Moments and Large Deviations of Linear Statistics of β-Ensembles

Francesco Mezzadri

University of BRISTOL

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Collaborators: F. Cunden (Bristol), A. Maltsev (Bristol), A. Reynolds (Bristol)

Motivations

Gaussian β-Ensemble, $\beta > 0$,

$$P(x_1, \ldots, x_N) \propto \exp\left(-\beta N \sum_{j=1}^{N} x_j^2\right) \prod_{1 \le j < k \le N} |x_k - x_j|^\beta$$

$$= \exp\left(-\beta H(x_1, \ldots, x_N)\right),$$

$$H(x_1, \ldots, x_N) = -N \sum_{j=1}^{N} x_j^2 + \sum_{1 \le j < k \le N} \log|x_k - x_j|$$

The j.p.d.f. of the eigenvalues can be interpreted as the Boltzmann factor of a one-component plasma of Coulomb particles confined on a line

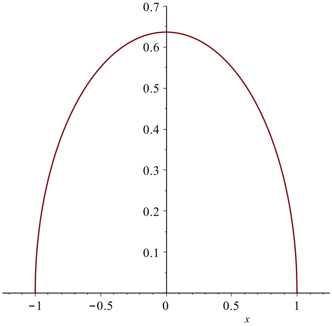
Motivations

In the limit as $N \to \infty$ the eigenvalue distribution is the unique Borel measure $d\mu$ that minimize the energy functional

$$I[\mu] = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log|t-u|\, d\mu(t)d\mu(u) + \int_{-\infty}^{\infty} t^2 d\mu(t)$$

Semicircle law:

$$d\mu(t) = \frac{2}{\pi}\sqrt{1-t^2}dt$$



Motivations

For "well-behaved" test functions f

$$\frac{1}{N}\sum_{j=1}^{N} f(x_i) \to \frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt \quad \text{a.s.}$$

Theorem (Johansson '98)

The linear statistics

$$\sum_{i=1}^{N} f(x_i) - \frac{2N}{\pi}\int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt$$

converges in distribution to a normal random variable with variance σ^2 which depends only on f.

- There is no $1/\sqrt{N}$ normalization!
- This behaviour of the global fluctuations of the spectra is a general phenomenon in random matrix ensembles!

Motivations

Question:

What can we say to the large by finite N corrections to CLT's for linear statistics of Random Matrices?

Applications to:

- Quantum Transport — weak localization corrections.
- QCD and Statistical Mechanics — Free energy asymptotic expansions.
- Combinatorics.
- Quantum Chaos.
- Fluid Dynamics.
- Log-correlated Gaussian processes

Outline

1. Tr X^k for finite N in β ensembles. (FM, Alexi Reynolds, arXiv:1510.02390)

2. Large Deviations for the 2d-OCP. (F. Cunden, A. Maltsev, FM, *Phys. Rev. E*, **91** (2015), 060105(R))

Tr X^k for finite N in β-Ensemble

- Classical β-ensemble,

$$P(x_1,\dots,x_N) = K_{\mathcal{E}} \prod_{j=1}^{N} w(x_j) \prod_{1\le j<k\le N} |x_k - x_j|^{\beta}, \ \beta > 0.$$

- Classical weights,

$$w(x) = \begin{cases} e^{-x^2/2}, & x \in \mathbb{R}, \quad \text{Gaussian.} \\ x^{\gamma} e^{-x/2}, & x \in \mathbb{R}_+, \quad \text{Laguerre.} \\ x^{\gamma_1}(1-x)^{\gamma_2}, & x \in [0,1], \quad \text{Jacobi.} \end{cases}$$

- What can we say about

$$M_k = \mathbb{E}\left\{\operatorname{Tr} X^k\right\}_{\mathcal{E}}, \qquad k \in \mathbb{N}, \ ?$$

- Recall that for the Gaussian β-ensemble

$$\frac{1}{N}\sum_{j=1}^{N} f(x_i) \to \frac{2}{\pi}\int_{-1}^{1} f(t)\sqrt{1-t^2}\,dt \quad \text{a.s.}$$

- This means that

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left\{\operatorname{Tr} X^{2k}\right\}_{\mathrm{G}\beta\mathrm{E}} = \frac{2}{\pi}\int_{-1}^{1} t^{2k}\sqrt{1-t^2}\,dt$$

$$= \left(\frac{1}{2}\right)^{2k} \underbrace{\frac{1}{k+1}\binom{2k}{k}}_{\text{Catalan numbers}}$$

- We look at the finite N version of these moments

Previous results

- For the GUE (Brézin et al. 1978)

$$\frac{1}{N}\mathbb{E}\left\{\operatorname{Tr} X^{2k}\right\}_{\mathrm{GUE}} = \sum_{g=0}^{\left[\frac{k}{2}\right]} a_g(k) N^{-2g}$$

- For the GβE first few terms in the asymptotic expansion as $N \to \infty$ (Forrester and Witte 2014)

- Maple Algorithm (Dumitriu and Edelman 2006)
- Bourger turbulence. Fyodorov, Le Doussal, Rosso (2010)
- Applications to log-correlated Gaussian processes. Fyodorov and Le Doussal, arxiv:1511.04258

Main Result 1

FM and A. Reynolds (2015)

The finite N moments of the classical (GβE, JβE and LβE) β-Ensemble are

$$\mathbb{E}\left\{\operatorname{Tr} X^k\right\}_{\mathcal{E}} = \frac{1}{\alpha^k k!}\sum_{\lambda\vdash k} j_\lambda \kappa^{\lambda}_{(k)}(\alpha)\,\mathbb{E}\left\{C^{(\alpha)}_\lambda\right\}_{\mathcal{E}} \qquad \alpha = 2/\beta$$

$$\lambda = (\lambda_1,\dots,\lambda_r), \qquad \lambda_1+\dots+\lambda_r = k$$

- The quantities $\mathbb{E}\left\{C^{(\alpha)}_\lambda\right\}_{\mathcal{E}}$ are averages of Jack polynomials and can be expressed in terms of special values of multivariate classical OP's (Forrester and Baker 1998)
- The coefficients $\kappa^{\lambda}_{(k)}(\alpha)$ are

$$\kappa^{\lambda}_{(k)}(\alpha) = (-1)^{|\lambda|-\lambda_1}\frac{\alpha^k k\,(\lambda_1-1)!}{j_\lambda\,(|\lambda|-1)!}\prod_{i=1}^{\ell(\lambda)-1}\binom{\frac{i}{\alpha}}{\lambda_i+1}\lambda_{i+1}!$$

Tr X^k for finite N in β-Ensemble

For example, for the LβE we have

$$\mathbb{E}\left\{\operatorname{Tr} X^k\right\}_{\mathrm{L}\beta\mathrm{E}} = \frac{1}{\alpha^k k!}\sum_{\lambda \vdash k} j_\lambda \kappa^{\lambda}_{(k)}(\alpha) L^{\alpha,\gamma}_{\lambda}(\mathbf{0})$$

$$L^{\alpha,\gamma}_{\lambda}(\mathbf{0}) = \left(\gamma + \frac{N-1}{\alpha} + 1\right)^{(\alpha)}_{\lambda} C^{(\alpha)}_{\lambda}\left(1^N\right)$$

Note that the $\kappa^{\lambda}_{(k)}(\alpha)$ do not depend on the measure of the matrix ensembles.

The coefficients $\kappa^{\lambda}_{(k)}(\alpha)$ can be interpreted as inverse of Jack Characters

We can also compute averages of secular coefficients of characteristic polynomials

$\operatorname{Tr} X^k$ and Jack characters

- Jack polynomials are a basis in the ring of symmetric functions.
- p_λ are the power sum symmetric functions (traces)

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_r}, \quad p_j = x_1^j + \cdots + x_N^j.$$

- It turns out that

$$C_\lambda^{(\alpha)} = \frac{\alpha^k k!}{j_\lambda} \sum_{\mu \vdash k} \theta_\mu^\lambda(\alpha) p_\mu$$

- Jack Characters are **transition matrices** (Stanley 1989, Vassileva 2014).

Tr X^k and Jack characters

- For example when $\alpha = 1$ $C_\lambda^{(1)} = s_\lambda$ (Schur polynomials)

$$s_\lambda = \frac{1}{m!} \sum_{\mu \vdash m} g_\mu \chi_\mu^\lambda p_\mu \quad \text{and} \quad p_\mu = \sum_{\lambda \vdash m} \chi_\mu^\lambda s_\lambda,$$

where χ_μ^λ are the characters of the symmetric group.

- Since Tr $X^k = p_k(x_1, \dots, x_N)$ and the $\kappa_{(k)}^\lambda(\alpha)$ are constant,

$$p_k = \frac{1}{\alpha^k k!} \sum_{\lambda \vdash k} j_\lambda \kappa_{(k)}^\lambda(\alpha) C_\lambda^{(\alpha)}$$

- The $\kappa_{(k)}^\lambda(\alpha)$ are the inverse of the Jack characters.

Large Deviations for the 2d-OCP

- The Boltzmann-Gibbs canonical measure of the 2d One-Component-Plasma (OCP) model is

$$\mathbb{P}_\beta\left(\{\vec{r}_k\}\right) = \frac{1}{\mathcal{Z}_{N,\beta}} e^{-\beta H(\vec{r}_1,\ldots,\vec{r}_N)}$$

$$H\left(\{\vec{r}_k\}\right) = -\frac{q^2}{2}\sum_{i\neq j}\log\left(\frac{r_{ij}}{L}\right) + \frac{q^2 N}{2}\sum_k\left(\frac{r_k}{L}\right)^2,$$

$$\vec{r}_k = (x_k, y_k), \qquad r_k = |\vec{r}_k|, \qquad r_{ij} = |\vec{r}_j - \vec{r}_i|$$

- Model for vortex systems in superfluids, superconductor (Ginzburg-Landau model), Bose-Einstein condensates.
- $\beta = 2$ is the Ginibre Ensemble of complex random matrices

Mean radial displacement

$$\Delta_N = \frac{1}{N}\sum_{k=1}^{N} r_k.$$

Define

$$\mathcal{P}_{N,\beta}(x) = \langle \delta\left(x - \Delta_N\right)\rangle$$

F. Cunden, A. Maltsev, FM (2015)

$$\mathcal{P}_{N,\beta}(x) \asymp \begin{cases} \left(\frac{2x}{3}\right)^{\beta N^2/2} e^{-(2/3)\beta N^2\left(x^2-4/9\right)} & \text{for } x \ll 1 \\ \exp\left[-\beta N^2\left(x-\frac{2}{3}\right)^2\right] & \text{for } x \simeq \frac{2}{3} \\ \left(\frac{2x}{3}\right)^{\beta N^2/2} e^{-(1/2)\beta N^2\left(x^2-4/9\right)} & \text{for } x \gg 1. \end{cases}$$

Fourth order phase transition at $x = \frac{2}{3}$

Large Deviations for the 2d-OCP

- Consider the moment-generating function

$$\widehat{\mathcal{P}}_{N,\beta}(s) = \left\langle \exp\left(-\beta s N^2 \Delta_N\right) \right\rangle$$

- The average is taken with respect to the Boltzmann-Gibbs measure

$$\frac{1}{\mathcal{Z}_{N,\beta}} e^{-\beta H(\vec{r}_1,\ldots,\vec{r}_N)}$$

- From the general theory of large deviations for log-gases

$$J(s) = -\lim_{N\to\infty} \frac{1}{\beta N^2} \log \widehat{\mathcal{P}}_{N,\beta}(s)$$

exists and is everywhere differentiable

Large Deviations for the 2d-OCP

- The probability law of Δ_N satisfies a large deviation principle with
  - Speed βN^2
  - Rate function

$$\Psi(x) = -\lim_{N\to\infty} \frac{1}{\beta N^2} \log \mathcal{P}_{N,\beta}(x) = \inf_s \left[J(s) - sx\right]$$

- We can compute $J(s)$:

$$J(s) = \frac{1}{2}\operatorname{arsinh}\left(\frac{s}{2}\right) - \frac{s^2}{4} + \frac{s}{48}\left[\left(s^2+10\right)\sqrt{s^2+4} - |s|^3\right].$$

- Differentiating at $s = 0$

$$\kappa_1(\Delta_N) = \frac{2}{3};\ \kappa_2(\Delta_N) = \frac{1}{2\beta N^2};\ \kappa_3(\Delta_N) = \frac{1}{2\beta^2 N^4}.$$

Large Deviations for the 2d-OCP

- How do we compute $J(s)$?
- Minimize mean-field energy functional

$$I_s[\mu] = -\frac{1}{2}\iint_{\vec{r}\neq\vec{r}'} d\mu(\vec{r})d\mu(\vec{r'})\log|\vec{r}-\vec{r'}| + \int d\mu(\vec{r})V_s(|\vec{r}|)$$

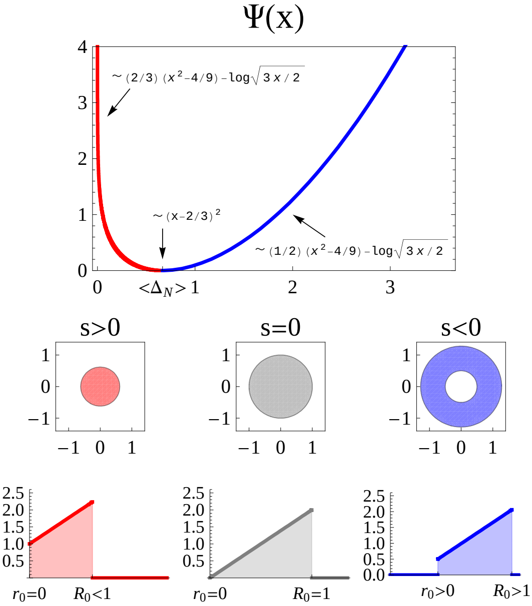
- Recall that

$$\mathcal{P}_{N,\beta}(s) = \left\langle \exp\left(-\beta s N^2 \Delta_N\right)\right\rangle,$$

where $V_s(|\vec{r}|) = \frac{1}{2}r^2 + sr$.

- $J(s)$ as excess free energy

$$J(s) = -\lim_{N\to\infty}\frac{1}{\beta N^2}\log\left[\frac{\int e^{-\beta H(\vec{r}_1,\ldots,\vec{r}_N;s)}}{\int e^{-\beta H(\vec{r}_1,\ldots,\vec{r}_N;0)}}\right]$$
$$= I_s[\mu_s^\star] - I_0[\mu_0^\star],$$



Top:rate function $\Psi(x)$. Centre: support of the equilibrium distribution. Bottom: radial distribution $\int_0^{2\pi} d\mu_s(|\vec{r}|,\phi)$.

Conclusions

- Explicit formulae for the averages of $\operatorname{Tr} X^k$ and of the secular coefficients for finite N in β-ensembles in terms of Jack characters.
- Large deviations in 2d-OCP for β-ensembles unveil a 4th order phase transition.