On distribution of algebraic numbers

Dmitry Zaporozhets
(St. Petersburg Department of Steklov Institute)

Joint work with Friedrich Götze
(Bielefeld University University)

and Denis Koleda
(Institute of Mathematics of NAS of Belarus)

BIELEFELD August 10, 2016

Basic definitions

Definition (basic objects)

An **integral polynomial** is a polynomial that has integer coefficients.

An **algebraic number** is a number α for which there exist an integral polynomial p such that $p(\alpha) = 0$.

The **minimal polynomial of an algebraic number** α is an integral polynomial p of the minimal degree such that $p(\alpha) = 0$ and its coefficients are coprime.

Distinct algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ are called **conjugate** if they have the same minimal polynomial.

Basic definitions

Definition (basic characteristics)

The **height of the polynomial** $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is

$$H(p) := \max_{0 \le i \le n} |a_i|.$$

The **degree of an algebraic number** $\deg \alpha$ is the degree of the minimal polynomial of α.

The **height of an algebraic number** $H(\alpha)$ is the height of its minimal polynomial.

Basic assumptions

- The degree of algebraic numbers and polynomials is fixed:

$$\deg = n.$$

- The heights of algebraic numbers and polynomials are bounded:

$$H \leq Q.$$

- We derive asymptotic formulas as

$$Q \to +\infty.$$

- All the implicit constants do **not** depend on Q, but **might** depend on n.

$n = 1$: rational numbers

Definition

The Farey sequence:

$$\mathcal{F}_Q := \left\{ \frac{a}{b} : a, b \in \mathbb{Z},\ 0 \le a \le b \le Q,\ \gcd(a, b) = 1 \right\}.$$

Theorem (Walfisz, 1963)

$$\#\mathcal{F}_Q = \frac{3}{\pi^2} Q^2 + O(Q(\ln Q)^{2/3}(\ln \ln Q)^{4/3}).$$

Property

$$\lim_{Q \to \infty} \frac{\#(\mathcal{F}_Q \cap I)}{\#\mathcal{F}_Q} = |I|, \quad I \subset [0, 1].$$

Algebraic numbers

Definition (Brown and Mahler, 1971)

The **n-th degree Farey sequence of order Q** is the sequence of all real roots of the set of integral polynomials of degree n and naive height at most Q.

Question (Mahler in a letter to Sprindžuk, 1985)

What is the distribution of the algebraic numbers for a fixed degree $n \geq 2$?

Conjecture (Bernik)

The algebraic numbers are distributed uniformly.

Algebraic numbers

Notation

$$\Phi(Q;I) = \#\{\alpha \in I \subset \mathbb{R} \,:\, \deg(\alpha) \le n, H(\alpha) \le Q\}.$$

Theorem (Koleda, 2012)

$$\Phi(Q;I) \overset{Q\to\infty}{=} \frac{(2Q)^{n+1}}{2\zeta(n+1)} \int_I \rho(x)\,dx + O\left(Q^n \log^{\mathbf{1}\{n=2\}} Q\right),$$

where

$$\rho(x) = 2^{-n-1} \int_{D_x} \left| \sum_{j=1}^{n} j t_j x^{j-1} \right| dt_1 \dots dt_n,$$

$$D_x = \left\{ (t_1, \dots, t_n) \,:\, \max_{1\le k\le n} |t_k| \le 1,\ |t_n x^n + \dots + tx| \le 1 \right\}.$$

Conjugate algebraic k-tuples ($k \leq n$)

Notation

$\Phi_k(Q;B) = \#\{\text{conjugate}\,(\alpha_1,\dots,\alpha_k) \in B \subset \mathbb{R}^k\ :$
$\deg(\alpha_i) \leq n, H(\alpha_i) \leq Q\}$.

Theorem (Götze, Koleda, and Zaporozhets 2015)

$$\Phi_k(Q;B) \overset{Q\to\infty}{=} \frac{(2Q)^{n+1}}{2\zeta(n+1)} \int_B \rho_k(\mathbf{x})d\,\mathbf{x} + O\left(Q^n \log^{\mathbf{1}\{n=2\}} Q\right),$$

$$\rho_k(\mathbf{x}) = 2^{-n-1} \prod_{1\leq i<j\leq k} |x_i - x_j| \int_{D_{\mathbf{x}}} \prod_{i=1}^{k} \left| \sum_{j=0}^{n-k} t_j x_i^j \right| dt_0 \dots dt_{n-k},$$

$$D_{\mathbf{x}} = \left\{ (t_0,\dots,t_{n-k}) \ :\ \max_{0\leq i\leq n} \left| \sum_{j=0}^{n-k} (-1)^{i-j} \sigma_{i-j}(\mathbf{x}) t_j \right| \leq 1 \right\}.$$

Complex algebraic numbers

$B \subset \mathbb{R}^k \times \mathbb{C}_+^l$, where $k + 2l \leq n$;
$\Phi_k(Q; B) = \#\{\text{conjugate}\,(\alpha_1, \ldots, \alpha_{k+l}) \in B\ :$
$\deg(\alpha_i) \leq n, H(\alpha_i) \leq Q\}$.

Theorem (Götze, Koleda, and Zaporozhets 2016)

$$\Phi_{k,l}(Q; B) \overset{Q \to \infty}{=} \frac{(2Q)^{n+1}}{2\zeta(n+1)} \times \int_B 2^l \rho_{k+2l}(x_1 \ldots, x_k, z_1, \bar{z}_1, \ldots, z_l, \bar{z}_l) d\,\mathbf{x} d\,\mathbf{z} + O\left(Q^n\right),$$

Corollary: $k = 0, l = 1, B \subset \mathbb{C}_+$

$$\Phi_{0,1}(Q; B) \overset{Q \to \infty}{=} \frac{(2Q)^{n+1}}{2\zeta(n+1)} \int_B 2\rho_2(z, \bar{z}) d\,z + O\left(Q^n\right),$$

Final slide

THANK YOU!