Asymptotic eigenvalue distributions of non-commutative polynomials and rational expressions in independent random matrices

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Contents

Random matrices and their eigenvalue distributions

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Definition (Random matrices)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Elements in the complex $*$-algebra

$$\mathcal{A}_N := M_N(L^{\infty-}(\Omega, \mathbb{P})), \quad \text{where} \quad L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathbb{P})$$

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Definition (Empirical eigenvalue distribution)

Given $X \in \mathcal{A}_N$, the empirical eigenvalue distribution of X is the random probability measure μ_X on $\mathbb{C}$ that is given by

$$\omega \mapsto \mu_{X(\omega)} = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j(\omega)},$$

where $\lambda_1(\omega), \ldots, \lambda_N(\omega)$ are the eigenvalues of $X(\omega)$ with multiplicities.

Self-adjoint Gaussian random matrices

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A self-adjoint Gaussian random matrix is a self-adjoint random matrix $X = (x_{k,l})_{k,l=1}^N \in \mathcal{A}_N$, for which

$$\{\Re(x_{k,l}) | \ 1 \leq k \leq l \leq N\} \cup \{\Im(x_{k,l}) | \ 1 \leq k < l \leq N\}$$

are independent Gaussian random variables, such that

$$\mathbb{E}[x_{k,l}] = 0 \quad \text{and} \quad \mathbb{E}[|x_{k,l}|^2] = N^{-1} \quad \text{for } 1 \leq k \leq l \leq N.$$

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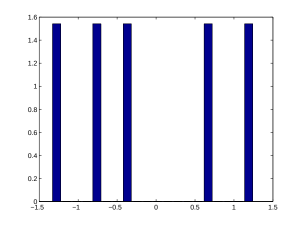
Example

$n = 5$



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Example

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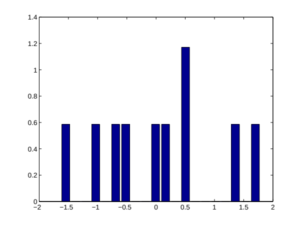

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Example

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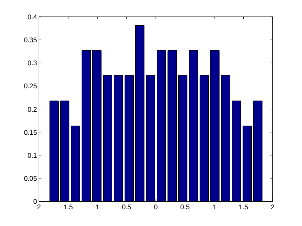

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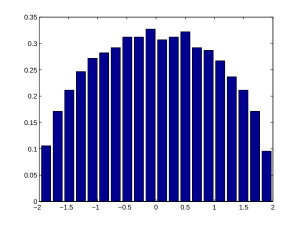

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Theorem (Wigner (1955/1958))

Let $(X^{(N)})_{N\in\mathbb{N}}$ be a sequence of self-adjoint Gaussian random matrices $X^{(N)} \in \mathcal{A}_N$. Then, for all $k \in \mathbb{N}_0$, it holds true that

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_{\mathbb{R}} t^k \, d\mu_{X_n}(t)\Big] = \int_{\mathbb{R}} t^k \, d\mu_S(t)$$

for the semicircular distribution $\quad d\mu_S(t) = \frac{1}{2\pi}\sqrt{4-t^2}\, 1_{[-2,2]}(t)\, dt.$

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“Functions” in independent random matrices

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Question

For each $N \in \mathbb{N}$, let independent Gaussian random matrices

$$X_1^{(N)}, \dots, X_n^{(N)} \in \mathcal{A}_N$$

be given and suppose that f is "some kind of non-commutative function". What can we say about the asymptotic behavior of the empirical eigenvalue distribution of

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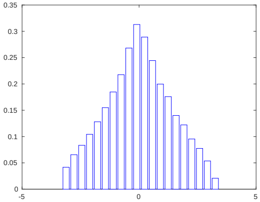

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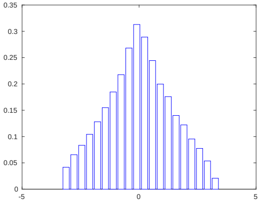

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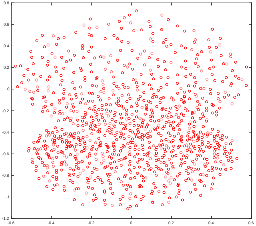

$f(x, y) = (x + i)^{-1}(x + iy)(x + i)^{-1}$

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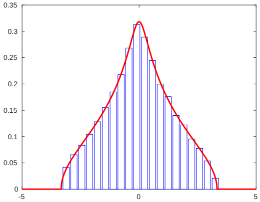
$\rightsquigarrow$ **Free Probability!**



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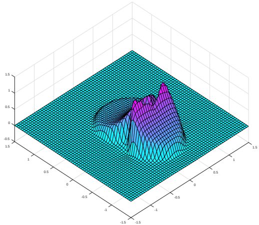

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Non-commutative probability spaces

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A non-commutative probability space $(\mathcal{A}, \phi)$ consists of

- a complex algebra $\mathcal{A}$ with unit $1_\mathcal{A}$ and
- a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ satisfying $\phi(1_\mathcal{A}) = 1$ (expectation).

Elements $X \in \mathcal{A}$ are called non-commutative random variables.

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- $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a classical probability space and $\mathbb{E}$ the usual expectation that is given by $\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$.

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- $(M_N(\mathbb{C}), \mathrm{tr}_N)$, where tr_N is the normalized trace on $M_N(\mathbb{C})$.
- $(\mathcal{A}_N, \phi_N)$, with $\mathcal{A}_N = M_N(L^{\infty-}(\Omega, \mathbb{P}))$ and expectation given by

$$\phi_N(X) := \mathbb{E}[\mathrm{tr}_N(X)] = \int_\Omega \mathrm{tr}_N(X(\omega))\, d\mathbb{P}(\omega).$$

Free independence

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Let $(\mathcal{A}, \phi)$ be a non-commutative probability space.

(i) Unital subalgebras $(\mathcal{A}_i)_{i \in I}$ of $\mathcal{A}$ are called freely independent (or just free), if

$$\phi(a_1 \cdots a_k) = 0$$

holds, whenever

- $a_j \in \mathcal{A}_{i(j)}$ with $i(j) \in I$ for all $j = 1, \dots, k$,
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Free probability theory is a highly non-commutative analogue of classical probability theory.

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We have the following multivariate version of Wigner's semicircle law.

Theorem (Voiculescu (1991))

For all $N \in \mathbb{N}$, realize independent self-adjoint Gaussian random matrices $X_1^{(N)}, \dots, X_n^{(N)} \in \mathcal{A}_N$. Then, for all $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$,

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}_N(P(X_1^{(N)}, \dots, X_n^{(N)}))] = \phi(P(S_1, \dots, S_n))$$

for freely independent semicircular elements $S_1, \dots, S_n$ in some non-commutative probability space $(\mathcal{A}, \phi)$.

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This means: Asymptotic freeness relates

- the limiting eigenvalue distribution of $Y^{(N)} = P(X_1^{(N)}, \dots, X_n^{(N)})$ and
- the distribution of $Y = P(S_1, \dots, S_n)$ for freely independent semicircular elements $S_1, \dots, S_n$.

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Goal

For the limiting object $Y := f(X_1, \ldots, X_n)$, we want to compute

- its analytic distribution in Case 1, [Belinschi, M., Speicher (2013)]

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- its analytic distribution in Case 1, [Belinschi, M., Speicher (2013)]
- its Brown measure in Case 2. [Belinschi, Sniady, Speicher (2015)] [Helton, M., Speicher (2015)]

C^*-probability spaces and analytic distributions

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A non-commutative probability space $(\mathcal{A}, \phi)$ is called C^*-probability space, if $\mathcal{A}$ is a unital C^*-algebra and ϕ a state on $\mathcal{A}$.

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Definition ("analytic distribution")

Let $(\mathcal{A}, \phi)$ be a C^*-probability space. The (analytic) distribution of $X = X^* \in \mathcal{A}$ is the unique Borel probability measure μ_X on $\mathbb{R}$ such that

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all } k \in \mathbb{N}_0.$$

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Example

For any $X = X^* \in M_N(\mathbb{C})$ with eigenvalues $\lambda_1, \dots, \lambda_N$, we have that

$$\mu_X = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}, \qquad \text{since} \qquad \mathrm{tr}_N(X^k) = \frac{1}{N} \sum_{j=1}^{N} \lambda_j^k = \int_{\mathbb{R}} t^k \, d\mu_X(t).$$

Cauchy-Stieltjes transforms of analytic distributions

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Let $(\mathcal{A}, \phi)$ be a C^*-probability space. For $X = X^* \in \mathcal{A}$, the holomorphic function

$$G_X : \mathbb{C}^+ \to \mathbb{C}^-, \ z \mapsto \phi\big((z - X)^{-1}\big) = \int_{\mathbb{R}} \frac{1}{z - t} \, d\mu_X(t)$$

is called the Cauchy transform of X.

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Theorem (Stieltjes inversion formula)

For each $\varepsilon > 0$, consider the absolutely continuous measure $\mu_{X,\varepsilon}$ given by

$$d\mu_{X,\varepsilon}(t) = \frac{-1}{\pi} \Im(G_X(t+i\varepsilon))\, dt.$$

Then $\mu_{X,\varepsilon} \to \mu_X$ weakly as $\varepsilon \searrow 0$.

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A non-commutative probability space $(\mathcal{A}, \phi)$ is called tracial W^*-probability space, if $\mathcal{A}$ is a von Neumann algebra and ϕ a faithful normal tracial state on $\mathcal{A}$.

W^*-probability spaces and Brown measures

Definition

A non-commutative probability space $(\mathcal{A}, \phi)$ is called tracial W^*-probability space, if $\mathcal{A}$ is a von Neumann algebra and ϕ a faithful normal tracial state on $\mathcal{A}$.

Definition (Brown measure)

Let $(\mathcal{A}, \phi)$ be a tracial W^*-probability space. The Brown measure of $X \in \mathcal{A}$ is defined (in distributional sense) by

$$\mu = \frac{2}{\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial \overline{z}} \log(\Delta(X - z)),$$

where Δ denotes the Fuglede-Kadison determinant, i.e.

$$\Delta(X) := \lim_{\varepsilon \searrow 0} \exp\left(\frac{1}{2}\phi(\log(XX^* + \varepsilon^2))\right)$$

regularized Cauchy transforms

regularized Cauchy transforms

Theorem ([Larsen (1999)], [Belinschi, Sniady, Speicher (2015)])

Let $(\mathcal{A}, \phi)$ be a tracial W^*-probability space and let $X \in \mathcal{A}$ be given. For each $\varepsilon > 0$, consider the regularized Brown measure $\mu_{X,\varepsilon}$ given by

$$d\mu_{X,\varepsilon}(z) = \frac{1}{\pi} \frac{\partial}{\partial \overline{z}} G_{X,\varepsilon}(z)\, d\lambda^2(z),$$

where $G_{X,\varepsilon}$ denotes the regularized Cauchy transforms of X,

$$G_{X,\varepsilon}(z) = \phi\Big((z-X)^*\big((z-X)(z-X)^* + \varepsilon^2\big)^{-1}\Big).$$

Then $\mu_{X,\varepsilon} \to \mu_X$ weakly as $\varepsilon \searrow 0$.

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hermitian reduction method [Janik, Nowak, Papp, Zahed (1997)]

$$G_{X,\varepsilon}(z) = \left[G_{\mathbb{X}}\left(\begin{bmatrix} i\varepsilon & z \\ \overline{z} & i\varepsilon \end{bmatrix}\right)\right]_{2,1} \quad \text{where} \quad \mathbb{X} := \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \in M_2(\mathcal{A})$$

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free probability theory $(\mathcal{A}, \phi)$	

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$\mathbb{C}1_{\mathcal{A}} \subseteq \mathcal{A}$	
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$\mathbb{C}^{\pm} = \{z \in \mathbb{C} \mid \pm \Im(z) > 0\}$	
$G_X : \mathbb{C}^{+} \to \mathbb{C}^{-}, \quad z \mapsto \phi((z - X)^{-1})$	

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A (non-commutative) rational expression r in n formal variables $x_1, \dots, x_n$ is a syntactically valid combination of

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Example

- $r(x_1, x_2) = (x_1 \cdot x_2 - 4)^{-1} \cdot x_1 \cdot (x_2 \cdot x_1 - 4)^{-1}$
- $r(x_1, x_2) = (i - x_1)^{-1} \cdot x_2 + x_1 \cdot (i - x_2)^{-1}$
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- $r_1(x_1, x_2) = 0^{-1}, \quad r_2(x_1, x_2) = (x_1 - x_1)^{-1}$

Self-adjoint formal linear representations

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Definition (Helton, M., Speicher (2015))

Let $\mathbb{r}$ be a self-adjoint $k \times k$ matrix of non-commutative rational expressions in formal variables $x_1, \dots, x_n$. A self-adjoint formal linear representation $\rho = (Q, v)$ of $\mathbb{r}$ consists of

- an affine linear pencil $Q = Q^{(0)} + Q^{(1)} x_1 + \dots + Q^{(n)} x_n$ with self-adjoint matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(n)} \in M_N(\mathbb{C})$,
- a matrix $v \in M_{N \times k}(\mathbb{C})$,

and satisfies the following property:
For any unital complex $*$-algebra $\mathcal{A}$ and each $X \in \mathcal{A}_{\text{sa}}^n$, for which $\mathbb{r}(X)$ is defined, $Q(X)$ is invertible in $M_N(\mathcal{A})$ and $\mathbb{r}(X) = -v^* Q(X)^{-1} v$ holds.

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Theorem (Helton, M., Speicher (2015))

Each self-adjoint matrix $\mathbb{r}$ of non-commutative rational expressions admits a self-adjoint formal linear representation $\rho = (Q, v)$.

The history of linearization

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From free probability theory ...

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... back to the famous ancestors.

- recognizable rational series: Schützenberger (1961)
- linear representations: Cohn (1985); Cohn and Reutenauer (1994); Malcolmson (1978)
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$\curvearrowright$ *Linearization even works for non-commutative rational expressions!*

Linearization meets operator-valued free probability

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Theorem

Given a self-adjoint $k \times k$ matrix $\mathbb{r}$ of non-commutative rational expression in $x_1, \dots, x_n$, we chose any self-adjoint formal linear representation $\rho = (Q, v)$ of size $N \times N$. Then, for any C^*-probability space $(\mathcal{A}, \phi)$ and any $X = (X_1, \dots, X_n) \in \mathcal{A}_{\mathrm{sa}}^n$, for which $\mathbb{r}(X)$ is defined, we have that

$$G_{\mathbb{r}(X)}(Z) = \lim_{\varepsilon \searrow 0} \left[G_{\hat{\mathbb{r}}(X)}(\Lambda_\varepsilon(Z)) \right]_{1,1} \quad \text{with} \quad \hat{\mathbb{r}}(X) := \begin{pmatrix} 0 & v^* \\ v & Q(X) \end{pmatrix}$$

holds with $\Lambda_\varepsilon(Z) := \begin{pmatrix} Z & 0 \\ 0 & i\varepsilon 1_N \end{pmatrix} \in \mathbb{H}^+(M_{N+k}(\mathbb{C}))$ for $Z \in \mathbb{H}^+(M_k(\mathbb{C}))$.

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Remark

We have $\hat{\mathbb{r}}(X) = b_0 + b_1 X_1 + \cdots + b_n X_n$ and $b_1 X_1, \dots, b_n X_n$ are freely independent in $(M_{N+k}(\mathcal{A}), \mathrm{id}_{M_{N+k}(\mathbb{C})} \otimes \phi, M_{N+k}(\mathbb{C}))$.

How to calculate the free additive convolution

Theorem (Belinschi, M., Speicher, 2013)

Assume that $(\mathcal{A}, \mathbb{E}, \mathcal{B})$ is an operator-valued C^*-probability space.

If $X, Y \in \mathcal{A}$ are free with respect to $\mathbb{E}$, then there exists a unique pair of (Fréchet-)holomorphic maps $\omega_1, \omega_2 : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$, such that

$$G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$$

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$$G_X(\omega_1(b)) = G_Y(\omega_2(b)) = G_{X+Y}(b), \quad b \in \mathbb{H}^+(\mathcal{B}).$$

Moreover, ω_1 and ω_2 can easily be calculated via the following fixed point iterations on $\mathbb{H}^+(\mathcal{B})$

$$\begin{aligned} w &\mapsto h_Y(b + h_X(w)) + b \qquad \text{for } \omega_1(b) \\ w &\mapsto h_X(b + h_Y(w)) + b \qquad \text{for } \omega_2(b) \end{aligned}$$

where we put $h_X(b) := G_X(b)^{-1} - b$ and $h_Y(b) := G_Y(b)^{-1} - b$, respectively.

Example I – Distributions

$$p(x_1, x_2) := x_1x_2 + x_2x_1$$

$$\rho = \left(\begin{pmatrix} 0 & x_1 & x_2 & -1 \\ x_1 & 0 & -1 & 0 \\ x_2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

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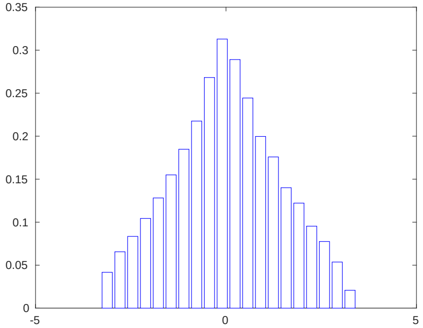
Eigenvalues of $p(X_1, X_2)$, where X_1, X_2 are independent self-adjoint Gaussian random matrices of size 1000×1000 ...



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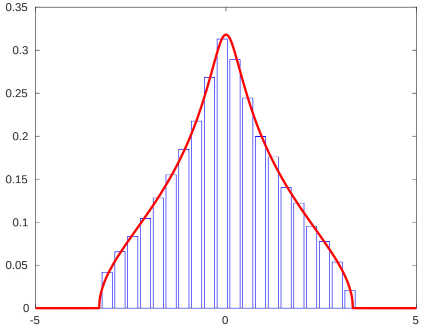

Example II – Distributions

$$r(x_1, x_2) := (4 - x_1)^{-1} + (4 - x_1)^{-1} x_2 \big((4 - x_1) - x_2 (4 - x_1)^{-1} x_2\big)^{-1} x_2 (4 - x_1)^{-1}$$

$$\rho = \left(\begin{pmatrix} -1 + \frac{1}{4}x_1 & \frac{1}{4}x_2 \\ \frac{1}{4}x_2 & -1 + \frac{1}{4}x_1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \right)$$

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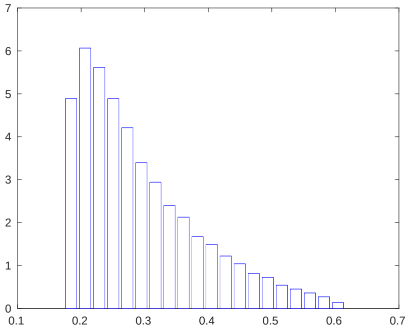

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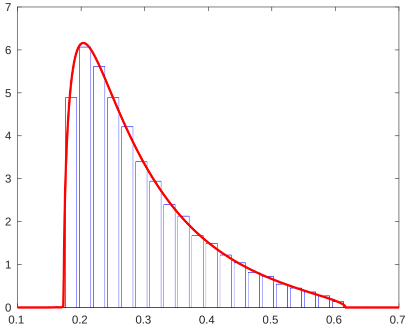

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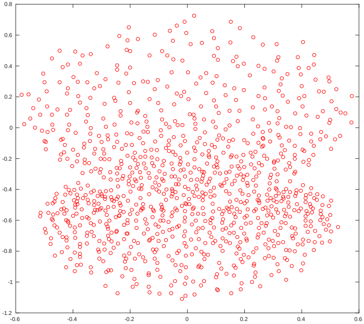

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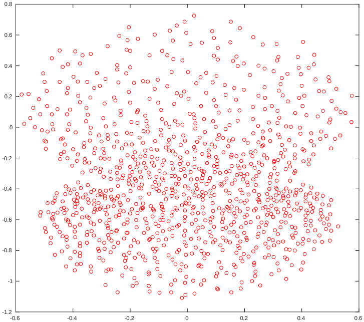

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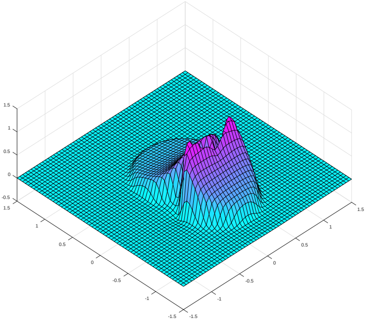

... compared to the Brown measure of $r(X_1, X_2)$, where X_1, X_2 are freely independent semicircular elements.

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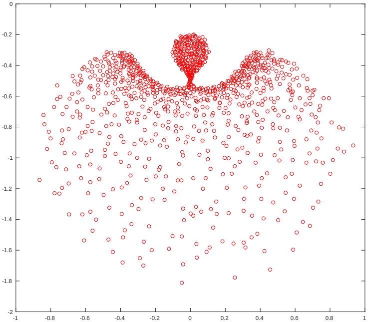

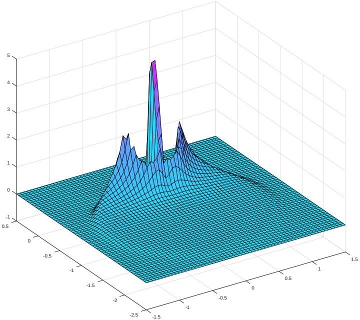

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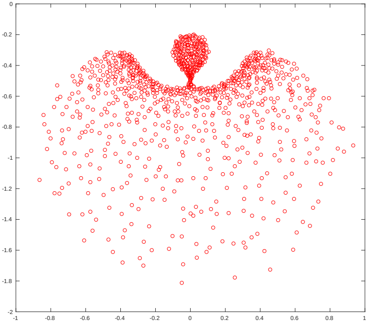

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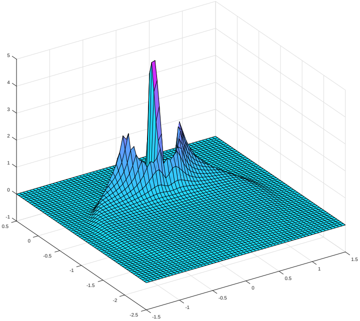

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Thank you!