Products of free random variables

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Lyapunov exponents for random matrices

Let X_t, $t = 1, 2, \ldots$ be a stationary sequence of identically distributed $n \times n$ random matrices and

$$\Pi_n = X_n ... X_1.$$

Assume that the matrices do not belong to a subgroup and $E \log^+ \|X_i\| < \infty$.

- Then the following limits exist and equal each other:

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_n\| = \lim_{n\to\infty} \frac{1}{n} \log \left(\|\Pi_n v\| \right),$$

  where v is an arbitrary unit vector.

- Let $\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq ...$ be eigenvalues of $\Pi_n^* \Pi_n$. Then limits

$$\lim_{n\to\infty} \frac{1}{n} \log \lambda_k^{(n)}$$

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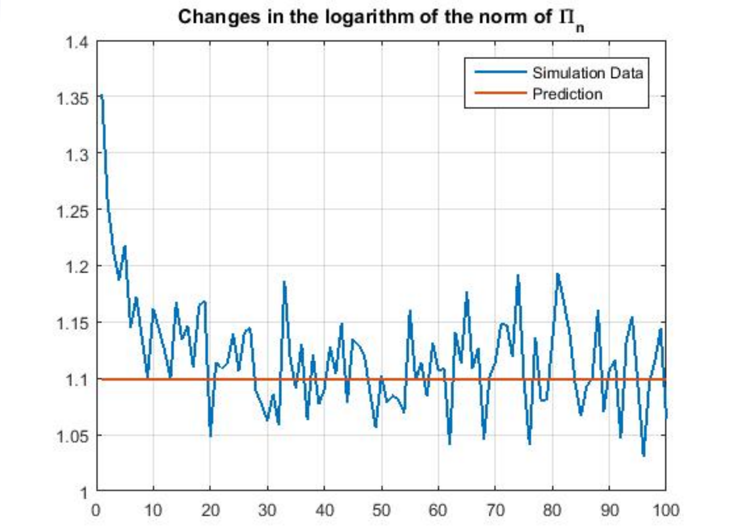

X_i are independent random 400 by 400 Gaussian matrices.

Norm of the product of free operators

Theorem If $X_1, X_2, \ldots$ is a sequence of free identically-distributed bounded operators from a non-commutative W^*-probability space $(\mathcal{A}, E)$, and $\Pi_n = X_n \ldots X_1$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_n^* \Pi_n\| = \log E\left(X_1^* X_1\right).$$

Proof #1 (direct)

Define

$$\psi(z) = E\left(\frac{zX_1^*X_1}{1 - zX_1^*X_1}\right) \text{ and } \psi_n(z) = E\left(\frac{z\Pi_n^*\Pi_n}{1 - z\Pi_n^*\Pi_n}\right).$$

Let $\psi^{(-1)}(z)$ and $\psi_n^{(-1)}(z)$ be functional inverses of $\psi(z)$ and $\psi_n(z)$, respectively.

Then $\|\Pi_n^*\Pi_n\|$ is related to the smallest positive critical value of $\log\psi_n^{(-1)}(x)$.

This critical value can be studied by using identity

$$\log\psi_n^{(-1)}(x) = (n-1)\log\left[\frac{1+z}{z}\right] + n\log\left[\psi^{(-1)}(x)\right].$$

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Proof #2

Suppose the distribution of X coincides with that of a polynomial of degree m in free group generators.

Then we can use freeness and Haagerup's inequality:

$$\begin{aligned} \left[E(X_1^* X_1)\right]^n &= E\left(\Pi_n^* \Pi_n\right) \leq \left\|\Pi_n^* \Pi_n\right\| \\ &\leq CnmE\left(\Pi_n^* \Pi_n\right) = Cnm\left[E\left(X_1^* X_1\right)\right]^n . \end{aligned}$$

(Haagerup's inequality is about the left-regular representation of the group algebra of the free group.

In a simplest form it says that if an element of the algebra is a product of no more than n generators (or their inverses), then its operator norm is bounded by its Frobenius norm multiplied by n.)

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What about

$$\frac{\|\Pi_n^*\Pi_n\|}{\|\Pi_n^*\Pi_n\|_2} \equiv \frac{\|\Pi_n^*\Pi_n\|}{E\left(\Pi_n^*\Pi_n\right)}?$$

According to Haagerup's inequality, this ratio is $O(n)$.

Theorem

$$\lim_{n\to\infty} \frac{\|\Pi_n^*\Pi_n\|}{E\left(\Pi_n^*\Pi_n\right)} = eVn,$$

where

$$V = \frac{E(X_1^*X_1)^2}{(EX_1^*X_1)^2} - 1$$

and $e = 2.7\ldots$ is the base of natural logarithms.

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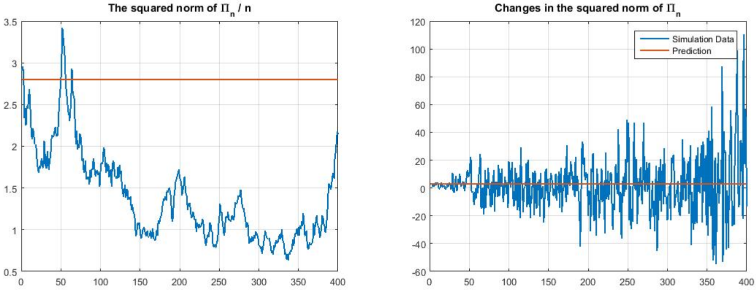


This result is not readily seen in the simulations of random matrix products.

Question

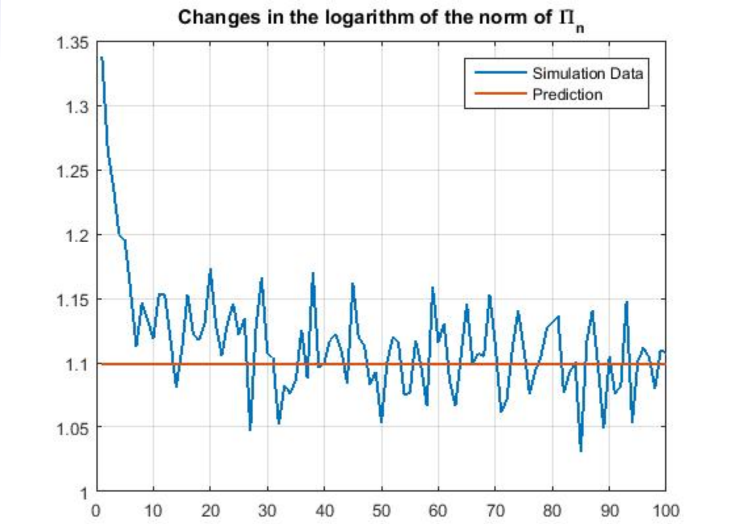
Does the formula

$$\lim_{n\to\infty} \frac{1}{n} \log \|\Pi_n^* \Pi_n\| = \log E\left(X_1^* X_1\right).$$

hold for block variables X_i which are free with amalgamation?

Example: Products of

$$X_i = \begin{bmatrix} A_i & A_i \\ A_i^t & B_i \end{bmatrix}.$$



The prediction here is

$$\frac{1}{2}\log\left(\frac{1}{2}\mathrm{Tr}\mathbb{E}X_1^*X_1\right).$$

Matrix-valued analogue of Haagerup's inequality

Let G be a free group with finitely many free generators, let H be a Hilbert space and let f be a function supported on $E_k(G)$ with values in $B(H)$.

$E_k(G)$ is the span of the group elements which are products of no more than k generators or their inverses.

What can we say about $\|\lambda(f)\|_{B(H \otimes l_2(G))}$?

Buchholz theorem (1999)

(1)

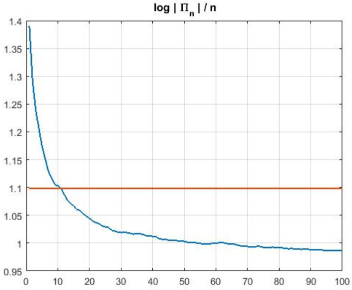
$$\|\lambda(f)\|_{B(H\otimes l_2(G))} \geq \max\{\|(f(pq))_{(p,q)\in E_i(G)\times E_{k-i}(G)}\|_{X_i} : 0 \leq i \leq k\},$$

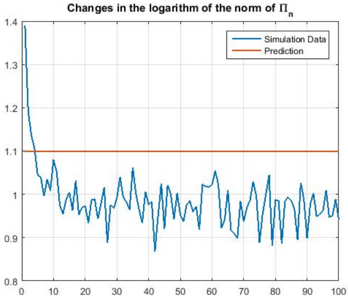
(2)

$$\|\lambda(f)\|_{B(H\otimes l_2(G))} \leq (k+1)\max\{\|(f(pq))_{(p,q)\in E_i(G)\times E_{k-i}(G)}\|_{X_i} : 0 \leq i \leq k\},$$

where $\|\cdot\|_{X_i}$ is the operator norm in the space $B(\bigoplus_{q\in E_{k-i}(G)} H, \bigoplus_{p\in E_i(G)} H)$.

Dependent variables





Does the limit of $n^{-1}\log(\|\Pi_n\|)$ exists for dependent block matrices?

The pictures are for products of

$$X_i = \begin{bmatrix} A_i & -B_{i-1} \\ B_i & A_i^t \end{bmatrix}$$

Lyapunov exponents

Let ν_n be the spectral probability measure of $n^{-1} \log \Pi_n^* \Pi_n$.

Definition #1: The distribution of Lyapunov exponents is defined as the following limit:

$$\nu := \lim_{n \to \infty} \nu_n,$$

if it exists.

Definition #2

Integrated Lyapunov exponent function:

$$\widetilde{\Lambda}(t) = \lim_{n} \frac{1}{n} \sup_{P_t \in \mathcal{A}} \log \det (\Pi_n P_t),$$

where P_t is a projection of dimension t.

Definition #3

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Theorem The limit

$$\Lambda(t) = \lim_{n\to\infty} \frac{1}{n} \log\det\left(\Pi_n P_t\right)$$

exists for each t.

Theorem *Let X_i be invertible. Then*

$$f(t) := \frac{d}{dt}\Lambda(t) = \frac{1}{2}\log\left[\frac{t}{1-t}\theta(t)\right],$$

where $0 \le t \le 1$, and $\theta(t)$ satisfies the equation:

$$E\left[\frac{1}{\theta(t) + X_1^* X_1}\right] = (1-t)\frac{1}{\theta(t)}.$$

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Relation with S-transform

Theorem. Let X_i be identically distributed free bounded invertible operators. Then the marginal Lyapunov exponent of $\{X_i\}$

$$f_X(t) = -\frac{1}{2}\log\left[S_{X_1^* X_1}(-t)\right]$$

where $S(t)$ denotes the S-transform.

Corollary If X is bounded then the largest Lyapunov exponent equals $\frac{1}{2}\log E(X^* X)$.

Corollary. Let X and Y be free, invertible, and bounded. Then

$$f_{XY}(t) = f_X(t) + f_Y(t).$$

Corollary If X is bounded and invertible, then

$$\det(X) = \exp\left\{-\frac{1}{2}\int_0^1 \log S_{X^* X}(-t)\,dt\right\}.$$

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Example: if X^*X has free Poisson (= Marchenko-Pastur) distribution with parameter $\lambda \geq 1$, then

$$f_X(t) = \frac{1}{2}\log(\lambda - t).$$

(Newman's triangle law is a particular case when $\lambda = 1$.)

Theorem (Tucci 2010) The distribution of $(\Pi_n^* \Pi_n)^{1/n}$ converges to a limit ν.

Theorem (Tucci) If X_i are invertible then the limit distribution ν is supported on $\left[(\|X_i^{-1}\|_2)^{-1}, \|X_i\|_2\right]$ and has the distribution function that satisfies $F\left(1/\sqrt{S_{X^*X}(t-1)}\right) = t$.

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Haagerup and Möller (2013) extended Tucci's result to unbounded X_i and gave a different proof.

The proof is based on identities

$$z + 1 = \mathbb{E}\left[\left(1 - \frac{z}{z+1} S_Y(z) Y\right)^{-1}\right],$$

valid for every self-adjoint Y, and

$$t = \int_0^\infty \left(1 + \frac{1-t}{t} S_{X^*X}(t-1)^n y^n\right)^{-1} d\nu_n(y),$$

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Open questions

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Are the measure of $(\Pi_n^* \Pi_n)^{1/n}$ related to the Brown measure of Π_n for non-R diagonal free operators.

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