

Sum of two complex correlated Wishart matrices

Tomasz Checinski



Random Product Matrices 2016

Gernot Akemann and Mario Kieburg

J. Phys. A: Math. Theor. **49** (2016) [[arXiv:1509.03466 \[math-ph\]](https://arxiv.org/abs/1509.03466)]

August 25, 2016

① Introduction

② Determinantal Point Process

③ Supersymmetry

④ Results

Two Wishart Matrices

- two complex Gaussians A & B :
 - rectangular: $N \times N_A$ & $N \times N_B$,
 - dimensions arbitrary, up to $N_A, N_B \geq N$,
 - complex entries $A_{ij}, B_{ij} \in \mathbb{C}$
- distribution

$$\mathcal{P}_W = C_A e^{-\text{Tr}\Sigma_A^{-1}AA^\dagger} C_B e^{-\text{Tr}\Sigma_B^{-1}BB^\dagger}, \quad C_k = \pi^{-NN_k} \det^{-N_k} \Sigma_k$$

- covariance matrices: $N \times N$ & positive definite,

$$\frac{1}{N_A} \left\langle AA^\dagger \right\rangle_{\mathcal{P}_W} = \Sigma_A, \quad \frac{1}{N_B} \left\langle BB^\dagger \right\rangle_{\mathcal{P}_W} = \Sigma_B$$

→ correlations among entries of A & B

- our interest: **cross correlations**

→ eigenvalues of $H = AA^\dagger + BB^\dagger$

Motivation: Time-Series Analysis

- **time-series** \equiv measurements performed in discrete time steps
- $\{N \text{ observables} \& N_W \text{ time steps}\} \hat{=} \text{rectangular data sample } W$
- **covariances:** estimator for correlations between entries s, t & s', t' ,

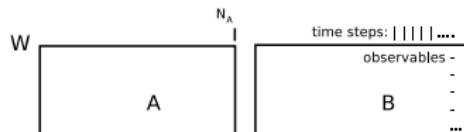
$$\langle W_{st} W_{s't'} \rangle_{\text{data}} = \Sigma_{ss',tt'},$$

s, s' label time series; t, t' label time steps

simplifications:

- $\Sigma_{ss',tt'} = \sigma \delta_{ss'} \delta_{tt'} \rightarrow$ no correlations (standard *Wishart-Laguerre Ensemble*)
- $\Sigma_{ss',tt'} = \Sigma_{ss'}^{(S)} \delta_{tt'} \rightarrow$ no time dependence, *one-sided correlated*
Marčenko, Pastur (1967); Muirhead (1982); Silverstein, Choi (1995); Recher, Kieburg, Guhr, Zirnbauer (2012); ...
- $\Sigma_{ss',tt'} = \Sigma_{ss'}^{(S)} \Sigma_{tt'}^{(T)} \rightarrow$ independent spatial & time correlations, *doubly correlated*
Simon, Moustakas (2004); Burda, Jurkiewicz, Waclaw (2005); Simon, Moustakas, Marinelli (2006); McKay, Grant, Collings (2007); Waltner, Wirtz, Guhr (2015); ...

Our Model: different spatial correlations in each of the **two epochs**,



Starting Point

Santosh Kumar, Europhysics Letters 107, 60002 (2014) [arXiv:1406.6638 [math-ph]]

- distribution of $H = AA^\dagger + BB^\dagger$

$$\mathcal{P}_H = C_H \det H^{N_W - N} e^{-\text{Tr} \Sigma_A^{-1} H} {}_1F_1 \left(N_B; N_W; \left(\Sigma_A^{-1} - \Sigma_B^{-1} \right) H \right),$$

${}_1F_1$ = confluent hypergeometric function with matrix arguments

- half degeneracy

$$\Sigma_A = \sigma_A \mathbb{1}_N, \quad \Sigma_B = \text{diag}(\sigma_{B1}, \sigma_{B2}, \dots, \sigma_{BN}),$$

⇒ joint probability distribution function

$$P_N(\lambda_1, \dots, \lambda_N) = \frac{C_\Lambda}{\Delta_N \left(\sigma_A^{-1} - \sigma_{Bi}^{-1} \right)} \Delta_N(\lambda_j) \det_{N \times N} [\varphi_j(\lambda_i)],$$

$$\varphi_j(\lambda) \equiv \lambda^{N_W - N} e^{-\sigma_A^{-1} \lambda} {}_1F_1 \left(N_B + 1; N_W + 1; \left(\sigma_A^{-1} - \sigma_{Bj}^{-1} \right) \lambda \right),$$

${}_1F_1$ = ordinary confluent hypergeometric function

- spectral density as $(N+1) \times (N+1)$ determinant

Goal: Determinantal Point Process

- k -point correlation function

$$\begin{aligned} R_k(\lambda_1, \dots, \lambda_k) &\equiv \frac{N!}{(N-k)!} \prod_{l=k+1}^N \int_0^\infty d\lambda_l P_N(\lambda_1, \dots, \lambda_N) \\ &= \det_{k \times k} [K_N(\lambda_i, \lambda_j)] \end{aligned}$$

→ What is the kernel $K_N(x, y)$?

- bi-orthogonal ensemble

A. Borodin, Nucl. Phys. B 536 (1998) [arXiv:math/9804027 [math.CA]]

$$P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \det_{N \times N} [\psi_j(\lambda_i)] \det_{N \times N} [\varphi_j(\lambda_i)]$$

1. Inversion of the Gram matrix

$$K_N(x, y) = \sum_{l,j=1}^N \psi_l(x)(g^{-1})_{lj}\varphi_j(y), \quad g_{kl} = \int_0^\infty d\lambda \psi_k(\lambda)\varphi_l(\lambda)$$

2. Construction of orthogonal functions

$$K_N(x, y) = \sum_{l=1}^N p_l(x)q_l(y), \quad \delta_{kl} = \int_0^\infty d\lambda p_k(\lambda)q_l(\lambda)$$

Polynomial Ensemble

- polynomial ensemble

A.B.J. Kuijlaars and D. Stivigny, Rand. Mat.: Theory and Applications 3 (2014)
[arXiv:1404.5802 [math.PR]]

$$\psi_j(\lambda) = \lambda^{j-1}, \quad \det_{N \times N} [\psi_j(\lambda_i)] = \Delta_N(\lambda_i)$$

→ orthogonal polynomials via Heine-like formula

$$p_N(\lambda) \sim \left\langle \det_{N \times N} [\lambda \mathbb{1}_N - H] \right\rangle_{\mathcal{P}_H}$$

- make use of a symmetry

$$\varphi_j(\lambda) = \varphi(\delta_j \lambda)$$

- $\delta_j = \sigma_A^{-1} - \sigma_{Bj}^{-1}$, thus:

$$P_N(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_N} \Delta_N(\lambda_i) \det_{N \times N} [\varphi(\delta_j \lambda_i)]$$

Our Approach

$$K_N(\textcolor{red}{x}, \textcolor{green}{y}) = \frac{N!}{(N-1)!} \int_0^\infty d\lambda_2 \dots d\lambda_N \frac{1}{Z_N} \Delta_N(\textcolor{red}{x}, \lambda_2, \dots, \lambda_N) \det_{N \times N} [\varphi_j(\textcolor{green}{y}), \varphi_j(\lambda_2), \dots, \varphi_j(\lambda_N)]$$

- $\det_{N \times N} [\varphi_j(\textcolor{green}{y}), \varphi_j(\lambda_2), \dots, \varphi_j(\lambda_N)] = \sum_{j=1}^N (-1)^{j-1} \varphi_j(\textcolor{green}{y}) \det_{(N-1) \times (N-1)} [\varphi(\delta_i \lambda_k)]^{(j)}$
- Vandermonde: $\Delta_N(\textcolor{red}{x}, \lambda_2, \dots, \lambda_N) = \prod_{k=2}^N (\lambda_k - \textcolor{red}{x}) \Delta_{N-1}(\lambda_2, \dots, \lambda_N)$
- each summand: $\sim \prod_{k=2}^N (\lambda_k - \textcolor{red}{x}) P_{N-1}^{(j)}(\lambda_2, \dots, \lambda_N)$

$$\Rightarrow K_N(x, y) = \sum_{j=1}^N p_{N-1}^{(j)}(x) \varphi_j(y), \quad p_{N-1}^{(j)}(x) = G_j \langle \det[x \mathbb{1}_{N-1} - H'] \rangle_{\sigma_A; \sigma_{B1}, \dots, \sigma_{BN}}^{(N-1, j)},$$

where

$$\delta_{kl} = \int_0^\infty d\lambda p_{N-1}^{(k)}(\lambda) \varphi_l(\lambda)$$

Generating Function $Z_{q|p}$

$$Z_{q|p}(X) \equiv \left\langle \frac{\prod_{j=1}^p \det[\mathbf{x}_j \mathbb{1}_N - H]}{\prod_{l=1}^q \det[\mathbf{y}_l \mathbb{1}_N - H]} \right\rangle_{\Sigma_A, \Sigma_B} = \left\langle \frac{\prod_{j=1}^p \det[\mathbf{x}_j \mathbb{1}_N - \mathbf{W} \mathbf{W}^\dagger]}{\prod_{l=1}^q \det[\mathbf{y}_l \mathbb{1}_N - \mathbf{W} \mathbf{W}^\dagger]} \right\rangle_{\mathcal{P}_W},$$

- $\det M^{-1} \sim \int d[z] e^{-z M z^\dagger}$, $\det M \sim \int d[\zeta] e^{-\zeta M \zeta^\dagger}$ with $z_j \in \mathbb{C}$, ζ_j Grassmann var.

⇒ characteristic function in superspace:

$$Z_{q|p}(X) = \text{Sdet}^{N_W - N} X \int d[\Phi] e^{i \text{Str} X \Phi^\dagger \Phi} \left\langle e^{-i \text{Str} \Phi^\dagger \mathbf{W}^\dagger \mathbf{W} \Phi} \right\rangle_{\mathcal{P}_W},$$

$$X = \text{diag} [y_1, \dots, y_q; x_1, \dots, x_p]$$

- supermatrix: $\begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix} = \begin{bmatrix} \text{boson-boson} & \text{boson-fermion} \\ \text{fermion-boson} & \text{fermion-fermion} \end{bmatrix}$
- supertrace: $\text{Str} \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix} = \text{Tr} a - \text{Tr} b$, cyclicity regained
- superdeterminant: $\ln [\text{Sdet}(\dots)] = \text{Str} [\ln(\dots)]$

Superbosonisation

duality: superspace \leftrightarrow ordinary space & superbosonisation formula on $\Phi^\dagger \Phi \sim U$:

$$Z_{q|p}(X) = C_{q,p}(N_A) \int d\mu(U_A) S\det^{N_A}(U_A) C_{q,p}(N_B) \int d\mu(U_B) S\det^{N_B}(U_B) \\ \times e^{-\text{Str}U_A - \text{Str}U_B} S\det^{-1}(1_N \otimes X - \Sigma_A \otimes U_A - \Sigma_B \otimes U_B),$$

- 'radial' supermatrices

$$U_{A/B} = \begin{bmatrix} \rho_{A/B} & \omega_{A/B} \\ -\omega_{A/B}^\dagger & z_{A/B} \end{bmatrix}, \quad \rho_{A/B} \in \text{positive definite, } q \times q \\ z_{A/B} \in U(p)$$

- $C_{q,p}(N_{A/B})$ normalization const.
- $d\mu(U_{A/B})$ Haar measure

Results

- general Σ_A & Σ_B

- $Z_{q|p}$ as 'two-epoch' supermatrix model:

$$Z_{q|p}(X) = C_{q,p}(N_A) \int d\mu(U_A) \text{Sdet}^{N_A}(U_A) C_{q,p}(N_B) \int d\mu(U_B) \text{Sdet}^{N_B}(U_B) \\ \times e^{-\text{Str}U_A - \text{Str}U_B} \text{Sdet}^{-1}(1_N \otimes X - \Sigma_A \otimes U_A - \Sigma_B \otimes U_B)$$

- spectral density via resolvent, $Z_{1|1}$:

$$R_1(y) = \frac{1}{\pi} \lim_{\text{Im}(y) \rightarrow 0^+} \text{Im} \partial_x Z_{1|1}(\text{diag}[y; x]) \Big|_{x=y}$$

- analogously k -point density correlation functions via $Z_{k|k}$

- half degeneracy: $\Sigma_A = \sigma_A \mathbb{1}$, $\Sigma_B = \text{diag}(\sigma_{B1}, \dots, \sigma_{BN})$
- kernel $K_N(x, y)$,

$$p_N(x) \sim N_A! N_B! \oint_0 \frac{dz_1}{2\pi i} \oint_0 \frac{dz_2}{2\pi i} \frac{1}{z_1^{N_A+1} z_2^{N_B+1}} e^{z_1 + z_2} \prod_{k=1}^N (x - z_1 \sigma_A - z_2 \sigma_B)$$

Spectral Density $R_I(x) = K_N(x, x)$

- analytic result: red curve $\leftrightarrow 10^6$ realisations: blue dashed area (histogram),
 $N = 9, \sigma_A = 1$ & $\Sigma_B = \text{diag}(0.02, 0.2, 0.3, 1.5, 2.01, 2.25, 2.27, 4.03, 4.05)$

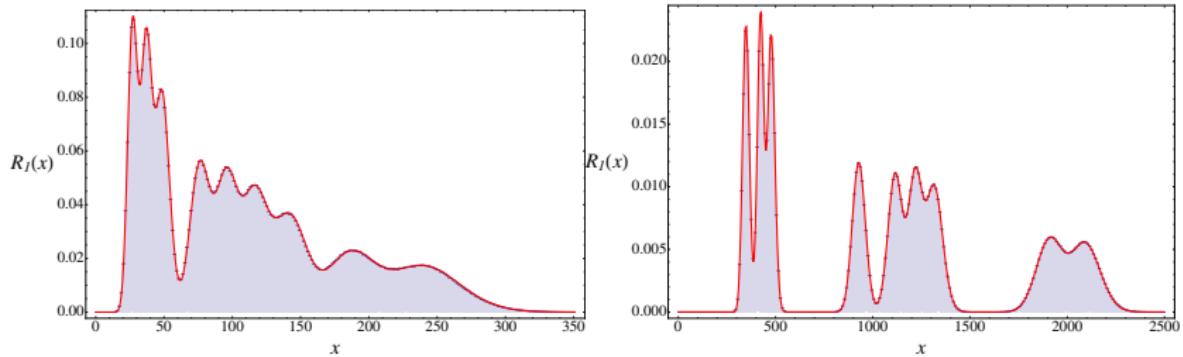


Figure : left: $N_A = 35$ & $N_B = 40$; right: $N_A = 350$ & $N_B = 400$

- for $N_A, N_B \gg N$: $R_I(x) \approx \frac{1}{N} \sum_{k=1}^N \delta(x - (N_A \sigma_A + N_B \sigma_B k))$

Outlook & Generalisations

- starting with more general \mathcal{P}_A & \mathcal{P}_B
- arbitrary number of epochs T

$$Z_{q|p}(X) = \prod_{j=1}^T C_{q,p}(N_{A_j}) \int d\mu(U_{A_j}) \text{Sdet}^{N_{A_j}}(U_{A_j}) \\ \times Q_{A_j}(U_{A_j}) \dots Q_{A_j}(U_{A_j}) \text{Sdet}^{-1} \left(\mathbb{1}_N \otimes X - \sum_{k=1}^T \Sigma_{A_k} \otimes U_{A_k} \right)$$

- rigorously exact results for finite N, N_A, N_B
- asymptotic analysis for $N, N_A, N_B \rightarrow \infty$
- asymptotic analysis with fixed ratios, e.g. $N_{A/B}/N$
- k -point correlation functions for real & quaternion time series

Thank you for your attention.

Gernot Akemann, Tomasz Checinski, Mario Kieburg,
Spectral correlation functions of the sum of two independent complex Wishart matrices with unequal covariances,
Journal of Physics A: Mathematical and Theoretical **49** (2016) 315201
[arXiv:1509.03466 [math-ph]]