_n(z)$;

Obvious observation

$$\mu_j(x) \mapsto \widetilde{\mu}_j(x) = \mu_j(x) \frac{\prod_{\ell=1}^{s_j}(z_{j,\ell} - x)}{\prod_{\ell=1}^{r_j}(w_{j,\ell} - x)} \qquad (\mu_j(x) = x^{a_j} \mathrm{e}^{-U_j(x)})$$

We need to compute the ratio of two partition functions with measures related as above

$$\Gamma_+(z) = \Gamma_-(z) \begin{pmatrix} 1 & \widetilde{\mu}_1(z)\chi_+ & 0 & & \\ 0 & 1 & \widetilde{\mu}_2(-z)\chi_- & 0 & \\ & & 1 & \dots & \\ & & & & 1 \end{pmatrix}$$



$$\Gamma(z) = \left(\mathbf{1} + \mathcal{O}\left(z^{-1}\right)\right) \operatorname{diag}\left[z^n, 1, \dots, 1, z^{-n}\right], \quad z \to \infty.$$

Discrete Schlesinger tranformations

The effect is the same as modification of the RHP; the jump matrix $M(z)$ is **conjugated** by a diagonal, rational, matrix $D(z)$. This was studied in **[B. Cafasso, 2014]** and corresponds to the "discrete Schlesinger transformations" in the theory of Painlevé equations.

The partition function $\mathcal{Z}$ is a *tau* function

$$\delta \ln \mathcal{Z} = \int_{\mathbb{R}} \mathrm{Tr}\left(\Gamma^{-1}(z)\Gamma'(z)\delta M(z)M^{-1}(z)\right)\frac{\mathrm{d}z}{2i\pi} =$$

$$= \sum_{j\geq 0}\int_{\mathbb{R}_+}\left(\Gamma^{-1}(z)\Gamma'(z)\right)_{2j+1,2j+2}\frac{\delta\mu_{2j+1}(z)\mathrm{d}z}{2i\pi} + \sum_{j\geq 1}\int_{\mathbb{R}_-}\left(\Gamma^{-1}(z)\Gamma'(z)\right)_{2j+1,2j}\frac{\delta\mu_{2j}(z)\mathrm{d}z}{2i\pi}$$

$$M(z;\vec{\mu}) = \begin{pmatrix} 1 & \mu_1(z)\chi_+ & 0 & & \\ 0 & 1 & \mu_2(-z)\chi_- & 0 & \\ & & 1 & \dots & \\ & & & & 1 \end{pmatrix}$$

Rational Diagonal Conjugations [B. Cafasso, "Darboux Transformations and Random Point Processes" '14]

If we maintain all boundary behaviours of the RHP for Γ but modify the jumps

$$\Gamma_+(z) = \Gamma_-(z)M(z) \quad \mapsto \quad \widetilde{\Gamma}_+(z) = \widetilde{\Gamma}_-(z)\widetilde{M}(z); \qquad M(z) := D^{-1}(z)M(z)D(z)$$

then the blue part gets an extra term

$$\delta \ln \widetilde{\mathcal{Z}} = \delta \ln \mathcal{Z} + \delta \ln \mathbb{G}$$

and $\mathbb{G}$ is a **finite determinant** of the size equal to the total degree of $D(z)$ whose entries are entirely read off $\Gamma(z)$, i.e. (in our case) orthogonal polynomials and associated functions.

After the dust settles we obtain all necessary formulas (and set on equal footing the various formulas used in Baik-Deift-Strahov '03, Fyodorov-Strahov '03).

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Example 14

If $D(z) = \mathrm{diag}\left(1, \ldots, 1, \frac{\prod^K (z-a_\ell)}{\prod^K (z-b_\ell)}, 1, \ldots, 1\right)$ then

$$\mathbb{G} = \frac{\det\left[\frac{\left(\Gamma^{-1}(a_j)\Gamma(b_k)\right)_{jj}}{a_j - b_k}\right]_{j,k=1}^K}{\det\left[\frac{1}{a_j - b_k}\right]_{j,k=1}^K} = \det\left[\frac{\left(\Gamma^{-1}(a_j)\Gamma(b_k)\right)_{jj}}{a_j - b_k}\right]_{j,k=1}^K \frac{\prod_{j,k}(a_j - b_k)}{\Delta(\vec{a})\Delta(\vec{b})}$$

Examples

$$\mathbb{K}_n(z, w) := \frac{\Gamma_n^{-1}(w)\Gamma_n(z)}{w - z}$$

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$$\left\langle \prod_{j=1}^{K} \det\left(z_j \mathbf{1} - M_1\right) \prod_{j=1}^{K} \det\left(w_j \mathbf{1} - M_p\right) \right\rangle \propto \left(\prod_{\ell=n}^{n+K-1} h_\ell \right) \frac{\det\left[\left(\mathbb{K}_{n+K}(z_k, (-)^p w_j) \right)_{p+1,1} \right]_{j,k=1}^{K}}{\Delta(\vec{z})\Delta(\vec{w})}$$

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(here $\propto$ means up to a constant depending on K only)

To do

Challenge: study the gap probabilities! (e.g. $p = 2$–matrix model)

In this case we can study **joint** gaps because we have the full control of scaling limit kernels;

$$\lim_{n\to\infty} \mathbb{P}_n \left(x_1 > \frac{s}{n^2} \right) = \det \left[Id_{L^2([0,s])} - \mathcal{R}_{++} \right] =: F_1^{(1)}(s)$$

(see recent **Forrester-Witte '16**)

$$\lim_{n\to\infty} \mathbb{P}_n \left(x_1 > \frac{s}{n^2}, y_1 > \frac{t}{n^2} \right) = \det \left[Id_{\mathcal{H}} - \mathcal{R} \right] =: F_2(s,t),$$

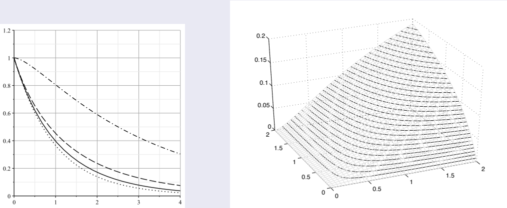

Figure: $F_2(s,s)$ (solid), $F_1^{(1)}(s)$ (dash), $F_1^{(2)}(s)$ (dash-dot) and $F_1^{(1)}(s)F_2^{(2)}(s)$ (dots) computed numerically as explained **[Bornemann '10]**, $a_1 = 0$, $a_2 = 1$. **Right**: plot of $1 - \frac{F_1^{(1)}(s)F_1^{(2)}(t)}{F_2(s,t)}$ on the domain $[0,2] \times [0,2]$. This quantity is a measure of deviation from independence, and it would be identically zero if the spectra were independent