Eigenvector statistics of the product of Ginibre matrices

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Outline

- Introduction
- Ginibre matrix
- Product of Ginibre matrices (finite N)
- Product of Ginibre matrices ($N \to \infty$)
- Summary

Eigenvector correlations Chalker and Mehlig '99

- Right/Left

$$X\left|R_\alpha\right\rangle = \lambda_\alpha\left|R_\alpha\right\rangle \ , \quad \left\langle L_\alpha\right| X = \left\langle L_\alpha\right| \lambda_\alpha$$

- Bi-orthogonality

$$\left\langle L_\alpha | R_\beta\right\rangle = \delta_{\alpha\beta} \ , \quad \sum_\alpha \left|L_\alpha\right\rangle\left\langle R_\alpha\right| = 1$$

- Scale transformation

$$\left|R_\alpha\right\rangle \to c_\alpha\left|R_\alpha\right\rangle \ , \quad \left\langle L_\alpha\right| \to \left\langle L_\alpha\right| c_\alpha^{-1}$$

- Overlap function:

$$O_{\alpha\beta} = \left\langle L_\alpha | L_\beta\right\rangle\left\langle R_\beta | R_\alpha\right\rangle$$

Local and global averages

- Diagonal

$$O(z) = \left\langle \frac{1}{N} \sum_{\alpha} O_{\alpha\alpha} \delta(z - \lambda_\alpha) \right\rangle$$

- Off-diagonal

$$O(z, w) = \left\langle \frac{2}{N(N-1)} \sum_{\alpha > \beta} O_{\alpha\beta} \delta(z - \lambda_\alpha) \delta(w - \lambda_\beta) \right\rangle$$

- Global overlap

$$O = \int d^2 z O(z) = \left\langle \sum_{\alpha} \frac{1}{N} O_{\alpha\alpha} \right\rangle$$

Ginibre ensemble

- Measure

$$d\mu(X) \propto e^{-N\mathrm{Tr}XX^\dagger} DX$$

- Schur decomposition: $X = U\tau U^\dagger$

$$\tau = \lambda + t = \begin{pmatrix} \lambda_1 & t_{12} & t_{13} & \dots & t_{1N} \\ 0 & \lambda_2 & t_{23} & \dots & t_{2N} \\ & & & \dots & \\ 0 & 0 & 0 & \lambda_{N-1} & t_{N-1N} \\ 0 & 0 & 0 & \dots & \lambda_N \end{pmatrix}$$

- Joint probability

$$d\mu(\lambda, t) = Z^{-1}|\Delta(\lambda)|^2 e^{-N\sum_\alpha |\lambda_\alpha|^2} \prod_\alpha d^2\lambda_\alpha \prod_{\beta<\gamma} \frac{N}{\pi} e^{-N|t_{\beta\gamma}|^2} d^2 t_{\beta\gamma}$$

Calculations of the overlap

- Calculate $O_{11} = \langle L_1 | L_1 \rangle \langle R_1 | R_1 \rangle$
- Use the basis of τ: $R_1 = (1, 0, 0, \ldots)^T$, $R_2 = (0, 1, 0, \ldots)^T$, $\ldots$
- Denote $L_1 = (b_1, b_2, b_3, \ldots)$

$$b_p = \frac{1}{\lambda_1 - \lambda_p} \sum_{q=1}^{p-1} b_q t_{qp}$$

$$b_1 = 1$$

$$b_2 = \frac{t_{12}}{\lambda_1 - \lambda_2}$$

$$b_3 = \frac{t_{13}}{\lambda_1 - \lambda_3} + \frac{t_{12} t_{23}}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \text{ , } etc.$$

- Finally:

$$O = \langle \frac{1}{N} \sum_{\alpha} O_{\alpha\alpha} \rangle = \langle O_{11} \rangle = \langle \sum_{p} |b_p|^2 \rangle$$

Integration

- First integrate t's out

$$\langle O_{11}\rangle = \int d\mu(\lambda,t) O_{11}(\lambda,t) = \int d\mu(\lambda) \underbrace{\int d\mu(t) O_{11}(\lambda,t)}_{O_t(\lambda)}$$

- Result

$$O_t(\lambda) = \prod_{q=2}^{N} \left(1 + \frac{1}{N|\lambda_1 - \lambda_q|^2}\right)$$

- Then integrate over λ's

$$O = Z^{-1} \int O_t(\lambda) |\Delta(\lambda)|^2 e^{-N\sum_\alpha |\lambda_\alpha|^2} \prod_\alpha d^2\lambda_\alpha$$

- Integrate over all but one

$$O(z) = Z^{-1} \int \delta(z-\lambda_1) O_t(\lambda) |\Delta(\lambda)|^2 e^{-N\sum_\alpha |\lambda_\alpha|^2} \prod_\alpha d^2\lambda_\alpha$$

Results Chalker and Mehlig '99

- $N = 2$

$$O(z) = \frac{2}{\pi}\left(1 + |z|^2\right) e^{-2|z|^2}$$

$$\int d^2 z O(z) = \frac{3}{2}$$

- Large N-limit

$$\frac{O(z)}{N} \longrightarrow \frac{1}{\pi}\left(1 - |z|^2\right)$$

$$\frac{O}{N} \longrightarrow \frac{1}{2}$$

Products of m Ginibre matrices

- Product iid Ginibre matrices X_i

$$P = X_1 X_2 \dots X_m$$

- Generalised Schur decomposition Akemann, Burda '12

$$X_i = U_{i-1} \tau_i U_i^\dagger \ , \quad U_m \equiv U_0$$

- $P = U_m \mathcal{T} U_m^\dagger$

$$\mathcal{T} = \tau_1 \tau_2 \dots \tau_m$$

- Measure

$$d\mu(\lambda, t) = Z^{-1} |\Delta(\lambda_1 \lambda_2 \dots \lambda_m)|^2 e^{-N \sum_{i,\alpha} |\lambda_{i\alpha}|^2} \prod_{i\alpha} d^2 \lambda_{i\alpha}$$

$$\prod_{j,\beta<\gamma} \frac{N}{\pi} e^{-N|t_{j\beta\gamma}|^2} d^2 t_{j\beta\gamma}$$

Examples

- $N = 2,\ m = 2$

$$\mathcal{T} = \begin{pmatrix} \lambda_{1,1} & t_{1,12} \\ 0 & \lambda_{1,2} \end{pmatrix} \begin{pmatrix} \lambda_{2,1} & t_{2,12} \\ 0 & \lambda_{2,2} \end{pmatrix} = \begin{pmatrix} \Lambda_1 & T_{12} \\ 0 & \Lambda_2 \end{pmatrix}$$

$\Lambda_\alpha = \lambda_{1,\alpha}\lambda_{2,\alpha}$
$T_{12} = \lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}$

- $N = 3,\ m = 2$

$$\mathcal{T} = \begin{pmatrix} \Lambda_1 & T_{12} & T_{13} \\ 0 & \Lambda_2 & T_{23} \\ 0 & 0 & \Lambda_3 \end{pmatrix}$$

$T_{12} = \lambda_{1,1}t_{2,12} + t_{1,12}\lambda_{2,2}$
$T_{13} = \lambda_{1,1}t_{2,13} + t_{1,12}t_{2,23} + t_{1,13}\lambda_{2,3}$
$T_{23} = \lambda_{1,2}t_{2,23} + t_{1,23}\lambda_{2,3}$

Strategy

- B-coefficients

$$B_p = \frac{1}{\Lambda_1 - \Lambda_p} \sum_{q=1}^{p-1} B_q T_{qp}$$

$$B_1 = 1$$
$$B_2 = \frac{T_{12}}{\Lambda_1 - \Lambda_2}$$
$$B_3 = \frac{T_{13}}{\Lambda_1 - \Lambda_3} + \frac{T_{12} T_{23}}{(\Lambda_1 - \Lambda_2)(\Lambda_2 - \Lambda_3)} \text{ , } etc.$$

- $O = \langle \sum_p |B_p|^2 \rangle_{t,\lambda}$
- $O(z) = \langle \delta(z - \Lambda_1) \sum_p |B_p|^2 \rangle_{t,\lambda}$ where $\Lambda_1 = \lambda_{1,1}\lambda_{2,1} \ldots \lambda_{m,1}$

Global overlap $O = \int d^2 z O(z)$

- Specific cases
- $m = 1 \longrightarrow O = 1 + \frac{1}{2}(N-1)$
- $N = 2 \longrightarrow O = 1 + \frac{m}{2}$
- $m = 2, N = 2, 3, 4, \ldots \longrightarrow O = 2, 3, 4, \ldots$
- $m = 3, N = 3 \longrightarrow O = 4$

Global overlap $O = \int d^2 z O(z)$

- Specific cases
- $m = 1 \longrightarrow O = 1 + \frac{1}{2}(N-1)$
- $N = 2 \longrightarrow O = 1 + \frac{m}{2}$
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- $m = 3, N = 3 \longrightarrow O = 4$

- Generally

$$O = 1 + \frac{m}{2}(N-1)$$

- Limit $N \to \infty$

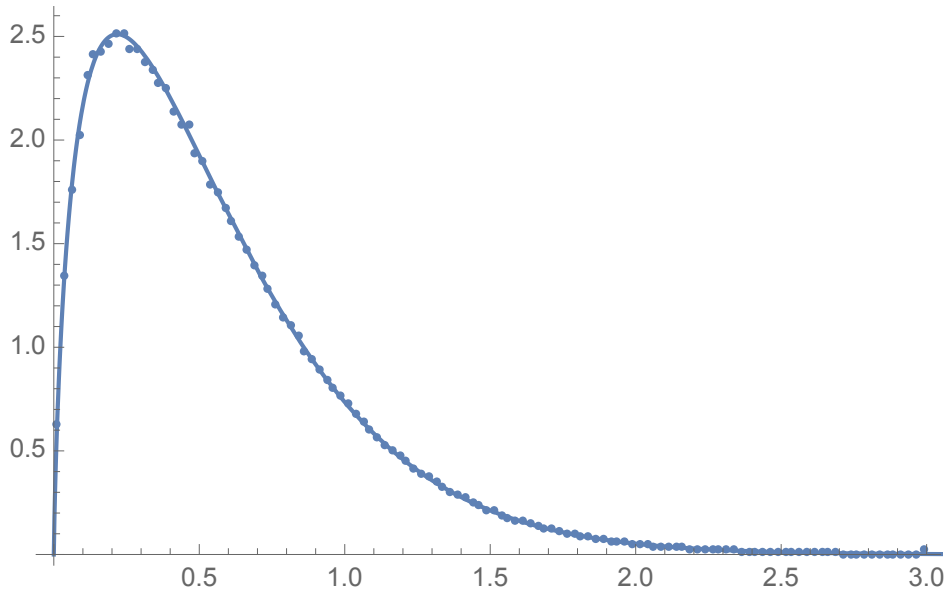
$$\frac{O}{N} \to \frac{m}{2}$$

Local overlap $O(z)$, $N = 2$, $m = 2$

- Radial profile: $O_{rad}(r) = 2\pi r O(r)$

$$O_{rad}(r) = 16r(1 + 2r^2)K_0(4r) + 16r^2 K_1(4r)$$

- Global overlap $\int O(z) d^2 z = 3/2$
- Monte Carlo check

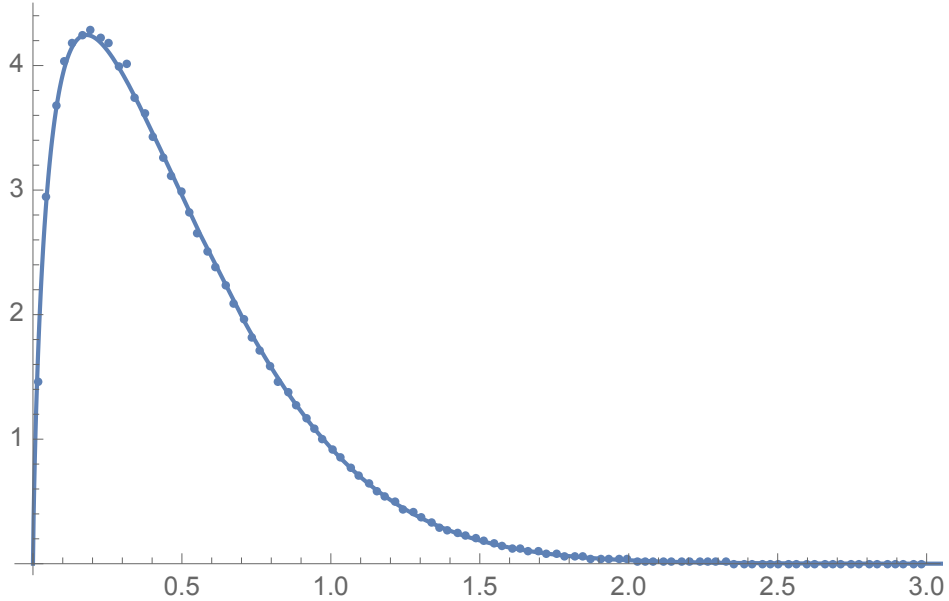


Local overlap $O(z)$, $N = 3$, $m = 2$

- Radial profile

$$O_{rad}(r) = 9r\left(4 + 24r^2 + 27r^4\right) K_0(6r) + 2\left(36r^2 + 81r^4\right) K_1(6r)$$

- Global overlap $\int O(z) d^2z = 3$
- Monte Carlo check

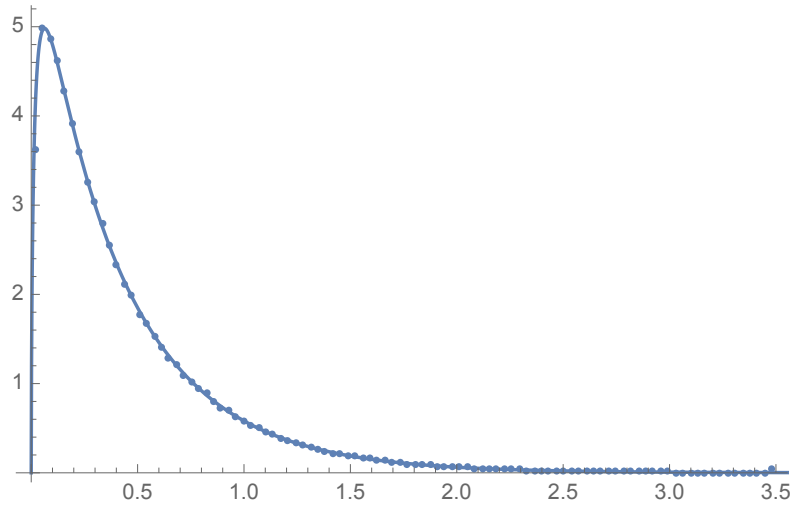


Local overlap $O(z)$, $N = 2$, $m = 3$

- Radial profile

$$O_{rad}(r) = 8r\left\{\sqrt{8}rG^{30}_{03}\left(\begin{matrix}-\\-\frac{1}{2},\frac{1}{2},\frac{1}{2}\end{matrix}\middle|8r^2\right)\right.$$
$$\left.+2G^{30}_{03}\left(\begin{matrix}-\\0,0,0\end{matrix}\middle|8r^2\right)+G^{30}_{03}\left(\begin{matrix}-\\0,0,1\end{matrix}\middle|8r^2\right)+G^{30}_{03}\left(\begin{matrix}-\\1,1,1\end{matrix}\middle|8r^2\right)\right\}$$

- Global overlap $\int O(z)d^2z = \frac{5}{2}$
- Monte Carlo check



Large N limit

- Local

$$\frac{O_{rad}(r)}{N} \longrightarrow 2r^{\frac{2}{m}-1}\left(1-r^{\frac{2}{m}}\right)\mathbb{1}_{[0,1]}(r)$$

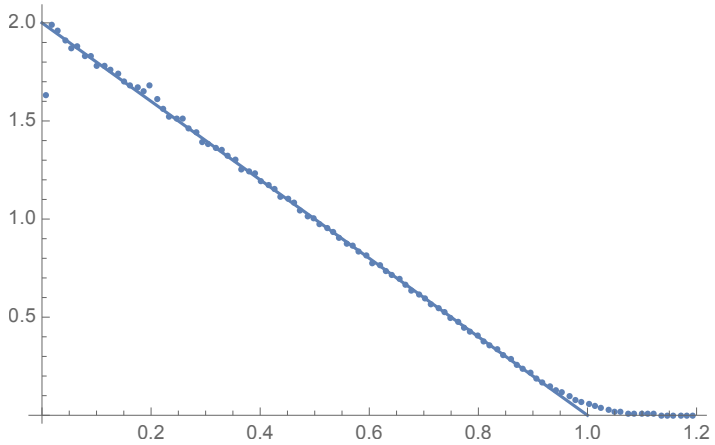
- Global

$$\frac{O}{N} \longrightarrow \frac{m}{2}$$

- Example $m=2$

$$\frac{O_{rad}(r)}{N} \longrightarrow 2\,(1-r)\,\mathbb{1}_{[0,1]}(r)$$

- MC check $N=100$



Planar combinatorics

- Planar limit $N \to \infty$ 't Hooft '74
- Planar diagrams enumeration Brezin, Itzykson, Parisi, Zuber '78
- Non-crossing partitions Speicher, '97
- Non-Hermitian matrices Janik, et al '97

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$$G = \left\langle (Q - X)^{-1} \right\rangle = Q^{-1} + \left\langle Q^{-1} X Q^{-1} \right\rangle + \left\langle Q^{-1} X Q^{-1} X Q^{-1} \right\rangle + \ldots$$

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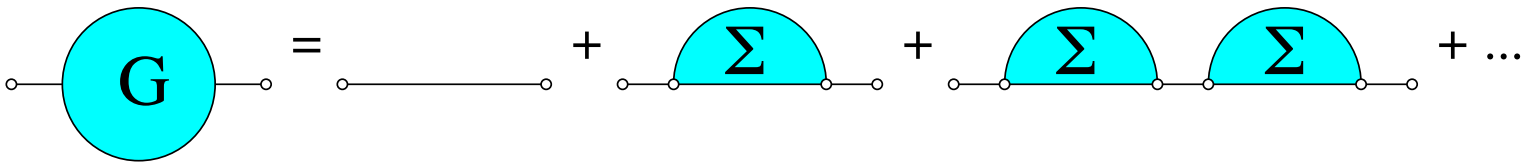
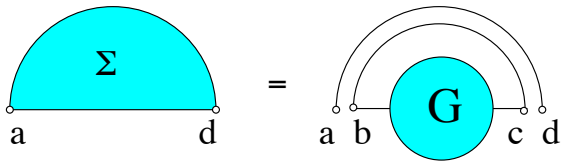
- Wick's theorem $\longrightarrow$ Feynman diagrams
- Dyson-Schwinger equations





Green's function for non-Hermitian matrices

- Matrix resolvent $2N \times 2N$ Janik, et al '97

$$\widehat{G} = \left\langle \left(\begin{pmatrix} z & \epsilon \\ -\bar{\epsilon} & \bar{z} \end{pmatrix} \otimes \mathbb{1}_N - \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix} \right)^{-1} \right\rangle$$

- Limits

$$\widehat{g} = \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \mathrm{Tr}_b \widehat{G} = \begin{pmatrix} g & \gamma \\ -\bar{\gamma} & \bar{g} \end{pmatrix}$$

- Density

$$\rho(z) = \frac{1}{\pi} \partial_{\bar{z}} g(z)$$

- Overlap Janik, et al '99

$$\frac{O(z)}{N} \longrightarrow \frac{1}{\pi} |\gamma(z)|^2$$

One Ginibre ensemble

- Green function

$$\widehat{g}_1(z) = \begin{pmatrix} g(z) & \gamma(z) \\ -\bar{\gamma}(z) & \bar{g}(z) \end{pmatrix}$$

- Solution Janik, et al '97

$$g(z) = \frac{1}{z}\,, \quad \gamma(z) = 0 \quad \text{for } |z| \geq 1$$

and

$$g(z) = \bar{z}\,, \quad \gamma(z) = \sqrt{1 - |z|^2} \quad \text{for } |z| \leq 1$$

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- Overlap

$$\frac{O(z)}{N} \longrightarrow \frac{1}{\pi}\left(1-|z|^2\right) \mathbb{1}_{|z|\leq 1}(z)$$

- Radial profile

$$\frac{O_{rad}(r)}{N} \longrightarrow 2r(1-r^2)\mathbb{1}_{[0,1]}(r)$$

Products of Ginibre matrices Burda, Janik, Waclaw, '09

- Root matrix $2N \times 2N$

$$R = \begin{pmatrix} 0 & X_1 \\ X_2 & 0 \end{pmatrix} \longrightarrow R^2 = \begin{pmatrix} X_1 X_2 & 0 \\ 0 & X_2 X_1 \end{pmatrix}$$

- Matrix resolvent $4N \times 4N$

$$\widehat{G}_2 = \left\langle \left(\begin{pmatrix} z & \epsilon \\ -\bar{\epsilon} & \bar{z} \end{pmatrix} \otimes \mathbb{1}_{2N} - \begin{pmatrix} R & 0 \\ 0 & R^\dagger \end{pmatrix} \right)^{-1} \right\rangle$$

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- $m = 3$

$$R = \begin{pmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ X_3 & 0 & 0 \end{pmatrix} \longrightarrow R^3 = \begin{pmatrix} X_1 X_2 X_3 & 0 & 0 \\ 0 & X_2 X_3 X_1 & 0 \\ 0 & 0 & X_3 X_1 X_2 \end{pmatrix}$$

Overlap for the product of m Ginibre matrices

- Resolvent for R:

$$\widehat{g}_m = \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \mathrm{Tr}_b \widehat{G}_m$$

- Solution

$$\widehat{g}_m = \widehat{g}_1 \otimes \mathbb{1}_m$$

- Overlap for the product ($R \to R^m$)

$$O_{rad,m}(r)dr = O_{rad,1}(s)ds, \quad s = r^m$$

- Result

$$\frac{O_{rad,m}(r)}{mN} \longrightarrow \frac{2}{m} r^{\frac{2}{m}-1} \left(1 - r^{\frac{2}{m}}\right) \mathbb{1}_{[0,1]}(r)$$

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- or

$$\frac{O_{rad}(r)}{N} \longrightarrow 2r^{\frac{2}{m}-1} \left(1 - r^{\frac{2}{m}}\right) \mathbb{1}_{[0,1]}(r)$$

Summary

- Finite N (generalised Schur decomposition)
- Global overlap $O_{N,m} = 1 + \frac{m}{2}(N-1)$
- Large N limit

$$\frac{O_{rad}(r)}{N} \longrightarrow 2r^{\frac{2}{m}-1}\left(1 - r^{\frac{2}{m}}\right)\mathbb{1}_{[0,1]}(r)$$

- The same result: Belinschi, Nowak, Speicher, Tarnowski, '16

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Thank you!