Moments and Spectral Densities of Singular Value Distributions for Products of Gaussian and Truncated Unitary Random Matrices

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Introduction and known results

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Special case $\alpha = r + 1$ and $\beta = 1$: Fuss-Catalan distributions of order r

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$$x \mapsto \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}},$$

  if $\ell_j - 2n$ is independent of n. This distribution coincides with $R_{1,1/2}$.

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- $s > 1$: ?

Results - Moments

Theorem (Neuschel, Stivigny, Gawronski, 2014)

Let $r, s \in \mathbb{N}_0$ such that $s < r$ and let m be a positive real number. Then there exists a unique measure $J_{r,s,m}$ supported on a compact interval $[0, x^*]$ such that its moments are given by the sequence $(J_{r,s,m}(k))_k$ with

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and for $k \geq 1$

$$J_{r,s,m}(k) = \frac{m}{k} \left(\frac{m^r}{(1+m)^s} \right)^k P_{k-1}^{(\alpha_{k-1}, \beta_{k-1})} \left(\frac{1-m}{1+m} \right),$$

where $P_k^{(\alpha_k, \beta_k)}(x)$ *are the Jacobi polynomials with varying parameters* $\alpha_k = rk + r + 1$ *and* $\beta_k = -(r+1-s)k - (r+2-s)$.

Theorem (cont.)

Moreover, for $m = 1$ we have

$$J_{r,s,1} = R_{r-s+1,1} \boxtimes R_{1,\frac{1}{2}}^{\boxtimes s}.$$

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Moreover, for $m = 1$ we have

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$$J_{r,s,1}(k) = \frac{1}{k2^{ks}} P_{k-1}^{(\alpha_{k-1},\beta_{k-1})}(0), \qquad k \geq 1,$$

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- The case $s = 1$: $\mu_{r,1} = R_{r,1} \boxtimes R_{1,\frac{1}{2}} = R_{\frac{r+1}{2},\frac{1}{2}}$.

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Theorem (Forrester, Wang, 2015)

The measure $\mu_{r,r}$ *is supported on the interval* $\left[0, \frac{(r+1)^{r+1}}{2^{r+1} r^r}\right]$ *and has the density*

$$\frac{d\mu_{r,r}}{dx}(x) = \frac{2^{r+2}\sin(\varphi)^2 \sin(r\varphi)^{r+1}}{\pi \sin((r+1)\varphi)^r \left(4\sin(\varphi)^2\sin(r\varphi)^2 + \sin((r-1)\varphi)^2\right)}$$

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$\rightarrow$ Global density of the Jacobi Muttalib–Borodin ensemble!

Results

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and w is the unique solution analytic on $\mathbb{C} \cup \{\infty\} \backslash [0, x^*]$ taking the value 1 at infinity.

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$$\rightarrow v^{r+1} - 2^{r+1} z(v-1) = 0.$$

This equation corresponds to the Fuss-Catalan case.

Results

The boundary values of v on the branch cut $(0, x^*)$ are given by

$$v_+(x) = \frac{\sin(r+1)\varphi}{\sin(r\varphi)} e^{-i\varphi}$$

and

$$v_-(x) = \frac{\sin(r+1)\varphi}{\sin(r\varphi)} e^{i\varphi},$$

if we choose the parameterization

$$x = x(\varphi) = \frac{\sin((r+1)\varphi)^{r+1}}{2^{r+1}\sin(\varphi)\sin(r\varphi)^r}, \quad 0 < \varphi < \frac{\pi}{r+1}.$$

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$$= \frac{1}{\pi x(\varphi)}\Im\left(\frac{\sin((r+1)\varphi)e^{i\varphi}}{2\sin(r\varphi) - \sin((r+1)\varphi)e^{i\varphi}}\right).$$

Results - Densities

Remark. The behavior at the endpoints of the support:

$$\frac{d\mu_{r,r}}{dx}(x) \sim \frac{\sin\frac{\pi}{r+1}}{\pi} x^{-r/(r+1)}, \quad x \to 0+,$$

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and, provided that $r > 1$,

$$\frac{d\mu_{r,r}}{dx}(x) \sim \frac{2^{r+2+1/2}}{\pi} \frac{r^{r+1/2}}{(r+1)^{r+1/2}(r-1)^2} \sqrt{1 - \frac{2^{r+1}r^r}{(r+1)^{r+1}}x},$$

as $x \to \frac{(r+1)^{r+1}}{2^{r+1}r^r} -$.

Results - Densities

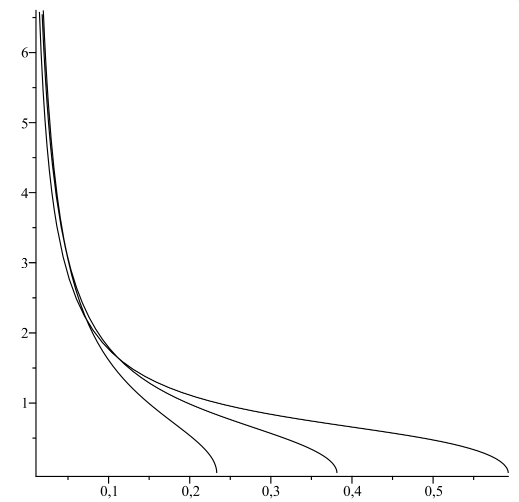

Figure: Densities of $\mu_{r,r}$ for $r = 3, 4, 5$ (from right to left).

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Theorem

The measure $\mu_{r,r-1}$ is supported on the interval $\left[0, x^*_{r,r-1}\right]$ and has the density

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and

$$\rho_r(\varphi) = \frac{3\sin(r\varphi)}{2\sin(r-1)\varphi} - \sqrt{\left(\frac{3\sin(r\varphi)}{2\sin(r-1)\varphi}\right)^2 - \frac{2\sin(r+1)\varphi}{\sin(r-1)\varphi}}.$$

Results - Densities

Remark. The behavior at the endpoints of the support:

$$\frac{d\mu_{r,r-1}}{dx}(x) \sim a x^{-r/(r+1)}, \quad x \to 0+,$$

and at the right endpoint of the support

$$\frac{d\mu_{r,r-1}}{dx}(x) \sim b \sqrt{1 - \frac{x}{x^*_{r,r-1}}}, \quad x \to x^*_{r,r-1},$$

with positive constants a and b.

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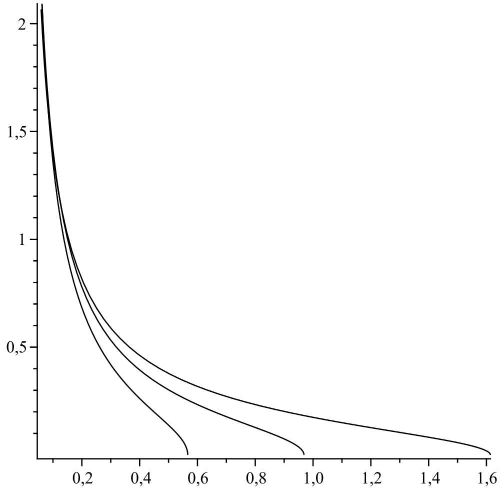

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Results - Densities

The case $r \geq s + 2$: Consider the product

$$\frac{1}{n^{r-s}} \left(G_r \cdots G_{s+1} T_s \cdots T_1 \right)^* \left(G_r \cdots G_{s+1} T_s \cdots T_1 \right),$$

then the limit distribution of the eigenvalues is given by

$$\mu_{r,s} = R_{r-s+1,1} \boxtimes R_{1,\frac{1}{2}}^{\boxtimes s}.$$

Results - Densities

Theorem

Let μ be a probability measure supported on the compact interval $[0, x^*]$ with $x^* > 0$ and let its Stieltjes transform be denoted by

$$F(z) = \int_0^{x^*} \frac{1}{z-t} d\mu(t).$$

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where P and Q are real polynomials with gcd(P,Q)=1 and $P(t) > 0$ for $t \in (0,1]$, $Q'(1) > 0$ and $\deg P \geq \deg Q + 2$ such that $\lim_{t\to+\infty} P(t)/Q(t) = +\infty$.

Results - Densities

Moreover, suppose that for all $x > 0$ the polynomial $w \mapsto P(w) - xQ(w)$ has exactly two roots (counted with multiplicities) inside the sector

$$S_\alpha = \{z \in \mathbb{C} \mid z = te^{is}, t \geq 0, -\alpha \leq s \leq \alpha\},$$

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Then the measure μ is absolutely continuous with respect to the Lebesgue measure with a strictly positive density on $(0, x^*)$ given by

$$\frac{d\mu}{dx}(x) = \frac{1}{2\pi^2 x} \Re \int\limits_{\gamma_\alpha} \log\left(1 - x\frac{Q(t)}{P(t)}\right) dt,$$

where the path of integration γ_α is given as the concatenation of two semi-infinite rays $\gamma_\alpha^{(1)} \oplus \gamma_\alpha^{(2)}$ with $\gamma_\alpha^{(1)}$ defined as the path $t \mapsto e^{i\alpha}/t, t > 0$, and $\gamma_\alpha^{(2)}$ is defined as the positive real axis.

Results - Densities

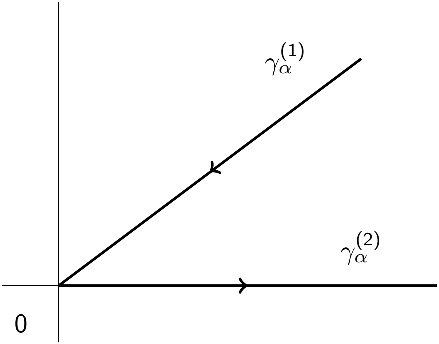

Figure: The path of integration γ_α.

Results - Densities

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$\rightarrow$ the density behaves like $x^{-(\ell-1)/\ell}$ as $x \to 0$.

Results - Densities

Define two quantities:

$$w_{r,s}^* = \frac{1 - s + \sqrt{(1-s)^2 + 4(r+1)(r-s)}}{2(r-s)} > 1$$

and

$$x_{r,s}^* = \frac{r+1}{s+1} \frac{(w_{r,s}^*)^r}{(w_{r,s}^*+1)^{s-1}\left(w_{r,s}^* - \frac{s-1}{s+1}\right)} > 0.$$

Results - Densities

Theorem

The measure $\mu_{r,s}$ is supported on the interval $[0, x^*_{r,s}]$ and has a strictly positive density on the interval $(0, x^*_{r,s})$ given by

$$\frac{d\mu_{r,s}}{dx}(x) = \frac{1}{2\pi^2 x} \Re \int\limits_{\gamma_{2\pi/(r+1)}} \log\left(1 - x\frac{(t-1)(t+1)^s}{t^{r+1}}\right) dt,$$

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where the path $\gamma_{2\pi/(r+1)}$ and the branch of the logarithm are defined as in the preceeding Theorem.
Moreover, the density behaves like $x^{-r/(r+1)}$ as $x \to 0+$ and it vanishes like a square root as $x \to x^*-$. Hence, only the boundary behavior at the origin depends on the number of matrices involved in the product.

Results - Densities

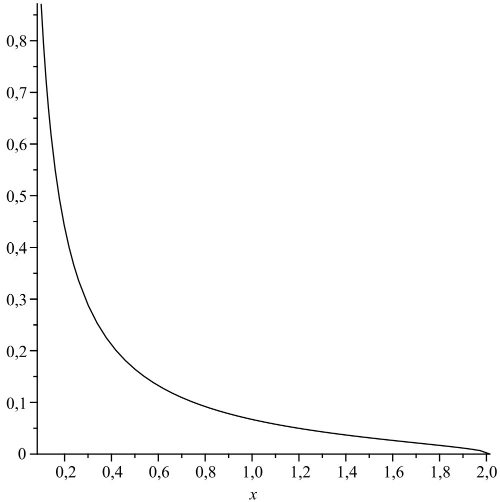

Figure: Plot of $\mu_{7,3}$ on its entire support.

Results - Densities

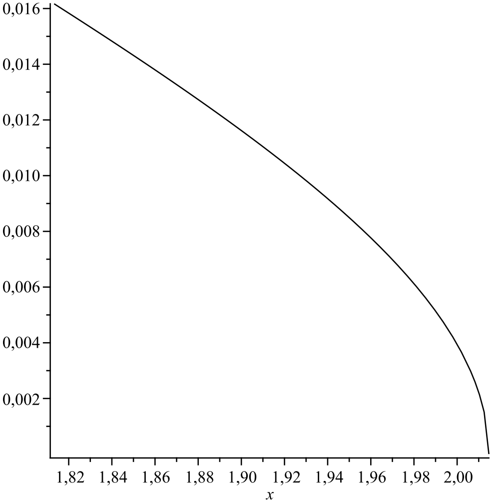

Figure: Plot of $\mu_{7,3}$ in the neighborhood of the right endpoint of its support.

Results - Densities

Remark. The special case $s = 0$ gives a new representation for the densities of the Fuss-Catalan distributions of order $r > 1$

$$\frac{d\mu_{r,0}}{dx}(x) = \frac{1}{2\pi^2 x}\Re \int\limits_{\gamma_{2\pi/(r+1)}} \log\left(1 - x\frac{t-1}{t^{r+1}}\right) dt,$$

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Thank You