(Pseudo) Entropy Methods for CMB Analysis

And Recent Results with Multipole Vectors

(https://i.redd.it/dahqe4rh56n01.jpg)

Marvin Pinkwart

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Jacobs University Bremen

RESEARCH TRAINING GROUP
Models of Gravity

Based on

- MP, P. Schupp (to appear) → entropy methods
- MP, D.J. Schwarz (arXiv:1803.07473 [astro-ph.CO]; sent to PhysRevD) → MPVs

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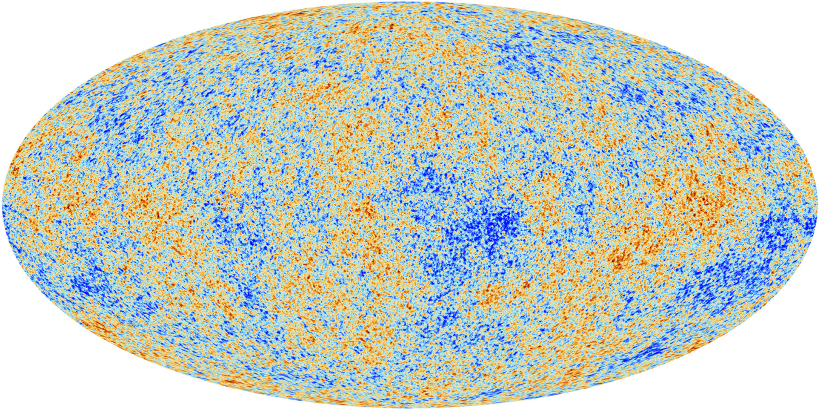
The CMB: Basic Questions

The CMB



https://www.esa.int/var/esa/storage/images/esa_multimedia/images/2013/03/planck_cmb/12583930-4-eng-GB/Planck_CMB.jpg

Cosmological Principle, Gaussianity

Statistical Cosmological Principle

The Universe is spatially statistically isotropic around each point in spacetime. (Observed statistical isotropy + Copernican Principle)

For CMB this implies: For all quantities $\mathcal{O}$ measured on the celestial sphere $C_n(\mathcal{O}, \vec{e}_1, \dots, \vec{e}_n) = C_n(\mathcal{O}, \hat{R}(\vec{e}_1), \dots, \hat{R}(\vec{e}_n))$ for all rotations $\hat{R}$ and points on the sphere $\vec{e}_i$ and for all n (implies isotropy in prob. dist. of a_{lm}, in particular $\langle a_{lm} a_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'}$).

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Gaussianity

δT is a Gaussian random field. This implies $p_l(\{a_{lm}\}) \sim \exp(-\sum_{m,m'} a^*_{lm'} \mathcal{D}_{l,m'm} a_{lm}/2)$,

where $\delta T = (T - T_0)/T_0 = \sum_l \sum_m a_{lm} Y_{lm}$. If Gaussian AND isotropic: $\mathcal{D}^{-1}_{l,mm'} = C_l \delta_{mm'}$.

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Questions

- Cosmological Principle more or less ad hoc assumption. Really true? Maybe small anisotropies from primordial effects? Or apparent anisotropies from foreground effects? Largest scales: Do we understand the Dipole correctly?
- Gaussianity implied by simplest single-field inflationary models + linear perturbation theory: Sizeable non-Gaussian effects? Primordial or disturbance during evolution?
- Furthermore: Compare data processing algorithms (full sky maps)

Here: Mainly introduce methods for data analysis. Null hypothesis: Gaussian, isotropic random field

Coherent States and Wehrl Entropy

Coherent States

Most General Definition (Neumaier, Farashi '18)

Coherent space: Set Z with nondegenerate, conjugate symmetric, positive semidefinite coherent product $K: Z \times Z \to \mathbb{C}$. Corresponding quantum Space $\mathbb{Q}(Z)$ is Euclidean space algebraically spanned by *coherent states* $\{|z\rangle | z \in Z\}$ such that $\langle z|z'\rangle = K(z,z')$.

Special Cases

- Perelomov '71: Coherent states by application of Lie group on "ground state" $|z(g)\rangle = \mathcal{D}(g)|0\rangle$, $g \in G$.

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- Rotation group $\to$ Bloch coherent states: $|\Omega\rangle_l = \sum_{m=-l}^{l} \sqrt{\binom{2l}{l+m}} p^{(l+m)/2}(1-p)^{(l-m)/2} e^{-im\phi/2} |l, m\rangle = \mathcal{R}_l(\Omega)|l, l\rangle$ with $p = \cos^2(\theta/2)$ and $\Omega = (\phi, \theta)$.

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Usual Properties

- Resolution of identity: $\int_Z \mathrm{d}\mu(z)|z\rangle\langle z| = \alpha(Z)\mathbb{1}$ if Z is furnished with a measure μ.
- Minimal uncertainty/closest to classical "points": $\Delta_z x \leq \Delta_\Psi x \ \forall |\Psi\rangle \in \mathcal{H}$.
  - $|\Omega\rangle \,\hat{=}$ quantum point on the sphere
- Connection to MPVs: MPVs are the zeros of ${}_l\langle\Omega|l,l\rangle(\phi,\theta)$.

Wehrl Entropy

Problem

How to define semi-classical phase space entropy? Classical phase space entropy may be negative.

Solution: Wehrl Entropy (Wehrl '78)

$$S_{\mathrm{W}}^{Z}(\rho) = -\int_{Z} \mathrm{d}\mu(z)\rho(z)\log(\rho(z))$$

with $\rho(z) = \langle z|\rho|z\rangle$ lower symbol of density operator ρ (uniquely determines ρ). Z is a coherent space and the states $|z\rangle$ are coherent states. "Most classical quantum entropy".

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Properties

- Positivity: $S_Q > S_N \geq 0$ and Concavity: $S_W(\lambda\rho + (1-\lambda)\rho') \geq \lambda S_W(\rho) + (1-\lambda)S_W(\rho')$
- Monoticity: $S_W(\rho_{12}) \geq S_W(\rho_1)$ (not satisfied by von Neumann)
- Strong subadditivity: $S_W(\rho_{123}) + S_W(\rho_2) \leq S_W(\rho_{12}) + S_W(\rho_{23})$
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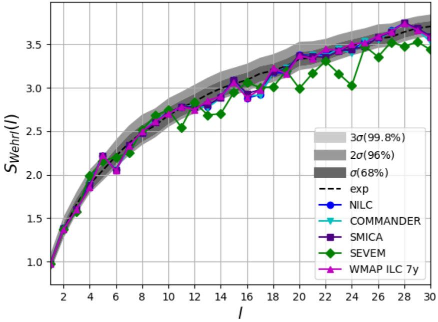
Lieb's Conjecture

Wehrl entropy minimized for pure coherent states $\rho = |z_0\rangle\langle z_0| \rightarrow$ coherent states "most classical".

Proven for symmetric $SU(N)$ coherent states (Lieb, Solovej '16). Includes Bloch. Earlier already proven for Glauber.

Application to CMB Analysis

- After normalization $\delta T_l(\phi,\theta) = \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi)$ can be identified with spin l state $|\Psi\rangle_l = \sum_{m=-l}^{l} a'_{lm}|l,m\rangle$ with $a'_{lm} = a_{lm}/\sum_{m'=-l}^{l}|a_{lm'}|^2$.
- Usual von Neumann entropy is trivially zero on pure states $S_N(\rho_\Psi) = -\mathrm{Tr}(\rho_\Psi \log(\rho_\Psi)) = 0$ with $\rho_\Psi = |\Psi\rangle\langle\Psi|$ $\Rightarrow$ no information gained
- Use Wehrl entropy $\Rightarrow$ not trivial.
- Coherent state (preferred direction) minimizes S_W & S_W is rotationally invariant $\rightarrow$ measure of anisotropy
- First application to CMB in (Helling, Schupp, Tesileanu '06)



(MP, P. Schupp to appear)

Nice: Can also be modified to test ranges of scales

Problem: numerically expensive $\rightarrow$ try numerical more tractable types of (pseudo) emtropies

Various Pseudo Entropies

Angular Pseudo Entropy

Note: Related, but slightly different methods (also with different results) applied independently in (Rath, Kumar Samal '15 and earlier contributions)

Motivation

In QM stat. physics: Evolution $\rho_{\text{ini}} \to \rho_{\text{fin}}$ of system coupled to environment most generally given by completely positive trace preserving map. Canonical (Kraus) form $\rho_{\text{fin}} = \sum_{i=1}^{r} A_i^{\dagger} \rho_{\text{ini}} A_i$ with $\sum_{i=1}^{r} A_i A_i^{\dagger} = d\mathbb{1}$ $\to$ mix pure state and then apply von Neumann.

Define Angular pseudo entropy: $S_{\text{ang}}(\rho_\Psi) = S_{\text{N}}(\rho_{\text{ang},\Psi})$ with $\rho_{\text{ang},\Psi} = \frac{1}{2l+1} \sum_{i=1}^{3} L_i \rho_\Psi L_i$.

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Properties

- Pseudo entropy: Not all properties of proper entropies satisfied
- Conjecture: Minimum for coherent states (proven only for lowest l).
- Maximum is $\log(3)$, obtained for maximally mixed ρ_{ang} → still open: to what kind of map does this correspond (if at all)?
- Rotationally invariant

Quadratic and Projections Pseudo Entropies

Define j-Projection pseudo entropy: $S_{\text{proj},j}(\rho_\Psi) = S_{\mathrm{N}}(\rho_{\text{proj},j,\Psi})$ with $\rho_{\text{proj},j,\Psi} = \frac{2l+1}{2(l+j)+1} P_{j+l}(\rho_\Psi \otimes \mathbb{1}_j) P_{j+l}$, where P_{j+l} denotes the projection onto $\mathcal{H}_{l+j}$, such that $\rho_{\text{proj},j,\Psi}$ is a $(l+j, l+j)$-matrix. The projection involves Clebsch-Gordan coefficients.

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Properties

- Pseudo entropy
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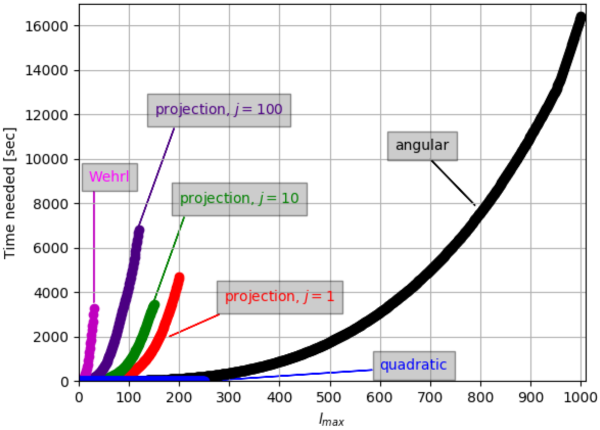
Define the quadratic entropy: $S_{\text{quad}}(\Psi) = -|P_{2l}(|\Psi\rangle \otimes |\Psi\rangle)|^2$

Properties

- Fastest to compute, but least suited $\to$ always negative, x^2-behavior instead of $x\log(x)$. Results differ from the other pseudo entropies

Numerical Expense

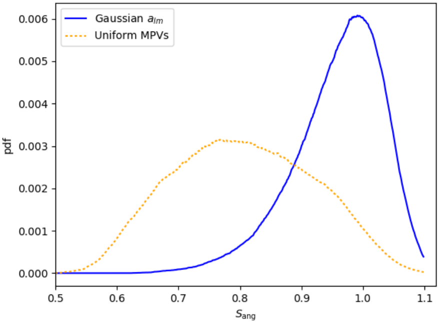
Time needed for one run up to l



Quadratic by far the fastest, but only approximate values for $l \gtrsim 250$, because $\binom{2l}{l+m}$ blows up.

Sensitivity

$l = 6$ comparison of probability distribution of angular entropy between isotropic, Gaussian map and map corresponding to uniformly distributed MPVs; computed with 10000 random maps

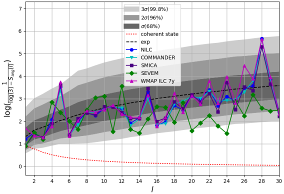


Deviation increases with growing $l \rightarrow$ better sensitivity at smaller angular scales. Both preferred direction and uniform MPVs have smaller entropy than Gaussian, isotropic map.

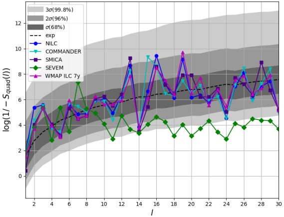
Results

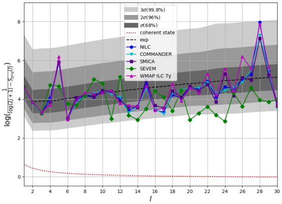
Results I – Comparison of Entropies

angular

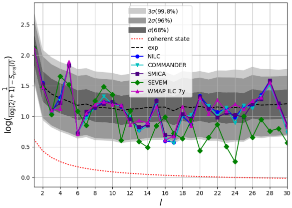


quadratic



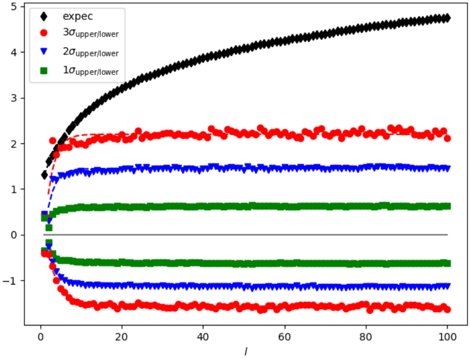


projection $j = 1$



projection $j = 100$

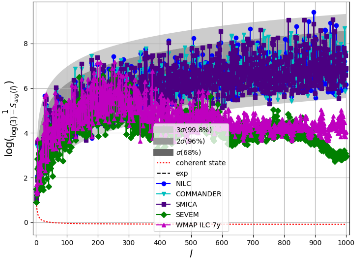
Results II - Fitting

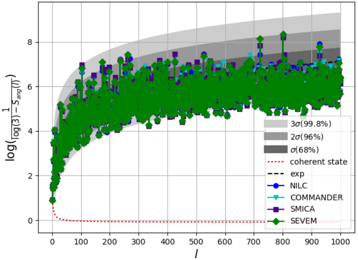


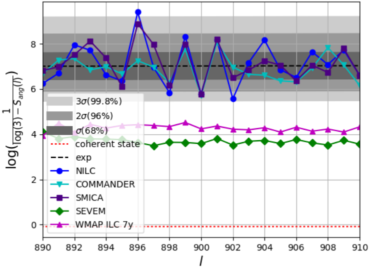
- Confidence levels: $a(1-\exp(-b(x-1)))$. Approx. constant from $l \approx 25$ on.
- Expectation value: $a\log(bx+c)$. Trustworthy fit for all l.

- For $\log\left(\frac{1}{\log(3)-S_{\text{ang}}}\right)$ we can fit the expectation value and the upper/lower sigma boundaries for isotropic, Gaussian a_{lm} with simple functions
- Can be used to compare null hypothesis and data up to very high l (≥ 1000).

Results III - Small Angular Scales





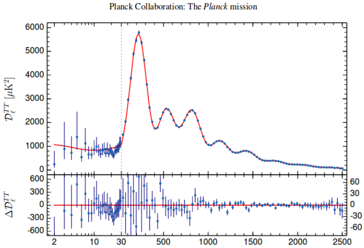


- As expected WMAP data bad from $l \approx 250$ on
- SEVEM extremely polluted from $l \approx 250$
- Masking removes strong deviation of SEVEN from other Planck maps
- Very low p-value for range $890 \leq l \leq 910$, except for COMMANDER.
- $\mathbb{P}(NILC; 896 \leq l \leq 902) \leq \sum_{i=4}^{7} B(i, 7; 0.04) \lessapprox 0.1\%$

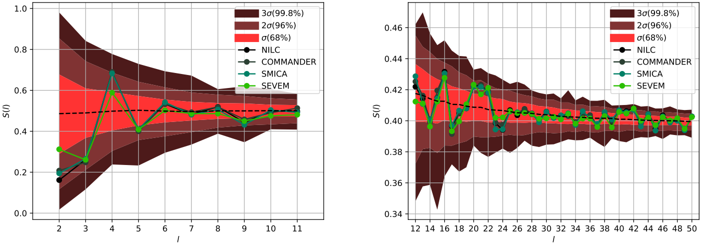
Sidekick: Recent Results with MPVs

Largest and Intermediate Angular Scales

Cosmic Dipole effect at same scales as pure power spectrum effect



Ade et al. '16 (Planck 2015 – I)



MP, Schwarz '18

$l = 2, 3, 4, 5$ and $l \approx 20$ in both even though MPV statistics independent of C_l!

Summary of MPV results in MP, D.J. Schwarz

- SEVEM strongly influenced by Galactic Plane $\rightarrow$ not suited for full sky analysis of isotropy
- COMMANDER, NILC and SMICA remarkably consistent
- $l \leq 5$ and l around 20: anomalous behavior in both power spectrum and MPV statistics. BUT: MPV statistics do not care for C_l but for only relative behavior of different m for given $l \rightarrow$ indication of statistical anisotropy?
- No unusual global behavior visible on range $2 \leq l \leq 50$
- Unusual correlation only w.r.t physical directions (mainly Cosmic Dipole), no intra-multipole correlations
- BUT: Lack of intra-multipole correlation
- Slight hint towards connection between parity anomaly and galactic foreground
- *Tasks for future:*
  - Investigate also cross-multipole correlations on extended range of scales
  - Consider polarization data (upcoming Planck release should yield pol-data for $l < 20$)

- Motivated by statistical QM, rotationally invariant pseudo entropies were constructed
- Allow for comparing hypothesis of Gaussian and isotropic δT up to $l \geq 1000$
- Methods are sensitive to deviation from null hypothesis