Long time behavior of the free Fokker-Planck equation

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The free Fokker-Planck equation

Given a (polynomial) potential $V : \mathbb{R} \to \mathbb{R}$, the process $(\mu_t)_{t\geq 0}$ satisfies the free Fokker-Planck equation if

$$\frac{\partial \mu_t}{\partial t} = \frac{\partial}{\partial x}\left[\mu_t \left(\frac{1}{2}V' - H\mu_t\right)\right],$$

where

$$H\mu_t(x) = \mathrm{vp}\int \frac{1}{x-y}\,d\mu_t(y), \forall x \in \mathbb{R}.$$

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For any regular function φ,

$$\frac{d}{dt}\int \varphi(x)d\mu_t(x) = -\frac{1}{2}\int V'(x)\varphi'(x)d\mu_t(x) + \iint \frac{\varphi'(x)-\varphi'(y)}{x-y}d\mu_t(x)d\mu_t(y)$$

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What can we say about the asymptotic behavior of μ_t as $t \to +\infty$?

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- Conclusion

First motivation : granular media equation

Several equations arising from physics can be stated under the following form :

$$\frac{\partial \mu_t}{\partial t} = \nabla . \left[\mu_t \nabla (\mathcal{U}'(\mu_t) + \mathcal{V} + \mathcal{W} * \mu_t) \right] .$$

where $\mathcal{U}$ is an internal energy, $\mathcal{V}$ a (confining) external potential and $\mathcal{W}$ a self-interaction energy.

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Examples/references : linear Fokker-Planck equation ($\mathcal{U}(s) = s \ln s$, $\mathcal{W}(s) = 0$), equation for porous media ($\mathcal{U} = 0$, $\mathcal{W}$ polynomial or convex) [Benedetto-Caglioti-Carrillo-Pulvirenti, Malrieu, Carrillo-McCann-Villani, Bolley-Gentil-Guillin etc.].

The free Fokker-Planck equation is given by $\mathcal{U} = 0, \mathcal{W} = -\ln$ [see also Carrillo-Castorina-Volzone for $d = 2$].

Remark : we are used in RMT to encountering the logarithmic interaction in dimension 1.

Motivation : a more probabilistic point of view

If the process $(X_t)_{t\geq 0}$ satisfies the classical SDE (in dimension 1)

$$dX_t = dB_t - \frac{1}{2}V'(X_t)dt,$$

its law $(\mu_t)_{t\geq 0}$ satisfies the linear Fokker-Planck equation

$$\frac{\partial \mu_t}{\partial t} = \Delta \mu_t + \frac{1}{2}\frac{\partial}{\partial x}\mu_t V'.$$

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If we want to get not trivial interactions $\mathcal{W}$, one can consider the empirical measure $L_N(t) = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i^N(t)}$ of a particle system of the form

$$d\lambda_i^N(t) = \frac{1}{\sqrt{N}}dB_i(t) + \frac{1}{N}\sum_{j\neq i}\mathcal{W}'\left(\lambda_i^N(t) - \lambda_j^N(t)\right)dt - \frac{1}{2}V'(\lambda_i^N(t))dt.$$

The generalized Dyson Brownian motion

Let $(\tilde{B}_t^N)_{t\geq 0}$ an $N \times N$ Hermitian matrix with Brownian coefficients. The process of the eigenvalues $(\lambda_1^N(t), \ldots, \lambda_N^N(t))_{t\geq 0}$ of $(\tilde{B}_t^N/\sqrt{N})_{t\geq 0}$ is called Dyson Brownian motion.

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Similarly, the process $(\lambda_1^N(t), \ldots, \lambda_N^N(t))_{t\geq 0}$ satisfying the system of SDEs

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is called generalized Brownian motion.

$$\begin{array}{ccc} L_N(t) & \rightarrow & \mu_t \\ \downarrow & & \downarrow ? \\ L_N & \rightarrow & \mu_V \end{array}$$

where

$$L_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i^N}, \quad (\lambda_1^N, \ldots, \lambda_N^N) \sim \frac{1}{Z_N}\prod_{i<j}|x_i - x_j|^2 \exp\left(-\sum_{i=1}^{N}V(x_i)\right) dx_1 \ldots dx_N$$

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and μ_V is the equilibrium measure associated to the potential V, that is the unique minimizer of

$$\Sigma_V : \mu \mapsto -\iint_{\mathbb{R}^2} \ln|x-y|\, d\mu(x)d\mu(y) + \int_{\mathbb{R}} V(x)\, d\mu(x).$$

Known results

- For a locally Lipschitz potential V growing fast enough at infinity, existence and unicity are well understood for a compactly supported initial condition [Biane-Speicher 2001, Li-Li-Xie 2014].

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- In some cases, the convergence towards μ_V cannot hold unless the initial condition μ_0 has the same filling fractions as μ_V.

We are interested in the quartic potential

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$$

Asymptotic behavior when $c \geq -2$

Théorème

(Donati-Martin, Groux, M. 2016)
Let $c \geq -2$. Let us assume that μ_0 is compactly supported. Then the solution $(\mu_t)_{t \geq 0}$ of the free Fokker-Planck equation satisfies

$$\lim_{t \to +\infty} W_p(\mu_t, \mu_V) = 0$$

for all $p \geq 1$, where μ_V is given by

$$d\mu_V(x) = \frac{1}{\pi} \left(\frac{1}{2} x^2 + b_0 \right) \sqrt{a^2 - x^2} \mathbf{1}_{[-a,a]}(x)\, dx$$

with

$$a^2 = \frac{2}{3} \left(\sqrt{c^2 + 12} - c \right), \qquad b_0 = \frac{1}{3} \left(c + \sqrt{\frac{c^2}{4} + 3} \right).$$

A quick free probability reminder

The free Brownian is a process $(s_t)_{t\geq 0}$ of self-adjoint operators such that

- for all $t \geq 0$, the distribution of s_t is the semi-circular law with variance t,
- the increments $s_t - s_u$, $0 \leq u \leq t$, are free and their distribution is the semi-circular law with variance $t - u$.

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$\to$ free stochastic calculus developed by Biane-Speicher, free diffusions etc.

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Proposition

Let $(x_t)_{t \geq 0}$ be the solution of this free SDE. If, for any $t \geq 0$, we denote by μ_t the law of x_t, then $(\mu_t)_{t \geq 0}$ satisfies the free Fokker-Planck equation

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The logarithmic singularity has been swept away.

Consequence : nice properties of $\{\mu_t, t \geq 0\}$

Proposition

(Biane-Speicher) Let $(\mu_t)_{t\geq 0}$ be the solution of the free Fokker-Planck equation starting from a compactly supported measure μ_0.

- *There exists $M > 0$ such that for all $t \geq 0$, $supp(\mu_t) \subset [-M, M]$*
- *There exists $K_1, K_2 > 0$ depending only on V such that for all $t \geq 0$, the density p_t of μ_t satisfies*

$$\|p_t\|_\infty \leq \frac{K_1}{\sqrt{t}} + K_2, \qquad \|D^{1/2} p_t\|_2 \leq \frac{K_1}{t} + K_2.$$

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- the free entropy Σ_V is decreasing and continuous along the trajectories
- any accumulation point μ has bounded density and is a solution of the Euler-Lagrange equation : $H\mu = \frac{1}{2} V'$ μ-a.e.

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  Global minimizer of the free entropy Σ_V. Unique measure for which there exists a constant C such that

  - for any $z \in \mathrm{supp}(\mu)$, we have $U^\mu(z) + \frac{1}{2}V(z) = C$,
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Equivalence between stationary measure and critical measure supported on $\mathbb{R}$.

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Proposition

[Kuijlaars-Silva, Huybrechs-Kuijlaars-Lejon] Let V be a polynomial and μ a critical measure for Σ_V supported on $\mathbb{R}$.

- *There exists a polynomial R of degree $2\deg(V) - 2$ such that*

$$R(z) = \left(\int \frac{1}{z-x} d\mu(x) + \frac{1}{2} V'(z) \right)^2$$

Moreover,

$$R(z) = \frac{1}{4} V'(z)^2 - \int_{\mathbb{R}} \frac{V'(x) - V'(z)}{x - z} \, d\mu(x).$$

- *Any non-real root of R has even multiplicity.*
- *The support of μ is a finite union of intervals connecting real zeros of R.*

Application to the quartic potential

For the potential

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$$

we have

$$R(z) = \frac{1}{4}z^6 + \frac{c}{2}z^4 + \frac{1}{4}(c^2 - 4)z^2 - \int x\, d\mu(x).z - \int x^2\, d\mu(x) - c\,.$$

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Elementary arguments allow to conclude that for $-2 \leq c$, any critical measure has a connected support.

We then use standard analytic tools to show that the only critical measure with connected support is the equilibrium measure, and we can then conclude the proof.

Conclusion

We obtain the first convergence result for the granular media equation in dimension 1, with a double-well potential $\mathcal{V}$ and a singular self-interaction $\mathcal{W}$.

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Open questions :

- Description of critical measures for $c < -2$? Description of the basins of attraction for $c < -2$?
- Generalization of the method to other families of potentials (higher degree, higher dimension, non-confining potential [Allez-Dumaz]...)
- Biane and Speicher's conjecture for the potential $V(x) = \frac{1}{2}x^2 + \frac{g}{4}x^4$, with $-\frac{1}{12} < g < 0$?
- Extension to the multiplicative setting ?

Thank you for your attention.