Correlation functions for products of coupled random matrices

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Based on joint work with G. Akemann

1. Sum and product matrix processes

(Motivated by recent works on products of random matrices: **Akemann, Ipsen, and Kieburg; Kuijlaars and Stivigny; Clayes, Kuijlaars, and Wang**)

Sum matrix processes

- $H_1, \ldots, H_m$-**random Hermitian matrices of the same size** $N \times N$.
- **Set**

$$X(k) = H_k + \ldots + H_1, \quad k \in \{1, \ldots, m\}.$$

- x_j^k-j**th largest eigenvalue of** $X(k)$. **The configuration**

$$\left\{(k, x_j^k) | k = 1, \ldots, m; j = 1, \ldots, N\right\}$$

forms a point process on $\{1, \ldots, m\} \times \mathbb{R}$. **It is called a sum matrix process.**

Product matrix processes

- G_1, G_2, ...,G_m**-random complex matrices.**
- G_1 **is of size** $N_1 \times N_0$, G_2 **is of size** $N_2 \times N_1$, ..., G_m **is of size** $N_m \times N_{m-1}$.
- **Set**

$$X(k) = G_k \dots G_1, \quad k \in \{1, \dots, m\}.$$

For each k, $X(k)^*X(k)$ **are random matrices of the same size** $N_0 \times N_0$.

- x_j^k-j**th largest eigenvalue of** $X(k)^*X(k)$. **The configuration**

$$\left\{(k, x_j^k) | k = 1, \dots, m; j = 1, \dots, N_0\right\}$$

forms a point process on $\{1, \dots, m\} \times \mathbb{R}_{>0}$. **It is called a product matrix process.**

Questions

- **Are there product matrix processes (or sum matrix processes) whose correlation functions can be explicitly computed?**
- **Can we choose complex random matrices G_1, G_2, $\ldots$, G_m such that the resulting product matrix process will be a determinantal point process on $\{1, \ldots, m\} \times \mathbb{R}_{>0}$?**
- **Can we choose Hermitian random matrices H_1, H_2, $\ldots$, H_m such that the resulting sum process will be a determinantal point process on $\{1, \ldots, m\} \times \mathbb{R}$?**

The Ginibre product process with polynomial ensemble initial conditions

- G_1, ...,G_m**-independent.**
- G_2, ...,G_m**-complex Ginibre matrices.**
- **The squared singular values** $\left(x_1^1,\ldots x_{N_0}^1\right)$ **of** G_1 **has density proportional to**

$$\triangle\left(x_1^1,\ldots,x_{N_0}^1\right)\det\left[f_i(x_j^1)\right]_{i,j=1}^{N_0}.$$

Theorem

The Ginibre product process with polynomial ensemble initial conditions is a determinantal point process on $\{1,\ldots,m\}\times\mathbb{R}_{>0}$.

Sketch of the proof

- **Step 1. Let $\underline{x}=(x^1,\ldots,x^m)$ be the vector of the squared singular values. The probability density of $\underline{x}$ is proportional to a product of determinants:**

$$\det\left[\left(x_j^m\right)^{k-1}\right]_{j,k=1}^{n}\prod_{l=1}^{m-1}\det\left[\frac{\left(x_j^{l+1}\right)^{\nu_{l+1}}}{\left(x_k^l\right)^{\nu_{l+1}+1}}e^{-\frac{x_j^{l+1}}{x_k^l}}\right]_{j,k=1}^{n}\det\left[f_k(x_j^1)\right]_{j,k=1}^{n},$$

where $n=N_0$, $\nu_j=N_j-N_0$.

- **Step 2. Apply the Eynard-Mehta theorem.**

Example 1. The initial matrix G_1 is a complex Ginibre matrix

Proposition

The correlation kernel is given by

$$K_{n,m}(r,x;s,y) = -\frac{1}{x} G^{s-r,0}_{0,s-r}\left(\begin{array}{c} - \\ \nu_{r+1},\ldots,\nu_s \end{array} \Bigg| \frac{y}{x} \right) \mathbf{1}_{s>r}$$

$$+ \frac{1}{(2\pi i)^2} \int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} du \oint\limits_{\Sigma_n} dt \frac{\prod_{j=0}^{s}\Gamma(u+\nu_j+1)}{\prod_{j=0}^{r}\Gamma(t+\nu_j+1)} \frac{\Gamma(t-n+1)}{\Gamma(u-n+1)} \frac{x^t y^{-u-1}}{t-u},$$

where Σ_n is a closed contour going around $0,1,\ldots,n$ in the positive direction and such that $t > -\frac{1}{2}$ for $t \in \Sigma_n$.

Example 2. The initial matrix is a truncation of a random unitary matrix

Proposition

The correlation kernel is given by

$$K_{n,m}(r,x;s,y)$$

$$= -\frac{1}{x} G^{s-r,0}_{0,s-r}\left(\begin{matrix} - \\ \nu_{r+1},\ldots,\nu_s \end{matrix} \middle| \frac{y}{x} \right) \mathbf{1}_{s>r} + \frac{1}{(2\pi i)^2} \int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} du$$

$$\oint\limits_{\Sigma_n} dt \frac{\prod_{j=0}^{s}\Gamma(u+\nu_j+1)}{\prod_{j=0}^{r}\Gamma(t+\nu_j+1)} \frac{\Gamma(t-n+1)\Gamma(t+l-n+1)}{\Gamma(u-n+1)\Gamma(u+l-n+1)} \frac{x^t y^{-u-1}}{t-u}.$$

Sums of independent GUE matrices

Proposition

If $H_1, \ldots, H_m$ are independent GUE matrices, then the sum matrix process is a determinantal point process on $\{1, \ldots, m\} \times \mathbb{R}$. Its correlation kernel can be written as

$$K_{N,m}(r, x; s, y) = -\frac{1}{2^{\frac{1}{2}}(s-r)^{\frac{1}{2}}\pi^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{2(s-r)}} \mathbf{1}_{s>r} + \frac{e^{-\frac{y^2}{2s}}}{(2\pi s)^{\frac{1}{2}}} \sum_{i=0}^{N-1} \left(\frac{r}{4s}\right)^{\frac{i}{2}} H_i\left(\frac{x}{2^{\frac{1}{2}} r^{\frac{1}{2}}}\right) H_i\left(\frac{y}{2^{\frac{1}{2}} s^{\frac{1}{2}}}\right) \frac{1}{i!},$$

where $H_i(x)$ are the Hermite polynomials.

Formulation of the problem

- **Drop the assumption that the involved matrices are independent.**
- **Extend the results to coupled random matrices.**

2. Coupled matrices

Products of two coupled matrices

- G_1, G_2-random complex matrices; G_1 is of size $M \times N$, G_2 is of size $N \times M$; $\nu = M - N \geq 0$.
- The joint distribution of G_1, G_2 is defined by

$$\frac{1}{Z} e^{-a \sum_{l=1}^{2} Tr\left[G_l^* G_l\right] + b\left(Tr[G_2 G_1] + Tr[(G_2 G_1)^*]\right)} \prod_{l=1}^{2} dG_l.$$

  Here $a > 0$, $b > 0$.
- Set

$$X(2) = G_2 G_1, \quad X(1) = G_1.$$

- x_j^k-jth largest eigenvalue of $X(k)^* X(k)$. The configuration

$$\left\{(k, x_j^k) | k = 1, 2; j = 1, \ldots, N\right\}$$

  is called the **Ginibre product process with coupling**.

The Ginibre product process with coupling as a determinantal process

Theorem

The Ginibre product process with coupling is a determinantal point process on $\{1,2\}\times\mathbb{R}_{>0}$. *Its correlation kernel is given by*

$$K_N(r,x;s,y) = -\frac{1}{x}e^{-\frac{ay}{x}}1_{s>r} + \sum_{p=0}^{N-1} P_{r,p}(x;a,b)Q_{s,p}(y;a,b),$$

where $1\leq r,s\leq 2$, *and the functions* $P_{r,p}(x;a,b)$ *and* $Q_{s,p}(y;a,b)$ *can be written explicitly.*

3. Particular case: the singular values of the product matrix.

We consider in detail the determinantal process on $R_{>0}$ formed by the squared singular values of $Y = G_2 G_1$.

Proposition

The density of the squared singular values of $Y = G_2 G_1$ is given by

$$P(y_1, \ldots, y_N; a, b) = \frac{1}{Z_N} \det \left[y_j^{\frac{k-1}{2}} I_{k-1}\left(2 b y_j^{\frac{1}{2}}\right) \right]_{j,k=1}^{N} \times \det \left[y_j^{\frac{k+\nu-1}{2}} K_{j+\nu-1}\left(2 a y_j^{\frac{1}{2}}\right) \right]_{j,k=1}^{N}.$$

- **Set** $a(\mu)=\frac{1+\mu}{2\mu}$, $b(\mu)=\frac{1-\mu}{2\mu}$.
- **We have**

$$(a)\ \lim_{\mu\to 0} P(y_1,\ldots,y_N;a(\mu),b(\mu)) = P_{Laguerre}(y_1,\ldots,y_N),$$

where $P_{Laguerre}(y_1,\ldots,y_N)$ **is proportional to**

$$\left(\det\left[y_i^{\frac{j-1}{2}}\right]_{i,j=1}^{N}\right)^2 \prod_{i=1}^{N} y_i^{\frac{\nu-1}{2}} \exp\left[-2y_i^{\frac{1}{2}}\right].$$

$$(b)\ \lim_{\mu\to 1} P(y_1,\ldots,y_N;a(\mu),b(\mu)) = P_{Indep}(y_1,\ldots,y_N),$$

where $P_{Indep}(y_1,\ldots,y_N)$ **is the density of the squared singular values of the product of two independent Ginibre matrices.**

The squared singular values $y_1, \ldots, y_N$ of the product $Y = X_1X_2$ of two coupled matrices form a determinantal point process. The correlation kernel of this process can be written as

$$K_N(x, y; \mu) = \sum_{n=0}^{N-1} P_n(x)Q_n(y),$$

where

$$P_n(x) = \frac{(-1)^n(\nu+n)!n!}{\nu!\mu^{\frac{1}{2}}} \sum_{k=0}^{n} \left(\frac{2x^{\frac{1}{2}}}{1-\mu}\right)^k \frac{(-n)_k}{(\nu+1)_k k!} I_k\left(\frac{1-\mu}{\mu}x^{\frac{1}{2}}\right),$$

and

$$Q_n(y) = \frac{2}{(-1)^n(n!)^2\nu!\mu^{\frac{1}{2}}} \sum_{l=0}^{n} \left(\frac{2y^{\frac{1}{2}}}{1+\mu}\right)^{l+\nu} \frac{(-n)_l}{(\nu+1)_l l!} K_{l+\nu}\left(\frac{1+\mu}{\mu}y^{\frac{1}{2}}\right).$$

Functions P_n, Q_n

Proposition

The functions $P_n(x)$, $Q_n(x)$ satisfy the biorthogonality condition:

$$\int_0^\infty P_n(x)Q_m(x)dx = \delta_{n,m}, \quad n,m = 0,1,2,\ldots.$$

Both $P_n(x)$ and $Q_n(y)$ satisfy five term recurrence relations,

$$\begin{aligned} xP_n(x) =& a_{2,n}P_{n+2}(x) + a_{1,n}P_{n+1}(x) + a_{0,n}P_n(x) \\ &+ a_{-1,n}P_{n-1}(x) + a_{-2,n}P_{n-2}(x), \end{aligned}$$

$$\begin{aligned} yQ_n(y) =& b_{2,n}Q_{n+2}(y) + b_{1,n}Q_{n+1}(y) + b_{0,n}Q_n(y) \\ &+ b_{-1,n}Q_{n-1}(y) + b_{-2,n}Q_{n-2}(y). \end{aligned}$$

The recurrence coefficients can be written explicitly.

- **Whenever the recurrence coefficients are available a Central Limit Theorem for the linear statistics can be derived (Breuer and Duits, 2013).**
- **The recurrence relations lead to a Christoffel-Darboux type formula for the correlation kernel (suitable for a subsequent asymptotic analysis).**

The CLT for the linear statistics

Theorem

Let $y_1, \ldots, y_N$ be the squared singular values of the product $Y = G_2 G_1$ of two coupled matrices. Set $Y_f^{(N)} = \sum_{i=1}^{N} f(y_i)$, where f is a polynomial with real coefficients. If $a(\mu) = N^{\frac{1+\mu}{2\mu}}$, $b(\mu) = N^{\frac{1-\mu}{2\mu}}$, then

$$Y_f^{(N)} - \mathbb{E} Y_f^{(N)} \to \mathcal{N}\left(0, \sum_{k=1}^{\infty} k \hat{f}_k \hat{f}_{-k}\right)$$

in distribution, where

$$\begin{aligned} \hat{f}_k &= \frac{1}{2\pi i} \oint_{|w|=1} f\left(s(w;\mu)\right) w^k \frac{dw}{w}, \\ s(w;\mu) &= \frac{1}{4w^2}(w+1)^3 \left(w(1-\mu)^2 + (1+\mu)^2\right). \end{aligned}$$

The Christoffel-Darboux type formula for $K_N(x, y; \mu)$

$$K_N(x,y;\mu)$$
$$= -\frac{a_{-2,N}P_{N-2}(x)Q_N(y) + a_{-2,N+1}P_{N-1}(x)Q_{N+1}(y) + a_{-1,N}P_{N-1}(x)Q_N(y)}{x-y}$$
$$+\frac{a_{1,N-1}P_N(x)Q_{N-1}(y) + a_{2,N-2}P_N(x)Q_{N-2}(y) + a_{2,N-1}P_{N+1}(x)Q_{N-1}(y)}{x-y},$$

where

$$a_{2,N} = \frac{(1-\mu)^2}{4(N+2)(N+1)},\ a_{1,N} = \mu + \frac{(1-\mu)^2(2N+\nu+2)}{2(N+1)},$$
$$a_{-1,N} = \mu N^2(N+\nu)(3N+\nu) + \frac{(1-\mu)^2}{2}N^2(\nu+2N)(\nu+N),$$
$$a_{-2,N} = \mu N^2(N-1)^2(N+\nu)(N+\nu-1)$$
$$+\frac{(1-\mu)^2}{4}(\nu+N)(\nu+N-1)N^2(N-1)^2.$$

4. Hard edge limit

Different asymptotic regimes

- **(A) The Bessel regime** $(\mu(N) = gN^{-\kappa},\ \kappa \geq 2,\ g > 0)$
- **(B) The strongly correlated regime** $(\mu(N) = gN^{-1},\ g > 0)$
- **(C) The weakly correlated regime** $(\mu = g,\ g \in (0,1))$

(A) The Bessel regime

Theorem

Assume that $\mu(N) = gN^{-\kappa}$, *where* $g > 0$. *Assume that* x, y *take values in a compact subset of* $(0, +\infty)$. *Then for* $\kappa \geq 2$ *we have*

$$\lim_{N\to\infty}\left[\frac{1}{N^2}K_N\left(\frac{x^2}{4N^2},\frac{y^2}{4N^2};\mu=\frac{g}{N^\kappa}\right)^{-\frac{x}{2N\mu(N)}+\frac{y}{2N\mu(N)}}\right]$$
$$= S^{\sqrt{Bessel}}(x,y),$$

where

$$S^{\sqrt{Bessel}}(x,y)$$
$$= -\frac{2x^{\frac{1}{2}}J_{\nu-1}\left(2x^{\frac{1}{2}}\right)J_\nu\left(2y^{\frac{1}{2}}\right)-2y^{\frac{1}{2}}J_{\nu-1}\left(2y^{\frac{1}{2}}\right)J_\nu\left(2x^{\frac{1}{2}}\right)}{(x-y)x^{\frac{1}{2}}y^{\frac{1}{2}}}\left(\frac{y}{x}\right)^{\frac{\nu}{2}}.$$

(B) The strongly correlated regime

Theorem

We have

$$\lim_{N\to\infty}\left[\frac{1}{N^2}K_N\left(\frac{x^2}{4N^2},\frac{y^2}{4N^2};\mu=\frac{g}{N}\right)\right]=\mathbb{S}(x,y;g),$$

where the kernel $\mathbb{S}(x,y;g)$ *is defined by*

$$\mathbb{S}(x,y;g)=\frac{4}{(x^2-y^2)g}\frac{1}{(2\pi i)^2}\oint_{\Sigma}dt\oint_{\Sigma}ds\frac{\Gamma(-t)\Gamma(-s)x^t y^{s+\nu}}{\Gamma(t+\nu+1)\Gamma(s+\nu+1)}$$
$$\times\left[\mathcal{P}(s,t,\nu)-g(s^2+t^2+\nu s-st)\right]I_t\left(\frac{x}{2g}\right)K_{s+\nu}\left(\frac{y}{2g}\right).$$

The contour Σ *starts at* $+\infty$ *in the upper half plane, encircles the positive real axis, and returns to* $+\infty$ *in the lower half plane. Here* $\mathcal{P}(s,t,\nu)$ *is a polynomial (can be written explicitly).*

(C) The weakly correlated regime

Theorem

Assume that $0 < g < 1$, *and that* x,y *take values in a compact subset of* $(0, +\infty)$. *We have*

$$\lim_{N\to\infty}\left[\frac{g}{N}K_N\left(\frac{gx}{N},\frac{gy}{N};\mu=g\right)\right]=S^{Ind}_{\nu,0}(x,y),$$

where

$$S^{Ind}_{\nu,0}(x,y)=\int_0^1 du\; G^{1,0}_{0,3}\left(\begin{array}{ccc} & - & \\ 0, & -\nu, & 0\end{array}\middle| ux\right) G^{2,0}_{0,3}\left(\begin{array}{ccc} & - & \\ \nu, & 0, & 0\end{array}\middle| uy\right).$$

$S^{Ind}_{\nu,0}(x,y)$ *is the Meijer G-kernel describing the hard edge limit for the product of two independent Ginibre matrices (obtained by Kuijlaars and Zhang).*

5. The interpolating process.

Interpolation between the Bessel-kernel process, and the Meijer G-kernel process

Theorem

The determinantal point process on $(0,+\infty)$ *defined by the kernel* $\mathbb{S}(x,y;g)$ *interpolates between the Bessel-kernel process, and the Meijer G-kernel process. Namely, for* x, y *taking values in a compact subset of* $(0,+\infty)$ *we have*

$$\lim_{g\rightarrow+\infty}\left[2g\mathbb{S}\left(2(gx)^{\frac{1}{2}},2(gy)^{\frac{1}{2}};g\right)\right]=S_{\nu,0}^{Ind}(x,y).$$

Furthermore,

$$\lim_{g\rightarrow 0}\mathbb{S}\left(x,y;g\right)=S^{\sqrt{Bessel}}(x,y).$$

Deformation of the Bessel-kernel process

- **Since** $\lim_{g\to 0}\mathbb{S}(x,y;g)=S^{\sqrt{Bessel}}(x,y)$, **the determinantal process defined by** $\mathbb{S}(x,y;g)$ **is a deformation of the Bessel-kernel process, and** g **is a deformation parameter.**
- **The local density of the Bessel-kernel process is defined by**

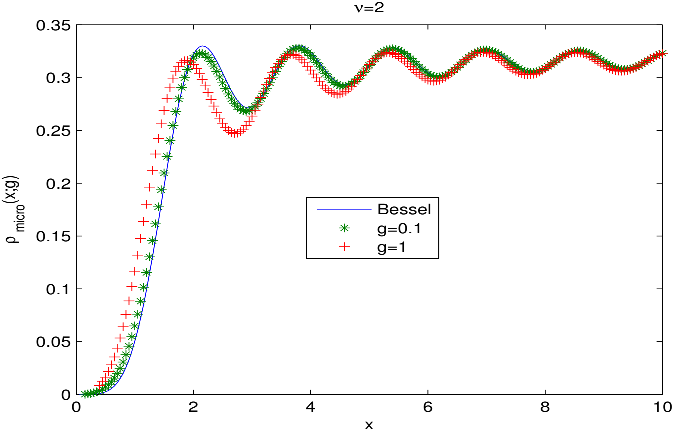
$$\rho_{\mathrm{micro}}^{Bessel}(x)=x^3S^{\sqrt{Bessel}}(x^2,x^2)=x\left(J_\nu(2x)^2-J_{\nu-1}(2x)J_{\nu+1}(2x)\right)$$

Note that $\lim_{x\to+\infty}\rho_{\mathrm{micro}}^{Bessel}(x)=\frac{1}{\pi}$

- **The local density of the interpolating process is defined by**

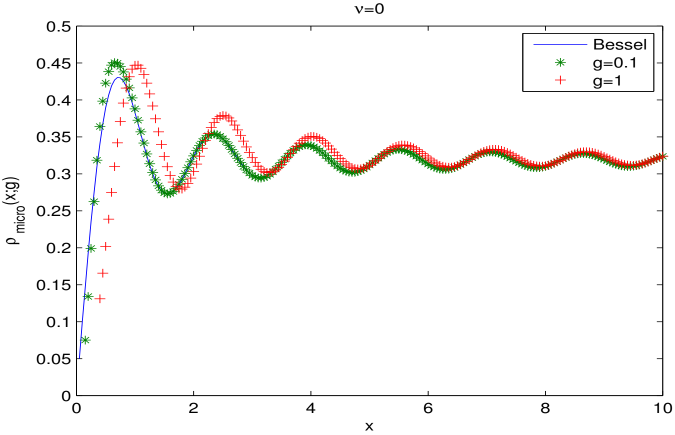
$$\rho_{\mathrm{micro}}(x,g)=x^3\mathbb{S}\left(x^2,x^2;g\right)$$

The local density of the interpolation process as a deformation of the Bessel density



Plot of $\rho_{\mathrm{micro}}(x;g)$ for different values of $g = 0.1$, and 1, together with $\rho_{\mathrm{micro}}^{Bessel}(x)$ corresponding to $g = 0$, at $\nu = 2$.

The local density of the interpolation process as a deformation of the Bessel density



Plot of $\rho_{\mathrm{micro}}(x; g)$ for different values of $g = 0.1$, and 1, together with $\rho_{\mathrm{micro}}^{\mathit{Bessel}}(x)$ corresponding to $g = 0$, at $\nu = 0$.

5. Comparison with the Borodin deformation of the Bessel-kernel process.

The Muttalib-Borodin ensemble

- **The Muttalib-Borodin ensemble is defined by**

$$P_{\mathrm{MB}}(y_1,\ldots,y_N) = const \prod_{j=1}^{N} y_j^{\alpha-y_j} \det\left[y_k^{l-1}\right]_{k,l=1}^{N} \det\left[y_k^{\theta(l-1)}\right]_{k,l=1}^{N}$$

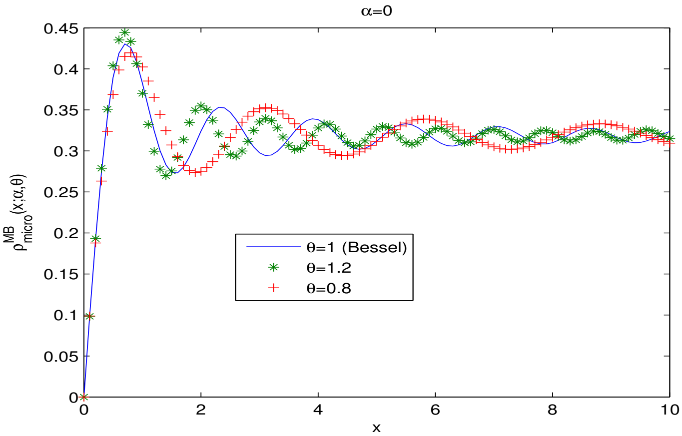
where $\alpha > -1$, $\theta > 0$.

- $\theta = 1$ **gives the standard Laguerre ensemble.**
- **The hard edge scaling limit is a deformation of the Bessel-kernel process defined by the Borodin correlation kernel**

$$K^{(\alpha,\theta)}(x,y) = \theta x^{\alpha} \int_0^1 du J_{(\alpha+1)\theta^{-1},\theta^{-1}}(xu) J_{\alpha+1,\theta}\left((yu)^{\theta}\right) u^{\alpha}$$

Here $J_{a,b}(x)$ is Wright's generalisation of the Bessel function

The Borodin density as a deformation of the Bessel density



Plot of $\rho^{\mathrm{MB}}_{\mathrm{micro}}(x;\alpha,\theta) = xK^{(\alpha,\theta)}\left(x^{\theta+1}, x^{\theta+1}\right)\frac{\theta^{-1/(1+\theta)}}{\sin\left(\frac{\pi}{\theta+1}\right)}$