How Many Stable Equilibria Will a Large Complex System Have?

[1]

Yan V Fyodorov

Department of Mathematics

KING'S College LONDON

Project supported by the EPSRC grant EP/N009436/1

[1]Based on: **YVF** & **B.A. Khoruzhenko** *Proc. Natl. Acad. Sci. USA* **113**,(25): 6827–6832 (2016); **G. Ben Arous**, **YVF** & **B.A. Khoruzhenko**, in progress

May-Wigner Instability Scenario :

"Will a Large Complex System be Stable?"

Robert May, Lord May of Oxford, president of the Royal Society (2000-2005); Chief Scientific Adviser to UK Government.

Will diversity make a food chain more or less stable?

The prevailing view in the mid-20th century was that diverse ecosystems have more resilience to recover from events displacing the system from equilibrium and hence are more stable. This 'ecological intuition' was challenged by Robert May in his article in *NATURE* **238**, 413 (1972) where he introduced a toy model for (in)stability of a large ecological system.

May-Wigner Instability Scenario :

May suggested to consider a toy **linear** model for the dynamics of many interacting species represented by a state vector $\mathbf{x} \in \mathbb{R}^N$:

$$\dot{\mathbf{x}} = -\mu \mathbf{x} + J\mathbf{x}, \quad \mu > 0, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \dots \\ x_N \end{pmatrix}$$

Without interactions the part $\dot{\mathbf{x}} = -\mu \mathbf{x}$ describes a **simple exponential** relaxation of N uncoupled degrees of freedom x_i with the same rate $\mu > 0$ towards the **stable equilibrium** $\mathbf{x} = 0$.

A complicated interaction between dynamics of different degrees of freedom is mimicked by a general **real asymmetric** $N \times N$ **random** **community matrix** J with mean zero and prescribed variance α^2 of all entries. As a typical eigenvalue of J with the largest real part grows as $\alpha\sqrt{N}$ the equilibrium at $\mathbf{x} = 0$ becomes **unstable** as long as $\mu < \mu_c = \alpha\sqrt{N}$.

This scenario is known in the literature as the "**May-Wigner** instability" and despite its oversimplifying and schematic nature attracted very considerable attention in mathematical ecology and complex systems theory over the years.

Limitations of May-Wigner Instabilty Scenario:

- Typical evolution equations are generically **nonlinear**. The May's analysis is essentially based on a **linearization** around a given equilibrium (set at $\mathbf{x} = 0$) , and hence tells us only about **local** asymptotic stability. It therefore does not allow to describe what may happen with the system when it does become unstable.

- The model has only limited bearing for dynamics of populations operating **out-of-equilibrium**. An instability does not necessarily imply lack of persistence: populations could coexist thanks to limit cycles or chaotic attractors, which typically originate from unstable equilibrium points. Interesting questions then relate to classification of equilibria by stability, studying the basins of attraction, and other features of global dynamics.

It is therefore desirable to have a generic nonlinear model which would be rich enough to allow description of May-Wigner instability as a feature of its **global phase portrait**, yet simple enough to allow analytical insights.

A Nonlinear Analogue of May-Wigner model:

We suggest a natural **nonlinear extension** of the May's model to a system of N coupled **nonlinear autonomous** random ODE's:

$$\dot{x}_i = -\mu x_i + f_i(x_1, \ldots, x_N), \quad i = 1, \ldots, N$$

where couplings $f_i(\mathbf{x})$ represent components of an $N-$dimensional vector field and are chosen as a sum of a "**gradient**" and "**solenoidal**" contributions:

$$f_i(\mathbf{x}) = -\frac{\partial V(\mathbf{x})}{\partial x_i} + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\partial A_{ij}(\mathbf{x})}{\partial x_j}, \quad i = 1, \ldots, N$$

where we require the fields $A_{ij}(\mathbf{x})$ to be antisymmetric: $A_{ij} = -A_{ji}$. To make the model as simple as possible and amenable to a detailed mathematical analysis we choose the scalar potential $V(\mathbf{x})$ and the fields $A_{ij}(\mathbf{x})$ to be independent mean zero **Gaussian** random fields, with additional assumptions of **stationarity** and **isotropy** w.r.t. the variables $\mathbf{x} = (x_1, \ldots, x_N)^T$ reflected in the covariance structure

$$\mathbb{E}\{V(\mathbf{x}_1)V(\mathbf{x}_2)\} = v^2 \Gamma_V \left(|\mathbf{x}_1 - \mathbf{x}_2|^2\right)$$

$$\mathbb{E}\{A_{ij}(\mathbf{x}_1)A_{nm}(\mathbf{x}_2)\} = a^2 \Gamma_A \left(|\mathbf{x}_1 - \mathbf{x}_2|^2\right) (\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn})$$

assuming certain smoothness of $\Gamma_{V,A}$ normalized such that $\Gamma''_V(0) = \Gamma''_A(0) = 1$.

Counting equilibria:

The standard analysis of autonomous ODE's starts with finding **equilibrium points** and classifying them by **stability** properties.

Ideally, we would like to have the full statistical characterization of the **total number** $\mathcal{N}_{tot}(D)$ of all possible **equilibria** in a domain D of $\mathbb{R}^N$ for the system of **nonlinear autonomous** random ODE's :

$$\dot{x}_i = -\mu x_i + f_i(x_1, \ldots, x_N), \quad i = 1, \ldots, N$$

and further of the number $\mathcal{N}_{st}(D)$ of **stable** equilibria attracting the dynamics in their vicinity.

Control parameters of the model are:
i) the '**May ratio**' of the relaxation rate to the characteristic value set by interaction:

$$m = \mu/\mu_c. \quad \mu_c = \sqrt{N(a^2 + v^2)}$$

ii) The 'non-potentiality' parameter

$$\tau = v^2/(v^2 + a^2)$$

characterizing the ratio of variances of **gradient** and **solenoidal** components of the field such that $\tau = 0$ corresponds to **purely solenoidal**, and $\tau = 1$ to purely **gradient descent** dynamics.

Mean number of equilibria and the Elliptic Ensemble:

Using Kac-Rice approach we are able to count the **mean values** $\mathbb{E}\{\mathcal{N}_{tot}\}$ and $\mathbb{E}\{\mathcal{N}_{st}\}$ of **all possible** equilibria and of **all stable** equilibria. The first one turns out to be given by the following integral:

$$\mathbb{E}\{\mathcal{N}_{tot}\} = \frac{1}{m^N}\int_{-\infty}^{\infty} \langle |\det(x-\mathbf{X})|\rangle_X \frac{e^{-\frac{Nt^2}{2}}dt}{\sqrt{2\pi/N}}$$

where $x = m + t\sqrt{\tau}$, and the brackets $\langle ... \rangle_X$ denote the averaging over the ensemble of random **real asymmetric** matrix $\mathbf{X}$ known as the **Gaussian Elliptic Ensemble**:

$$\mathcal{P}(\mathbf{X}) = C_N(\tau) e^{-\frac{N}{2(1-\tau^2)}\left[\mathrm{Tr}\mathbf{X}\mathbf{X}^T - \tau \mathrm{Tr}\mathbf{X}^2\right]}, \quad \tau \in [0,1]$$

One can see that the **real Ginibre ensemble** corresponds to **purely solenoidal** dynamics with $\tau = 0$, whereas **GOE** with $\tau = 1$ corresponds to **purely gradient descent** flow.

Similarly, the mean number of **stable** equilibria is given by

$$\mathbb{E}\{\mathcal{N}_{st}\} = \frac{1}{m^N}\int_{-\infty}^{\infty} \langle \det(x-\mathbf{X})\, \chi_{x-\mathbf{X}}\rangle_X \frac{e^{-\frac{Nt^2}{2}}dt}{\sqrt{2\pi/N}}$$

where $\chi_A = 1$ if all eigenvalues of A have negative real parts, and $\chi_A = 0$ otherwise.

A Nonlinear Analogue of May-Wigner Instability as Topology Detrivialization:

By relating $\langle|\det(x-\mathbf{X})|\rangle_X$ to the mean density of **real** eigenvalues of the elliptic ensemble and using the work by **Forrester** $\&$ **Nagao** '07 one can find asymptotics of $\mathbb{E}\{\mathcal{N}_{tot}\}$ for $N \gg 1$. Such an analysis reveals a **topology detrivialization** transition, with the total number of equilibria **abruptly** changing from **a single equilibrium** for $\mu > \mu_c = \sqrt{N(a^2+v^2)}$ to **exponentially many** equilibria as long as $\mu < \mu_c$.

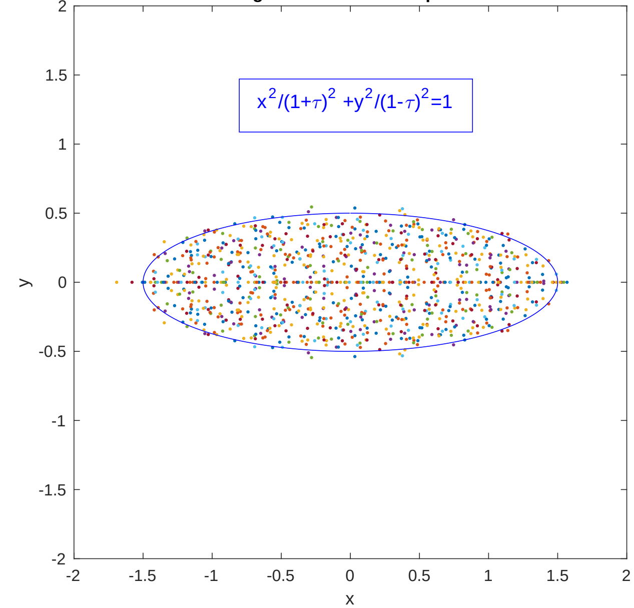
Namely, for $m = \frac{\mu}{\mu_c} > 1$ we have $\lim_{N\to\infty} \mathbb{E}\{\mathcal{N}_{tot}\} = 1$ for any τ, whereas for $m = \frac{\mu}{\mu_c} < 1$ and any $0 \le \tau \le 1$ we obtain instead

$$\mathbb{E}\{\mathcal{N}_{tot}\} \approx K(\tau,m)\, e^{N\Sigma_{tot}(m)}, \quad \Sigma_{tot}(m) = \frac{m^2-1}{2} - \ln m > 0$$

where the pre-exponential factor has the form

$$K(\tau,m) = \begin{cases} \sqrt{\frac{2(1+\tau)}{1-\tau}}, & \text{for} \quad 0 \le \tau < 1 \\ 4\sqrt{\frac{N}{\pi}} \int_0^{\sqrt{1-m^2}} e^{-t^2u^2}\, dt & \text{for } \tau = 1 - \frac{u^2}{N} \to 1 \end{cases}$$

A qualitatively similar transition was reported recently in a model of randomly coupled nonlinear ODE's describing neural networks (**G. Wainrib** $\&$ **J. Touboul** '13).



Large Deviation Approach:

The asymptotic behaviour of

$$\mathcal{D}_N(x) = \langle \det(x\,\delta_{ij} - X_{ij})\chi_{x-\mathbf{x}}\rangle$$

for large $N \gg 1$ can be extracted by exploiting the **large deviation** ideas, see e.g. **Ben Arous** $\&$ **Zeitouni** '98. One finds

$$\mathcal{D}_N(x) \approx \begin{cases} A_N(x)e^{N\Phi_1(x)}, & x > 1+\tau \\ B_N(x)e^{-N^2\mathcal{I}_\tau(x)+N\Phi_2(x)}, & x < 1+\tau \end{cases}$$

where

$$\Phi_1(x) = \frac{1}{8\tau}\left(x - \sqrt{x^2-4\tau}\right)^2 + \ln\frac{x+\sqrt{x^2-4\tau}}{2},$$

whereas the explicit form of the functions $A_N(x), B_N(x), \mathcal{I}_\tau(x)$ and $\Phi_2(x)$ is not actually needed for our purposes apart from the following facts:

(**i**) $\mathcal{I}_\tau(x)$ defined for all $x \leq 1+\tau$ has its **minimum** at $x = 1+\tau$ and at that point its value is **zero**: $\mathcal{I}_\tau(1+\tau) = 0$.

(**ii**) The functions $\Phi_1(x)$ defined for $x > 1+\tau$ and $\Phi_2(x)$ defined for $x < 1+\tau$ satisfy a **continuity** property $\lim_{x\to(1+\tau)+0}\Phi_2(x) = \lim_{x\to(1+\tau)-0}\Phi_1(x) = \frac{\tau}{2}$.

The mean number of stable equilibria in the topologically non-trivial phase:

Using the **large deviation** analysis, we were further able to show that the mean number of **stable equilibria** satisfies asymptotically:

$$\lim_{N\to\infty}\frac{1}{N}\ln\mathbb{E}\{\mathcal{N}_{st}\}=\Sigma_{st}(m;\tau)=-\ln m+(m-1)-\frac{(1-m)^2}{2\tau}$$

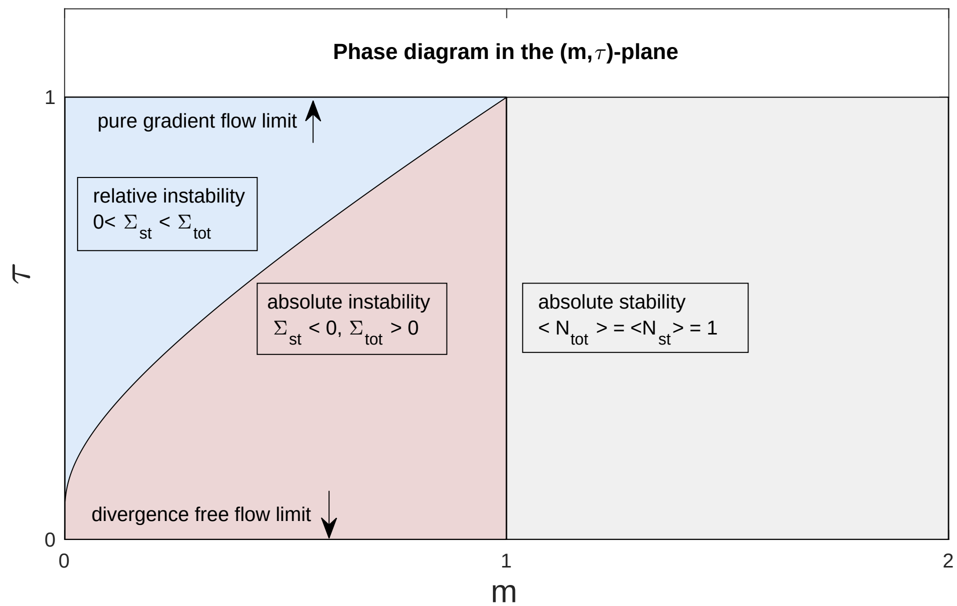
everywhere in the 'topologically non-trivial' quadrant $0<m<1, 0\le\tau\le 1$ of the (m,τ) plane. Moreover, there exists a curve $\tau_B(m)$ given explicitly by

$$\tau_B(m)=-\frac{(1-m)^2}{2(1-m+\ln m)}$$

such that for a given $m<1$ and $\tau<\tau_B(m)$ the 'complexity function' $\Sigma_{st}(m;\tau)$ is **negative**, implying that the mean number of stable equilibria is exponentially small. This in turn implies that for such a regime the probability of having **one or more** stable equilibria is exponentially small, so in a typical realization of random ODE's there are simply **no stable equilibria** at all, i.e. $\mathcal{N}_{st}=0$. It is natural to name this type of the phase portrait the '**absolute instability**' regime.

In contrast, for $\tau>\tau_B(m)$ the complexity function $\Sigma_{st}(m;\tau)$ is positive so that stable equilibria are exponentially abundant. Still, for any $m<1$, the stable equilibria are exponentially rare among all possible equilibria. One may call the associated type of the phase portrait as the '**relative instability**' regime.

Complexity of stable equilibria in the topologically non-trivial phase:



$\Sigma_{st} < 0$ below the line $\tau_B(m)$ implies that the probability of having **one or more** stable equilibria is exponentially small though the total number of equilibria is exponentially big, hence '**absolute instabilty**' regime. Above the line stable equilibria are exponentially abundant but still are only vanishing fraction among all equilibria.

Summary :

- As a generalization of the linear model by **Robert May** we suggest to consider an autonomous system of N nonlinear differential equations coupled by random Gaussian fields:
$$\dot{\mathbf{x}} = -\mu\mathbf{x} + \mathbf{f}(\mathbf{x})$$
- The problem of counting (on average) all possible equilibria, as well as of only stable equilibria can be mapped onto the problem of evaluating the expected value of the objects related to characteristic polynomial of random matrices from '**real elliptic ensemble**'. The asymptotics of those objects for $N \gg 1$ can be efficiently studied by either 'large deviation' techniques, or, in particular instances, by relating to real eigenvalues of elliptic matrices.
- The asymptotic analysis reveals that when the magnitude of random couplings increases with respect to the relaxation rate μ a **single stable equilibrium** is replaced by **exponentially many** equilibria via a sharp "**topology (de)trivialization**" transition. However, immediately after the transition **none** of those equilibria are stable, unless the dynamics is of purely gradient descent type. Further increase in random couplings gives rise to exponentially many stable equilibria, unless the the dynamics is purely divergence-free, or 'solenoidal'.

Open questions:

- It is certainly important to further classify equilibria by 'index' , that is to find how many equilibria with a given **number of unstable directions** exist on average.

- Fluctuations in the number of equilibria of a given 'index' is an interesting and difficult open problem.

- Universality of the emerging picture for similar types of models (e.g. ' non-relaxational spherical model dynamics' (**Cugliandolo et al.** '96), random neural networks (**Sompolinsky et al.** '88; **Weinrib and Touboul** '13)

- Completely open are issues related to clarifying the global dynamical behaviour for a generic non-potential random flow, existence & stability of limit cycles, emergence of chaotic dynamics and associated Lyapunov exponents, glassy and non-equilibrium effects like 'aging', etc.