Chapter 1

Random matrix theory and number theory

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Abstract

We review some of the connections between Random Matrix Theory and Number Theory. These include modelling the value distributions of the Riemann zeta-function and other L-functions, and the statistical distribution of their zeros.

1.1 Introduction

One of the more surprising recent applications of Random Matrix Theory (RMT) has been to problems in Number Theory. There it has been used to address seemingly disparate questions, ranging from modelling mean and extreme values of the Riemann zeta-function to counting points on curves. In order to cover the many aspects of this rapidly developing area, we will focus on a selection of key ideas and list specific references where more information can be found. In particular, we direct readers who seek a more in-depth overview to the book of lecture notes edited by Mezzadri and Snaith [MS05], or previous reviews [Con01, KS03, Kea05].

1.2 The number theoretical context

Although the applications of random matrix theory (RMT) to number theory appear very diverse, they all have one thing in common: L-functions. The statistics of the critical zeros of these functions are believed to be related to those of the eigenvalues of random matrices.

The L-functions we will discuss share various properties. They have a Dirichlet series representation,

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.2.1}$$

with coefficients a_n satisfying suitable growth conditions, and, more importantly, a representation as an Euler product over the primes. These representations converge in a half-plane $\mathrm{Re}s > s_1$. Each L-function then has an analytic continuation that extends it to a meromorphic function in the whole complex plane. A functional equation relates the value of the L-function at s to that of the same or a closely related L-function at $a - s$; that is, the functional equation is a reflection symmetry about a *critical line* in complex s-plane. The Generalized Riemann Hypothesis puts the non-trivial zeros of the L-function on this critical line.

The most well-known L-function is the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - 1/p^s)^{-1}, \quad \mathrm{Re}(s) > 1. \tag{1.2.2}$$

For the zeta function the functional equation is

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} \zeta(1-s), \tag{1.2.3}$$

or, equivalently,

$$\tilde{\zeta}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \tilde{\zeta}(1-s). \tag{1.2.4}$$

The critical line $\mathrm{Re}(s) = 1/2$, about which this formula effectively reflects, is where the Riemann Hypothesis places all the complex zeros of $\zeta(s)$.

We shall, in the remainder of this article, assume the truth of the Generalized Riemann Hypothesis. This is not strictly necessary, but it makes the connections with Random Matrix Theory more transparent.

1.3 Zero Statistics

Denoting the n^{th} zero up the critical line by $\rho_n = 1/2 + i\gamma_n$, there is a great deal of interest in studying the statistical distribution of the heights γ_n. The

Riemann zeros grow logarithmically more dense with height up the critical line, so defining

$$w_n = \gamma_n \frac{1}{2\pi} \log \tfrac{|\gamma_n|}{2\pi} \tag{1.3.1}$$

we have

$$\lim_{W \to \infty} \frac{1}{W} \# \{w_n < W\} = 1. \tag{1.3.2}$$

Montgomery [Mon73] studied the pair correlation, or two-point correlation function, of these scaled zeros and conjectured that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{1 \le n,m \le N} f(w_n - w_m) = \int_{-\infty}^{\infty} f(x) \big(R_2(x) + \delta(x) \big) dx, \tag{1.3.3}$$

where

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2. \tag{1.3.4}$$

He proved that (1.3.3) is true for $f(x)$ such that

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x \tau} dx \tag{1.3.5}$$

has support in $(-1, 1)$.

Dyson pointed out that (1.3.4) coincides with the two-point correlation function of random hermitian (GUE) and unitary (CUE) matrices in the limit as the matrix size tends to infinity; that is, it coincides with the two-point correlation function of the eigenvalues of matrices averaged with respect to Haar measure on $U(N)$ in the limit $N \to \infty$. Odlyzko [Odl89] later produced striking numerical support for Montgomery's conjecture, and for its generalization to other local statistics of the Riemann zeros.

Montgomery's theorem, that (1.3.3) is true for $\hat{f}(\tau)$ with support in $(-1, 1)$, follows directly from the prime number theorem, via the connection, known as the *explicit formula*, between the Riemann zeros and the primes. (The explicit formula is a consequence of the Euler product.) Montgomery's theorem generalizes to all n-point correlations between the zeros [RS96]. The extension to values of τ outside $(-1, 1)$ requires information about correlations between the primes, as embodied in the Hardy-Littlewood conjecture concerning the number of integers $n < X$ such that both n and $n + h$ are prime. See, for example [BK95, BK96a], where the the Hardy-Littlewood conjecture is used to show heuristically that all n-point correlations of the Riemann zeros coincide with the corresponding CUE/GUE expressions in the large-matrix-size limit.

It is not just in the zero statistics high on the critical line of an individual L-function, such as the Riemann zeta-function, that one sees random matrix statistics. Katz and Sarnak [KS99a, KS99b] proposed that the zero statistics of

L-functions averaged over naturally defined families behave like the eigenvalues of matrices from $U(N)$, $O(N)$ or $USp(2N)$. Which ensemble of matrices models a given family depends on details of that family.

In many statistics, the eigenvalues of matrices from each of the three compact groups above show the same behaviour in the large matrix limit. For example, the limiting two-point correlation function (1.3.4) is common to all three. However, the orthogonal and unitary symplectic matrices have eigenvalues appearing in complex conjugate pairs, making the points 1 and -1 symmetry points of their spectra. Katz and Sarnak showed that the distribution of the first eigenvalue (or more generally the first few eigenvalues) near to this symmetry point is ensemble-specific (i.e. group-specific). If we define the distribution of the k-th eigenvalue $e^{i\theta_k}$ of a matrix A varying over $G(N) = U(N),\ O(N)$ or $USp(2N)$,

$$\nu_k(G(N))[a,b] = \text{meas}\{A \in G(N) : \tfrac{\theta_k N}{2\pi} \in [a,b]\}, \tag{1.3.6}$$

then Katz and Sarnak show that the limit

$$\lim_{N\to\infty} \nu_k(G(N)) = \nu_k(G) \tag{1.3.7}$$

exists, but, in contrast to the two-point correlation function, depends on G.

To be more explicit, let $\mathcal{F}$ denote a family of L-functions. An individual L-function within the family is identified by f and has conductor c_f (the conductor often orders the L-functions within the family and it also features in the density of the zeros). If γ_f^j denotes the height up the critical line of the jth zero, then the zeros near $s = 1/2$ (the point corresponding to 1 on the unit circle in the RMT analogy) are normalized to have constant mean spacing by scaling them in the following way:

$$\frac{\gamma_f^j \log c_f}{2\pi}. \tag{1.3.8}$$

Let $\mathcal{F}_X$ denote the members of the family $\mathcal{F}$ with conductor less than X. Then Katz and Sarnak define the distribution of the jth zero above $s = 1/2$ as

$$\nu_j(X,\mathcal{F})[a,b] = \frac{\#\left\{f \in \mathcal{F}_X : \frac{\gamma_f^{(j)} \log c_f}{2\pi} \in [a,b]\right\}}{\#\mathcal{F}_X}. \tag{1.3.9}$$

It is then expected, and Katz and Sarnak provide analytical and numerical evidence for this, that $\nu_j(X,\mathcal{F})$ converges, as $X \to \infty$, to $\nu_j(G)$, where G represents the symmetry type of the family: unitary, orthogonal or symplectic. Similarly, other statistics of the lowest zeros are also expected, upon averaging over the family, to tend to the corresponding random matrix statistics of the appropriate symmetry type in this limit.

It should be noted that the influence of the symmetry points on nonlocal statistics may extend much further into the spectrum, even in the large-matrix-size limit. The nature of the transition in various statistics, both local and non-local, has been explored in [KO08].

The question of determining the symmetry type of a given family a priori is, in general, a difficult one. The method used by Katz and Sarnak is that for some families of L-functions a related family of zeta functions on finite fields can be defined (see section 1.6 for further details). In the case of these zeta functions the definition of families is straightforward, the Riemann hypothesis has been proven (in that all zeros lie on a circle) and the symmetry type is determined by the monodromy of the family (see [KS99a]). The symmetry type of the related family of L-functions is then assumed to be the same.

1.4 Values of the Riemann zeta-function

In many instances in number theory questions arise involving the values of L-functions on or near the critical line. There is now a substantial literature, starting with [KS00a, KS00b], providing evidence that these values can be modelled using characteristic polynomials of random matrices. The characteristic polynomial is zero at the eigenvalues of the matrix, so for an $N \times N$ unitary matrix A with eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_N}$ we define

$$\Lambda(s) = \Lambda_A(s) = \det(I - A^* s) = \prod_{n=1}^{N} (1 - s e^{-i\theta_n}). \tag{1.4.1}$$

Averaging functions of this characteristic polynomial over one of the classical compact groups is a straightforward calculation using Selberg's integral (see [Meh04], chapter 17). For example, averaging over the unitary group with respect to Haar measure, moments of the characteristic polynomial [KS00a] are

$$\begin{aligned} M_N(\lambda) := \int_{U(N)} |\Lambda_A(e^{i\theta})|^{2\lambda} dA_{Haar} &= \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(2\lambda + j)}{(\Gamma(j+\lambda))^2} \\ &= \frac{G(N+1)G(2\lambda+N+1)G^2(\lambda+1)}{G(2\lambda+1)G^2(\lambda+N+1)}, \end{aligned} \tag{1.4.2}$$

where in the last line the Barnes double gamma function [Bar00] is used to express more compactly products over gamma functions by invoking

$$\begin{aligned} G(1) &= 1, \\ G(z+1) &= \Gamma(z)\, G(z). \end{aligned} \tag{1.4.3}$$

For large N (1.4.2) is asymptotic to

$$M_N(\lambda) \sim \frac{(G(1+\lambda))^2}{G(1+2\lambda)} N^{\lambda^2}. \tag{1.4.4}$$

We compare this to the moments of the Riemann zeta-function averaged along the critical line. Number theorists have conjectured that these grow asymptotically like

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda}dt \sim c_\lambda(\log T)^{\lambda^2} \tag{1.4.5}$$

for large T. Numerical evidence [KS00a, CFK+05] supports the conjecture that

$$c_\lambda = a_\lambda \frac{(G(1+\lambda))^2}{G(1+2\lambda)}, \tag{1.4.6}$$

where a_λ is a product over primes

$$a_\lambda \quad = \quad \prod_p \left(1-\tfrac{1}{p}\right)^{\lambda^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)}\right)^2 p^{-m}. \tag{1.4.7}$$

This agrees with the known values, $c_1 = a_1$ and $c_2 = a_2 \frac{2}{4!}$, and those conjectured independently based on number theoretical heuristics, $c_3 = a_3 \frac{42}{9!}$ and $c_4 = a_4 \frac{24024}{16!}$.

In the previous section it was seen in the comparison of Montgomery's pair correlation of the Riemann zeros and Dyson's two-point correlation of eigenvalues of matrices from $U(N)$ that the limit of large height up the critical line, $T \to \infty$, is related to the limit $N \to \infty$. Here we see that in the range where they are large but finite, N plays the role of $\log T$. This is a correspondence that is observed throughout this subject. It has been tested numerically in many situations and is motivated by equating the density of zeros with the density of eigenvalues on the unit circle: $\frac{1}{2\pi}\log\frac{T}{2\pi} = \frac{N}{2\pi}$.

One striking feature of the leading order moment conjecture (1.4.6) is the factorization into an arithmetical component, a_λ, and a random-matrix component coming from (1.4.4). The arithmetical component may be understood by noting that it arises essentially from substituting in (1.4.5) the formula for the zeta-function as a prime product, ignoring the fact that this product diverges on the line $\mathrm{Re}s = 1/2$, and interchanging the t-average and the product (i.e. assuming independence of the primes). However, this fails to capture the random-matrix component. Alternatively, the random-matrix component may be understood by substituting for the zeta-function its (Hadamard) representation as a product over its zeros and assuming that the zeros are distributed like CUE eigenvalues *over the range of scales that contribute to this product*. This fails to capture the arithmetical component, essentially because the moments are not local statistics.

The resolution of this problem comes from using a "hybrid" representation of the zeta-function:

$$\zeta(s) = P_X(s)Z_X(s)(1+o(1)), \tag{1.4.8}$$

where $P_X(s)$ behaves like the prime product for the zeta-function truncated smoothly so that only primes $p < X$ contribute to it, and $Z_X(s)$ behaves like the Hadamard product over the zeros of the zeta-function, truncated smoothly so that only zeros within a distance of the order of $1/\log X$ from s contribute [GHK07]. If X is not too large compared to T, the moments of $P_X(1/2+it)$ can be determined rigorously in terms of a_λ. The moments of $Z_X(1/2+it)$ may be modeled by averaging correspondingly truncated characteristic polynomials over $U(N)$ as $N \to \infty$. The RMT average can be computed in this limit using the asymptotic properties of Toeplitz determinants. Assuming that the moments of $\zeta(1/2+it)$ split into a product of the moments of $P_X(1/2+it)$ and those of $Z_X(1/2+it)$, the dependence on X drops out, leading directly to (1.4.6) [GHK07].

There is a specific interest in the moments when λ a positive integer, k. On the random matrix theory side, the $2k$th moment is then a polynomial in N of degree k^2, as can be seen from (1.4.2). This polynomial can be represented as a contour integral (the equality follows by evaluating the integral using Cauchy's theorem):

$$\begin{aligned}\int_{U(N)} |\Lambda_A(e^{i\theta})|^{2k} dA_{Haar} &= \prod_{j=0}^{k-1} \left(\frac{j!}{(k+j)!} \prod_{i=1}^{k} (N+i+j) \right) \\ &= \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{G(z_1,\ldots,z_{2k})\Delta^2(z_1,\ldots,z_{2k})}{\prod_{j=1}^{2k} z_j^{2k}} \\ &\qquad \times e^{\frac{1}{2}N\sum_{j=1}^{k} z_j - z_{k+j}} dz_1 \cdots dz_{2k},\end{aligned} \tag{1.4.9}$$

where the Vandermonde determinant is defined as $\Delta(x_1,\ldots,x_n) = \prod_{1\le j<\ell\le N}(x_\ell - x_j)$,

$$G(z_1,\ldots,z_{2k}) = \prod_{i=1}^{k}\prod_{j=1}^{k}(1-e^{-z_i+z_{j+k}})^{-1}, \tag{1.4.10}$$

and the integration is on small contours around the origin. See [BH00] for earlier examples of such contour integrals for Hermitian matrices.

The final equality in (1.4.9) is directly analogous to a similar contour integral which is conjectured to give the full main term of the corresponding moment of the Riemann zeta-function:

$$\begin{aligned}\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k} dt = \frac{1}{T}\int_0^T & \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}} \\ & \times \oint \cdots \oint \frac{G_\zeta(z_1,\ldots,z_{2k})\Delta^2(z_1,\ldots,z_{2k})}{\prod_{j=1}^{2k} z_j^{2k}} \\ & \times e^{\frac{1}{2}\log\frac{t}{2\pi}\sum_{j=1}^{k} z_j - z_{k+j}}\, dz_1 \ldots dz_{2k}\, dt + o(1),\end{aligned} \tag{1.4.11}$$

where

$$G_\zeta(z_1,\ldots,z_{2k}) = A_k(z_1,\ldots,z_{2k}) \prod_{i=1}^{k}\prod_{j=1}^{k} \zeta(1+z_i-z_{k+j}), \tag{1.4.12}$$

and A_k is another Euler product which is analytic in the regions we are interested in (see [CFK+08] for more on the development of these integral formulae to moments with non-integer values of λ). A close inspection of the residue structure of the contour integral in (1.4.11) shows that it is a polynomial of degree k^2 in $\log \frac{t}{2\pi}$, in analogy with the random matrix case.

The integral expressions in (1.4.9) and (1.4.11) are remarkably similar. We note the identical role played by N in the random matrix formulae and $\log T$ in the number theory expression. Also, to a large extent the behaviour of (1.4.9) and (1.4.11) are dominated by the poles of G and G_ζ, which are the same due to $(1-e^{-x})^{-1}$ having a simple pole with residue 1 at $x=0$ exactly as $\zeta(1+x)$ does.

These contour integral formulae for moments generalize to averages of the forms

$$\int_0^T \prod_{\alpha\in A} \zeta(\tfrac{1}{2}+it+\alpha) \prod_{\beta\in B} \zeta(\tfrac{1}{2}-it+\beta)dt, \tag{1.4.13}$$

[CFK+05] and

$$\int_0^T \frac{\prod_{\alpha\in A} \zeta(\frac{1}{2}+it+\alpha)\ \prod_{\beta\in B} \zeta(\frac{1}{2}-it+\beta)}{\prod_{\gamma\in C} \zeta(\frac{1}{2}+it+\gamma)\ \prod_{\delta\in D} \zeta(\frac{1}{2}-it+\delta)} dt, \tag{1.4.14}$$

with $\Re\gamma, \Re\delta > 0$ [CFZ08]. Again there is a striking similarity with corresponding formulae for the characteristic polynomials of random matrices [CFK+03, CFZ]. A remarkable amount of information can be extracted from the resulting formulae about statistics of zeros and values of the zeta function and its derivatives [CS07]. For example, the ratios conjecture with two zeta functions in the numerator and two in the denominator yields a formula, first written down by Bogomolny and Keating [BK96b], that includes all the significant lower order terms for the two-point correlation function for which Montgomery conjectured the limiting ($T\to\infty$) form:

$$\begin{aligned}\sum_{\gamma,\gamma'\leq T} f(\gamma-\gamma') = \frac{1}{(2\pi)^2}\int_0^T \Bigg(2\pi f(0)\log\frac{t}{2\pi} + \int_{-T}^{T} f(r)\Bigg(\log^2\frac{t}{2\pi} \\ +2\Bigg(\left(\frac{\zeta'}{\zeta}\right)'(1+ir) + \left(\frac{t}{2\pi}\right)^{-ir}\zeta(1-ir)\zeta(1+ir)A(ir) - B(ir)\Bigg)\Bigg)\, dr\Bigg)\, dt \\ +O(T^{1/2+\epsilon});\end{aligned} \tag{1.4.15}$$

here the integral is to be regarded as a principal value near $r = 0$,

$$A(\eta) = \prod_p \frac{(1 - \frac{1}{p^{1+\eta}})(1 - \frac{2}{p} + \frac{1}{p^{1+\eta}})}{(1 - \frac{1}{p})^2}, \tag{1.4.16}$$

and

$$B(\eta) = \sum_p \left(\frac{\log p}{(p^{1+\eta} - 1)} \right)^2 . \tag{1.4.17}$$

The limit coincides with random matrix theory (taking $T \to \infty$ in (1.4.15) with $f(x)$ replaced by $g\big(x(\log \frac{T}{2\pi})/(2\pi)\big)/\big(\frac{T}{2\pi} \log \frac{T}{2\pi}\big)$ reproduces Montgomery's conjecture (1.3.3)), but the lower order contributions contain important arithmetic information specific to the Riemann zeta-function. See [BK99] for further details and numerical illustrations of the importance of these lower order terms and [CS08] for a generalization to all n-point correlations.

1.5 Values of L-functions

Many applications in number theory require one to consider mean values of L-functions evaluated at the critical point (the centre of the critical strip – where the critical line crosses the real axis) and averaged over a suitable family. Two examples of such families are given below and more details can be found in, for example, [CF00, KS00b, CFK+05].

The Dirichlet L-functions associated to real, quadratic characters form a family with symplectic symmetry:

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} = \prod_p \left[1 - \chi_d(p) p^{-s}\right]^{-1}, \tag{1.5.1}$$

where $\chi_d(n) = \left(\frac{d}{n}\right)$ is Kronecker's extension of Legendre's symbol which is defined for p prime,

$$\left(\frac{d}{p}\right) = \begin{cases} +1 & \text{if } p \nmid d \text{ and } x^2 \equiv d (\text{mod} p) \text{ is soluble} \\ 0 & \text{if } p|d \\ -1 & \text{if } p \nmid d \text{ and } x^2 \equiv d (\text{mod} p) \text{ is not soluble.} \end{cases} \tag{1.5.2}$$

The character χ_d exists for all fundamental discriminants d, and the L-functions attached to these characters are said to form a family as we vary d. The family can be partially ordered by the *conductor* $|d|$. The symplectic symmetry type can be seen in the statistics of the zeros, studied numerically by Rubinstein [Rub01] and theoretically in the limit of large d by Özlük and Snyder [ÖS99].

The symplectic symmetry type also determines the behaviour of the moments of these L-functions. A moment conjecture similar to that leading to (1.4.11) can be applied here to obtain a conjectural contour integral formula [CFK$^+$05] which implies:

$$\sum_{|d|\leq D}^{*} L(\tfrac{1}{2}, \chi_d)^k = \frac{6}{\pi^2} D\, \mathcal{Q}_k(\log D) + O(D^{\frac{1}{2}+\epsilon}), \tag{1.5.3}$$

where $\sum^*$ is over fundamental discriminants and so the sum is over all real, primitive Dirichlet characters of conductor up to D. Here $\mathcal{Q}_k$ is a polynomial of degree $k(k+1)/2$, with leading coefficient $f_k a_k$, where

$$a_k = \prod_p \frac{(1-\frac{1}{p})^{\frac{k(k+1)}{2}}}{1+\frac{1}{p}} \left(\frac{(1-\frac{1}{\sqrt{p}})^{-k} + (1+\frac{1}{\sqrt{p}})^{-k}}{2} + \frac{1}{p} \right) \tag{1.5.4}$$

and

$$f_k = \prod_{j=1}^{k} \frac{j!}{(2j)!}. \tag{1.5.5}$$

The crucial point to note is the degree of the polynomial $\mathcal{Q}_k$, which is $k(k+1)/2$, mirroring RMT calculations with matrices from $USp(2N)$ [KS00b]:

$$\int_{USp(2N)} \Lambda_A(1)^k dA_{Haar} = \left(2^{k(k+1)/2} \prod_{j=1}^{k} \frac{j!}{(2j)!} \right) \prod_{1\leq i\leq j\leq k} (N + \tfrac{i+j}{2}). \tag{1.5.6}$$

This is to be compared to the corresponding degree, k^2, for the unitary group. Also, in this case equating the density of zeros near the point $1/2$ with the density of eigenvalues gives an equivalence $2N = \log|d|$, for large $|d|$ and N. The main term of conjecture (1.5.3) for $k = 1, 2, 3$ has been proved in the number theory literature.

As for the Riemann zeta-function, there is a contour integral expression for shifted moments and averages of ratios. To give a specific example [CS07] (where the two terms below result from residues of the contour integral formula):

$$\begin{aligned}\sum_{d\leq X} \frac{L(1/2+\alpha,\chi_d)}{L(1/2+\gamma,\chi_d)} &= \sum_{d\leq X} \Bigg(\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)} A_D(\alpha;\gamma) \\ &\quad + \left(\frac{d}{\pi}\right)^{-\alpha} \frac{\Gamma(1/4-\alpha/2)}{\Gamma(1/4+\alpha/2)} \frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)} A_D(-\alpha;\gamma) \Bigg) + O(X^{1/2+\epsilon}),\end{aligned} \tag{1.5.7}$$

where

$$A_D(\alpha;\gamma) = \prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}}\right)^{-1} \left(1 - \frac{1}{(p+1)p^{1+2\alpha}} - \frac{1}{(p+1)p^{\alpha+\gamma}}\right). \tag{1.5.8}$$

From this conjecture one can determine the one-level density of the zeros of these Dirichlet L-functions and see the dependence of the lower order terms on arithmetical quantities (the Riemann zeta-function and products over primes) in exactly the same way as for the two-point correlation function (1.4.15):

$$\begin{aligned}\sum_{d\le X}\sum_{\gamma_d} f(\gamma_d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\sum_{d\le X}\Bigg(&\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4+it/2) \\ &+\frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4-it/2)+2\bigg(\frac{\zeta'(1+2it)}{\zeta(1+2it)}+A_D'(it;it) \\ &-\left(\frac{d}{\pi}\right)^{-it}\frac{\Gamma(1/4-it/2)}{\Gamma(1/4+it/2)}\zeta(1-2it)A_D(-it;it)\bigg)\Bigg)\,dt \\ &+O(X^{1/2+\epsilon}).\end{aligned} \tag{1.5.9}$$

Here $A_D'(it;it)$ is a derivative of A_D.

If the zeros γ_d in (1.5.9) are scaled by their mean density then in the limit as $X\to\infty$ the expression multiplying $f(t)$ in the integrand reduces to $1-[\sin(2\pi t)/(2\pi t)]^2$, which is the limiting one-level density of the group of unitary symplectic matrices $USp(2N)$. This is the limiting behaviour predicted by Katz and Sarnak (see section 1.3). However, one can see from (1.5.9) that the approach to the limit is determined by arithmetical structure.

A family with zero statistics corresponding to an orthogonal family of matrices is that of elliptic curve L-functions. An elliptic curve is represented by an equation of the form $y^2=x^3+ax+b$, where a and b are integers. Let

$$L_E(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}, \tag{1.5.10}$$

where $a(p)=p+1-\#E(\mathbb{F}_p)$ ($\#E(\mathbb{F}_p)$ being the number of points on E counted over $\mathbb{F}_p$), be the L-function associated with an elliptic curve E. A family can be created by "twisting" this L-function by χ_d and varying d:

$$L_E(s,\chi_d)=\sum_{n=1}^{\infty}\frac{a(n)\chi_d(n)}{n^s}. \tag{1.5.11}$$

This family is then ordered by d in exactly the same way as the symplectic family above (which, in fact, could be thought of as "twisting" the Riemann zeta-function by the real, quadratic characters). Similar calculations to those described for the family of Dirichlet L-functions and for the Riemann zeta-function can also be carried out in this case. In particular, one sees once again the symmetry type, in this case orthogonal, dictating the form of the moments. The moments of characteristic polynomials of matrices from $SO(2N)$

are [KS00b]

$$\int_{SO(2N)} \Lambda_A(1)^k dA_{Haar} = \left(2^{k(k-1)/2}\, 2^k \prod_{j=1}^{k-1} \frac{j!}{(2j)!}\right) \prod_{0\le i<j\le k-1} (N + \tfrac{i+j}{2}), \tag{1.5.12}$$

which is a polynomial in N of degree $k(k-1)/2$. From the elliptic curve family one can select the L-functions which have an even functional equation (forcing even symmetry of the zeros on the critical line around the point 1/2 - in analogy with selecting from $O(2N)$ the matrices $SO(2N)$ which have even symmetry of their eigenvalues around the point 1 on the unit circle). The average value of the kth power of these L-functions, with $|d| < D$, approximates a polynomial in $\log D$ of degree $k(k-1)/2$.

In this case the moment conjectures inspired by RMT have very significant applications to ranks of elliptic curves; that is, to the long-standing and important question of how likely it is that a given elliptic curve has a finite as opposed to an infinite number of rational solutions (points) [CKRS00, CKRS06]. The connection comes via the conjecture of Birch and Swinnerton-Dyer, which relates this number to the value of L_E at the centre of the critical strip. RMT predicts the value distribution of L_E at this point and hence the asymptotic fraction of curves with $d < X$ having an infinite number of solutions.

1.6 Further areas of interest

Many of the connections between RMT and number theory are still somewhat speculative, being a mixture of theorems proved in a limited class of cases, numerical computations, and heuristic arguments. The area in which they have the most solid foundations is that relating to zeta functions of curves over finite fields. Let $\mathbb{F}_q$ denote a finite field with q elements and $\mathbb{F}_q[x]$ polynomials $f(x)$ with coefficients in $\mathbb{F}_q$. A polynomial $f(x)$ in $\mathbb{F}_q[x]$ is *reducible* if $f(x) = g(x)h(x)$ with $\deg(g) > 0$ and $\deg(h) > 0$, and *irreducible* otherwise. One defines an extension field $\mathbb{E}$ of $\mathbb{F}_q$, of degree n, by adjoining to $\mathbb{F}_q$ a root of an irreducible polynomial of degree n. Polynomials in $\mathbb{F}_q$ play the role of integers, and monic irreducible polynomials play the role of the primes. One can define a zeta function for $\mathbb{F}_q[x]$ in terms of a product over the monic irreducible polynomials in direct analogy to the prime product for the Riemann zeta-function. More generally, one can associate a zeta function to a smooth, geometrically connected, proper curve $\mathcal{C}$ defined over $\mathbb{F}_q$ in the following way: let N_n denote the number of points of $\mathcal{C}$ over an extension field of degree n, then

$$Z_{\mathcal{C}}(u) = \exp\big(\sum_{n=1}^{\infty} \frac{N_n u^n}{n}\big). \tag{1.6.1}$$

It turns out that $Z_{\mathcal{C}}(u)$ is a rational function of u, of the form

$$Z_{\mathcal{C}}(u) = \frac{P_{\mathcal{C}}(u)}{(1-u)(1-qu)} \tag{1.6.2}$$

where $P_{\mathcal{C}}(u)$ is a monic integer polynomial of degree $2g$, where g is the genus of $\mathcal{C}$. $Z_{\mathcal{C}}(u)$ satisfies a functional equation connecting u with $1/(qu)$, and importantly, can be *proved* to obey a Riemann Hypothesis, namely all of the inverse roots of $P_{\mathcal{C}}(u)$ have absolute value $\sqrt{q}$. Of particular relevance is the fact that $P_{\mathcal{C}}(u)$ turns out to be the characteristic polynomial of a symplectic matrix of size $2g$.

The connection with RMT comes when the curve $\mathcal{C}$ is allowed to vary. Consider, for example, the moduli space of hyperelliptic curves $\mathcal{C}_Q$ of genus g over $\mathbb{F}_q$, namely curves of the form $y^2 = Q(x)$, where $Q(x)$ is a square-free monic polynomial of degree $2g+1$. Averaging $Q(x)$ uniformly over all possible square-free monic polynomials generates an ensemble of $(q-1)q^{2g}$ curves. The central result, which follows from work of Deligne, is that, when g is fixed, the matrices associated with $P_{\mathcal{C}}(u)$ become equidistributed with respect to Haar measure on $USp(2g)$ in the limit $q \to \infty$ [KS99a, KS99b]. The zeros of the zeta functions $Z_{\mathcal{C}}(u)$ are therefore distributed like the eigenvalues of random matrices associated with $USp(2g)$. For other families of curves, equidistribution with respect to the unitary or orthogonal groups can similarly be established. Examples have also been found relating to the exceptional group G_2 [KLR03]. It is believed that when q is fixed and $g \to \infty$ the local statistics of the eigenvalues of the matrices associated with $P_{\mathcal{C}}(u)$ have a limit which coincides with the corresponding limit for averages of $USp(2g)$ [KS99a, KS99b], but this has been established in only a few cases, principally relating to the value distribution of the traces of matrices in question [PZ03].

A number of further results have followed the initial work linking RMT and number theory. RMT calculations play various roles in these examples. Often rigorous calculations with random matrices suggest conjectures in number theory or that theorems hold more generally than can be proven number theoretically, as in the example of Hughes and Rudnick's mock Gaussian statistics [HR03b, HR02, HR03a]. They prove [HR02] the following theorem about linear statistics of the zeros of the Riemann zeta-function:

Theorem 1.6.1 *Define*

$$N_f(\tau) := \sum_{j=\pm1,\pm2,\ldots} f\Big(\frac{\log T}{2\pi}(\gamma_j - \tau)\Big), \tag{1.6.3}$$

and consider τ as varying near T in an interval of size $H = T^a$, $0 < a \leq 1$. With f a real-valued, even test function having Fourier transform $\hat{f} :=$

$\int_{-\infty}^{\infty} f(x)e^{-2\pi ixu}dx \in C_{\mathcal{C}}^{\infty}(\mathbb{R})$ *and* $\mathrm{supp}\hat{f} \subseteq (-2a/m, 2a/m)$*, then the first* m *moments of* N_f *converge as* $T \to \infty$ *to those of a Gaussian random variable with expectation* $\int_{-\infty}^{\infty} f(x)dx$ *and variance*

$$\sigma_f^2 = \int_{-\infty}^{\infty} \min(|u|, 1)\hat{f}(u)^2 du. \tag{1.6.4}$$

Interestingly, the higher moments for test functions with this same range of support are not Gaussian.

Hughes and Rudnick illustrate exactly the same behaviour for eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_N}$ of an $N \times N$ random unitary matrix U. They consider the 2π-periodic function

$$F_N(\theta) := \sum_{j=-\infty}^{\infty} f\left(\tfrac{N}{2\pi}(\theta + 2\pi j)\right) \tag{1.6.5}$$

and model N_f with

$$Z_f(U) := \sum_{j=1}^{N} F_N(\theta_j). \tag{1.6.6}$$

They find the same variance as (1.6.4), although, significantly, there is no restriction on the support of the Fourier transform of the test function in order for the variance to have this form. In the RMT calculation, they see the same mock-Gaussian behaviour for the moments up to the mth, defined as $\lim_{N\to\infty} \mathbb{E}\{(Z_f - \mathbb{E}\{Z_f\})^m\}$ however in the RMT setting they can prove that the mth moment is Gaussian if $\mathrm{supp}\hat{f} \subseteq [-2/m, -2/m]$. This leads them to conjecture that the more restrictive condition on the support is not necessary in the number theory case.

Random matrix theory has also played a part in the study of average values and the statistics of zeros of the derivative of the Riemann zeta-function. Methods for proving a lower bound for the fraction of zeros of the Riemann zeta-function lying on the critical line could be improved if more was known about the positions of the zeros of the derivative of the zeta function near the critical line. In fact, if the Riemann hypothesis is true then all the zeros of the derivative will have real part greater than or equal to 1/2. More delicate information about their horizontal distribution would allow for more accurate counting of zeros of zeta on the critical line. Inspired by this, Mezzadri [Mez03] conjectured and the team [DFF$^+$] confirmed the form for the leading order term near the unit circle of the radial distribution of the zeros of the derivative of the characteristic polynomial of a random unitary matrix. (Note that all the zeros of the derivative of a polynomial with roots on the unit circle lie *inside* the unit circle.) This led to a similar calculation for the distribution of the zeros of the derivative of the Riemann zeta-function near the critical line, and the prediction compares very well with the distribution computed numerically. The theorem of [DFF$^+$] is

Theorem 1.6.2 *Let $\Lambda(z)$ be the characteristic polynomial of a random matrix in $U(N)$ distributed with respect to Haar measure, and let $Q(s;N)$ be the probability density function of $S = N(1-|z'|)$ for z' a root of $\Lambda'(z)$. Then*

$$Q(s) = \lim_{N\to\infty} Q(s;N) \sim \frac{4}{3\pi} s^{1/2}. \tag{1.6.7}$$

Various moments of the derivative of characteristic polynomials have also been considered, with a view to providing insight into moments of the derivative of $\zeta(s)$, but this problem seems much less tractable than the moments of characteristic polynomials themselves. In particular, for averages over $U(N)$ at an arbitrary point on the unit circle of even integer powers of the derivative of the characteristic polynomial, only leading order results, for large matrix size, have been obtained. This is to be compared with the exact formula for the moments of $\Lambda(s)$ itself (1.4.2). In his thesis [Hug01] (Chapter 6), Hughes calculated the averages

$$\int_{U(N)} |\Lambda_A(1)|^{2k-2h} |\Lambda'_A(1)|^{2h} dA_{Haar} \sim F(h,k) N^{k^2+2h}, \tag{1.6.8}$$

and so conjectured the leading order form of joint moments of the Riemann zeta-function and its derivative:

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k-2h} |\zeta'(\tfrac{1}{2}+it)|^{2h} \sim F(h,k) a_k \Big(\log\frac{T}{2\pi}\Big)^{k^2+2h}. \tag{1.6.9}$$

Here a_k is given in (1.4.7) and $F(h,k)$ is a combinatorial sum. There is also a similar result for the $2k$th moment of the absolute value of the derivative of the characteristic polynomial [CRS06]. This case is covered by Hughes' work, but the latter paper gives a different form for the coefficient, the analog of $F(h,k)$, in terms of $k\times k$ determinants of Bessel functions. Looking to the future, a goal would be to derive lower order contributions to (1.6.9) and to be able to calculate non-integer moments.

Another moment calculation involving the derivative of the characteristic polynomial is the discrete moment, where the derivative is evaluated at the zeros of $\zeta(s)$. This was first modelled using RMT in [HKO00]. As can be seen below, in this case an exact formula for the RMT moment can be obtained, leading to an asymptotic conjecture for the analogous number theoretical quantity. Here the eigenvalues of the unitary matrix A are $e^{i\theta_1},\ldots,e^{i\theta_N}$, $G(z)$ is the Barnes double gamma function, and we require $k > -3/2$:

$$\begin{aligned}\int_{U(N)} \frac{1}{N}\sum_{n=1}^{N} |\Lambda'_A(e^{i\theta_n})|^{2k} dA_{Haar} &= \frac{G^2(k+2)}{G(2k+3)} \frac{G(N+2k+2)G(N)}{N\, G^2(N+k+1)} \\ &\sim \frac{G^2(k+2)}{G(2k+3)} N^{k(k+2)}, \quad \text{as } N\to\infty. \end{aligned}\tag{1.6.10}$$

This result led to the following conjecture for large T, where $N(T)$ is the number of zeros of the Riemann zeta-function with height γ_n between 0 and T,

$$\frac{1}{N(T)}\sum_{0<\gamma_n\le T}|\zeta'(\tfrac{1}{2}+i\gamma_n)|^{2k}\sim\frac{G^2(k+2)}{G(2k+3)}a_k\left(\log\frac{T}{2\pi}\right)^{k(k+2)}. \tag{1.6.11}$$

Lower order terms for the moments with $k=1$ and $k=2$ were determined in [CS07] (Sections 7.1 and 7.2) using the ratios conjectures mentioned at (1.4.14).

A significant recent development in the subject is that Bourgade, Hughes, Nikeghbali and Yor [BHNY08] have shown that averages of characteristic polynomials behave in the same way as products of independent beta variables:

Theorem 1.6.3 *Let $V_N\in U(N)$ be distributed with the Haar measure $\mu_{U(N)}$. Then for all $\theta\in\mathbb{R}$,*

$$\det(I-e^{i\theta}V_N)\overset{\text{law}}{=}\prod_{k=1}^{N}(1+e^{i\theta_k}\sqrt{\beta_{1,k-1}}) \tag{1.6.12}$$

with $\theta_1,\ldots,\theta_n,\beta_{1,0},\ldots,\beta_{1,n-1}$ independent random variables, the θ_k's uniformly distributed on $[0,2\pi]$, and the $\beta_{1,j}$'s $(0\le j\le N-1)$ being beta-distributed with parameters 1 and j. (By convention, $\beta_{1,0}$ is the Dirac distribution on 1.)

The probability density of a beta-distributed random variable, defined on $[0,1]$, with parameters α and β is

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}. \tag{1.6.13}$$

This leads to an alternative proof of (1.4.2) and a probabilistic proof of the Selberg integral formula. It also provides an explanation for the fact, which may be proved by analyzing the moment generating function for $\log\Lambda_A$ (given by a formula generalizing (1.4.2)), that the logarithm of the characteristic polynomial of a random $U(N)$ matrix satisfies a central limit theorem when $N\to\infty$ [BF97, KS00a]. The limiting Gaussian behaviour follows naturally from the probabilistic model because the logarithm of the characteristic polynomial may be decomposed into sums of independent random variables and then classical central limit theorems can be applied. This approach also leads to new estimates on the rate of convergence to the Gaussian limit. That $\log\Lambda_A$ satisfies a central limit theorem is relevant to number theory because of a theorem due to Selberg [Tit86, Odl89]:

Theorem 1.6.4 (Selberg)

For any rectangle $B\in\mathbb{C}$,

$$\lim_{T\to\infty}\frac{1}{T}\left|\left\{t:T\le t\le 2T,\frac{\log\zeta(1/2+it)}{\sqrt{\frac{1}{2}\log\log T}}\in B\right\}\right|=\frac{1}{2\pi}\int\int_B e^{-(x^2+y^2)/2}dx\,dy.$$

That is, in the limit as T, the height up the critical line, tends to infinity, the value distributions of the real and imaginary parts of $\log\zeta(1/2 + iT)/\sqrt{(1/2)\log\log T}$ each tend, independently, to a Gaussian with unit variance and zero mean. This coincides exactly with the central limit theorem for $\log\Lambda_A$, if, as in other calculations, N and $\log(T/2\pi)$ are identified.

The fact that $\log\Lambda_A$ models the value distribution of $\log\zeta(1/2+it)$ extends to the large deviations regime too. See [HKO01], where also the ergodicity of the Gaussian limit of the value distribution of $\log\Lambda_A$ is established.

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