Lyapunov exponents and eigenvalues of products of random matrices

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Lyapunov exponents

Consider a sequence $\{Y_n, n \geq 1\}$ of *i.i.d* $d \times d$ real random matrices with common distribution μ. Let $S_n = Y_n \cdots Y_1$ for all $n \geq 1$. Then the Lyapunov exponents are defined as follows.

Definition

If $\mathbf{E}(\log^+ \|Y_1\|) < \infty$ (we write a^+ for $\max(a, 0)$), the **first Lyapunov exponent** associated with μ is the element γ of $\mathbb{R} \cup \{-\infty\}$ defined by

$$\gamma_1 = \lim_{n\to\infty} \frac{1}{n} \mathbf{E}(\log \|S_n\|)$$

If $d = 1$, then $\gamma_1 = \mathbf{E}(\log |Y_1|)$

Fact

Let $Y_1, Y_2, \ldots$ *be i.i.d matrices in* $GL_d(\mathbb{R})$ *and* $S_n = Y_n \cdots Y_1$. *If* $\boldsymbol{E}(\log^+ \|Y_1\|) < \infty$, *then with probability one*

$$\lim_{n\to\infty} \frac{1}{n} \log \|S_n\| = \gamma_1.$$

Exterior powers

- The p-th exterior power $\wedge^p M$ $(1 \leq p \leq d)$ is $\binom{d}{p} \times \binom{d}{p}$ matrix.

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- Rows and columns are indexed by p-sized subsets of $\{1, 2 \dots d\}$ in dictionary order, such that I, J-th element is given by

$$\wedge^p M_{I,J} = [M]_{I,J},$$

where $[M]_{I,J}$ denotes the $p \times p$ minor of M that corresponds to the rows with index in I and the columns with index in J. ($[M]_I$ denotes $[M]_{I,I}$).

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- Let $M = U\Sigma V^*$ be a singular value decomposition of M and $\sigma_1(M) \ge \sigma_2(M) \cdots \ge \sigma_d(M)$ be singular values of M, then

$$\wedge^p M = (\wedge^p U)(\wedge^p \Sigma)(\wedge^p V^*)$$

is a singular value decomposition of $\wedge^p M$. Therefore

$$\| \wedge^p M\| = \sigma_1(M) \cdots \sigma_p(M).$$

Definition

The **Lyapunov exponents** $\gamma_1, \gamma_2 \dots \gamma_d$ associated with μ are defined inductively by $\gamma = \gamma_1$ and for $p \geq 2$

$$\sum_{i=1}^{p} \gamma_i = \lim_{n\to\infty} \frac{1}{n}\mathbf{E}(\log \| \wedge^p S_n\|).$$

If for some p $\sum_{i=1}^{p-1} \gamma_i = -\infty$, we put $\gamma_p = \gamma_{p+1} = \cdots \gamma_d = -\infty$.

Fact

Let $Y_1, Y_2, \dots$ be i.i.d matrices in $GL_d(\mathbb{R})$ and $S_n = Y_n \cdots Y_1$. Let $\sigma_1(n) \geq \sigma_2(n) \geq \cdots \geq \sigma_d(n) > 0$ are singular values of S_n. If $\boldsymbol{E}(\log^+ \|Y_1\|) < \infty$, then with probability one, for $1 \leq p \leq d$

$$\gamma_p = \lim_{n\to\infty} \frac{1}{n}\boldsymbol{E}(\log \sigma_p(n)) = \lim_{n\to\infty} \frac{1}{n} \log \sigma_p(n).$$

Definition

Let μ (resp. μ^*) be the probability measure of the invertible random matrix Y_1 (resp. Y_1^*) on $GL_d(\mathbb{R})$. Then T_μ is defined to be the smallest closed semigroup in $GL_d(\mathbb{R})$ which contains the support of μ in $GL_d(\mathbb{R})$. (Support of μ in $GL_d(\mathbb{R})$ is the set of all points in $GL_d(\mathbb{R})$ whose every open-neighbourhood has positive measure).

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Definition

Given a subset T of $GL_d(\mathbb{R})$, we define the index of T as the least integer r such that there exists a sequence $\{M_n, n \geq 0\}$ in T for which $\|M_n\|^{-1}M_n$ converges to a rank r matrix. We say that T is **contracting** when its index is one. T is said to be p**- contracting** if $\{\wedge^p M; M \in T\}$ is contracting.

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Definition

T is **strongly irreducible** if there does not exist a finite family of proper linear subspaces of $\mathbb{R}^d$, $V_1, V_2 \ldots V_k$ such that $M(V_1 \cup V_2 \cup \ldots V_k) = (V_1 \cup V_2 \cup \ldots V_k)$ for any M in T. T is p**-strongly irreducible** if $\{\wedge^p M; M \in T\}$ is strongly irreducible.

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Define $\ell(M) = \max(\log^+ \|M\|, \log^+ \|M^{-1}\|)$. Observe that $\frac{1}{p}|\log \| \wedge^p M\|| \leq \ell(M)$ for all $1 \leq p \leq d$.

Fact

Let μ be a probability measure on $GL_d(\mathbb{R})$ such that $\int \log^+ \|M\| d\mu(M)$ is finite. We suppose T_μ is irreducible. Then $\gamma_1 > \gamma_2$ if and only if T_μ is strongly irreducible and contracting. If T_μ is p-strongly irreducible and p-contracting for some $p \in \{1, \dots, d-1\}$, then $\gamma_p > \gamma_{p+1}$.

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Theorem

Let $Y_1, Y_2, \ldots$ be i.i.d random elements of $GL_d(\mathbb{R})$ with common distribution μ such that T_μ is strongly irreducible and contracting. Set $S_n = Y_n \cdots Y_1$ and let $\lambda_1(n)$ be an eigenvalue of S_n with maximum absolute value and $\sigma_1(n)$ be the maximum singular value of S_n, for all $n \geq 1$. If for some $\tau > 0$, $\boldsymbol{E}e^{\tau \ell(Y_1)}$ is finite, then for any $r > 0$ with probability one

$$\lim_{n\to\infty} \frac{1}{n^r} \log \left(\frac{|\lambda_1(n)|}{\sigma_1(n)} \right) = 0.$$

Theorem

Let $Y_1, Y_2, \ldots$ be i.i.d random elements of $GL_d(\mathbb{R})$ with common distribution μ such that T_μ is p-strongly irreducible and p-contracting for all $1 \le p \le d$. Set $S_n = Y_n \cdots Y_1$ and let $\lambda_1(n), \lambda_2(n), \ldots, \lambda_d(n)$ be the eigenvalues of S_n such that $|\lambda_1(n)| \ge |\lambda_2(n)| \ge \cdots \ge |\lambda_d(n)| > 0$ and $\sigma_1(n), \sigma_2(n), \ldots, \sigma_d(n)$ be the singular values of S_n such that $|\sigma_1(n)| \ge |\sigma_2(n)| \ge \cdots \ge |\sigma_d(n)| > 0$, for all $n \ge 1$. If for some $\tau > 0$, $\boldsymbol{E} e^{\tau \ell(Y_1)}$ is finite, then for any $r > 0$ with probability one

$$\lim_{n \to \infty} \frac{1}{n^r} \log \left(\frac{|\lambda_p(n)|}{\sigma_p(n)} \right) = 0,$$

for all $1 \le p \le d$.

Key idea

- $\mathrm{Tr}(S_n) = \mathrm{Tr}(U_n \Sigma_n V_n^*) = \sum_{i=1}^{d} \sigma_i(n) \langle V_n e_i, U_n e_i \rangle.$

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- $|\operatorname{Tr}(S_n)| \leq d|\lambda_1(n)| \leq d\sigma_1(n)$

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- $\mathrm{Tr}(S_n) = \mathrm{Tr}(U_n \Sigma_n V_n^*) = \sum_{i=1}^d \sigma_i(n) \langle V_n e_i, U_n e_i \rangle$.
- $|\mathrm{Tr}(S_n)| \leq d|\lambda_1(n)| \leq d\sigma_1(n)$
- $S_{n,k} = Y_n \cdots Y_{k+1}$. Then

$$\mathrm{Tr}(S_n) = \mathrm{Tr}(S_{n,k} S_{k,0})$$

$$= \sum_{i=1}^d \sum_{j=1}^d \sigma_i(n,k) \sigma_j(k,0) \langle V_{n,k} e_i, U_{k,0} e_j \rangle \langle V_{k,0} e_j, U_{n,k} e_i \rangle.$$

Lemma

Let $Y_1, Y_2, \ldots$ be i.i.d random elements of $GL_d(\mathbb{R})$ with common distribution μ such that T_μ is strongly irreducible and contracting and for some $\tau > 0$, $\boldsymbol{E} e^{\tau \ell(Y_1)}$ is finite. Set $S_n = Y_n \cdots Y_1$ and consider a Singular value decomposition $S_n = U_n \Sigma_n V_n^*$ with U_n and V_n in $O(d)$, for all $n \geq 1$. Then for any $r > 0$

(i) *for any sequence $\{u_n, n \geq 1\}$ of random unit vectors in $\mathbb{R}^d$ such that $S_n e_1$ and u_n are independent for each n, with probability one we have that*

$$\lim_{n\to\infty} \left| \frac{\langle S_n e_1, u_n \rangle}{\|S_n e_1\|} \right|^{\frac{1}{n^r}} = \lim_{n\to\infty} |\langle U_n e_1, u_n \rangle|^{\frac{1}{n^r}} = 1$$

(ii) *for any sequence $\{v_n, n \geq 1\}$ of random unit vectors in $\mathbb{R}^d$ such that $S_n^* e_1$ and v_n are independent for each n, with probability one we have that*

$$\lim_{n\to\infty} \left| \frac{\langle S_n^* e_1, v_n \rangle}{\|S_n^* e_1\|} \right|^{\frac{1}{n^r}} = \lim_{n\to\infty} |\langle V_n e_1, v_n \rangle|^{\frac{1}{n^r}} = 1.$$

Corollary

Let ξ be a real valued continuous random variable whose support contains an open set and there exists $\tau > 0$ such that

$$\boldsymbol{E}|\xi|^{\tau} < \infty \;\; \textit{and} \;\; \sup_{a\in\mathbb{R}} \boldsymbol{E}\,|\xi - a|^{-\tau} < \infty.$$

Let $X_1, X_2, \ldots$ be a sequence of $d \times d$ i.i.d random matrices whose elements are i.i.d random variables distributed like ξ. Then for any $r > 0$ with probability one for all $1 \leq p \leq d$,

$$\lim_{n\to\infty} \frac{1}{n^r} \log\left(\frac{|\lambda_p(n)|}{\sigma_p(n)}\right) = 0.$$

Remark

If the random variable ξ has bounded density, then $\sup_{a\in\mathbb{R}} \boldsymbol{E}|\xi - a|^{-\tau} < \infty$ for any $0 < \tau < 1$. Let us say f is the probability density of ξ ,bounded by K, then

$$\begin{aligned}
\boldsymbol{E}|\xi - a|^{-\tau} &= \sup_{a\in\mathbb{R}} \int |x-a|^{-\tau} f(x)dx \\
&= \int_{|x-a|<\epsilon} |x-a|^{-\tau} f(x)dx + \int_{|x-a|\geq\epsilon} |x-a|^{-\tau} f(x)dx \\
&\leq K \int_{|x-a|<\epsilon} |x-a|^{-\tau} dx + \epsilon^{-\tau} \\
&= K \frac{2\epsilon^{1-\tau}}{1-\tau} + \epsilon^{-\tau} < \infty.
\end{aligned}$$

Since ϵ, K, τ do not depend on a, we get that

$$\sup_{a\in\mathbb{R}} \boldsymbol{E}|\xi - a|^{-\tau} < \infty.$$

Fact

T_μ is p-strongly irreducible and p-contracting, for any $1 \leq p \leq d$, if there exists a matrix M such that

- *$\|M\|^{-1}M$ is not orthogonal,*
- *for any $K \in O(d)$, KMK^{-1} is in T_μ.*

Thank you!!