Komi Science Center of Ural Branch of RAS

On the local laws for product of non-Hermitian random matrices

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joint work with **F. Götze** and **A. Naumov**

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Random matrices

- $\mathbf{X}^{(q)} = [X_{jk}^{(q)}]_{j,k=1}^{n}, q = 1, \ldots, m, m \geq 1$ – independent random matrices
- Conditions $(\mathbf{C0})$:
  1. $X_{jk}^{(q)}, 1 \leq j, k \leq n, q = 1, \ldots, m,$ are independent (identically) distributed.
  2. $\mathbb{E}\, X_{jk}^{(q)} = 0, \quad \mathbb{E}\, |X_{jk}^{(q)}|^2 = 1.$

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  2. $\mathbb{E}\, X_{jk}^{(q)} = 0, \quad \mathbb{E}\, |X_{jk}^{(q)}|^2 = 1.$
- Define $\mathbf{X} := n^{-m/2} \prod_{q=1}^m \mathbf{X}^{(q)}$ and introduce its eigenvalues

$$\lambda_1(\mathbf{X}), ..., \lambda_n(\mathbf{X}).$$

- ESD: for any $A \subset \mathbb{C}$

$$\mu_n^{(m)}(A) := \frac{1}{n} \sum_{k=1}^n I[\lambda_k(\mathbf{X}) \in A]$$

.

Circular law and its extension

- Let $\xi \sim$ Uniform$(|z| \leq 1)$. Denote by $p^{(m)}(z)$ the density function of ξ^m:

$$p^{(m)}(z) = \frac{|z|^{\frac{2}{m}-2}}{\pi m} I[|z| \leq 1], \quad z \in \mathbb{C},$$

- Theorem. *Assume* $(\mathbf{C0})$. *In probability or a.s.*

$$\mu_n^{(m)} \xrightarrow{\text{w}} \mu^{(m)}, \quad n \to \infty,$$

where $d\mu^{(m)}(z) = p^{(m)}(z) dA(z)$.

Goetze and Tikhomirov (2010), Soshnikov and O'Rourke (2010).

In the case $m = 1$
– Ginibre (1965)
– Girko (1984)
– Bai (1997)
– Götze and Tikhomirov (2007)
–Pan and Zhou (2007)
– Götze and Tikhomirov (2010)
– Tao and Vu (2010)

Circular law

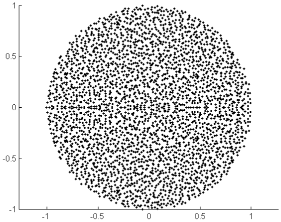

Figure: On the left: spectra of $\mathbf{X}$ with i.i.d. Gaussian entries. On the right: spectra of $\mathbf{X}$ with i.i.d. ± 1 entries.

Products of random matrices

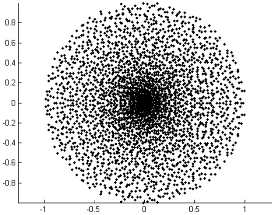

Figure: $n = 3000, m = 2$

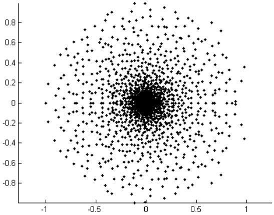

Figure: $n = 3000, m = 10$

Local law for products

- $$\frac{1}{\pi r^2}\mu_n(B(z_0,r)) = \frac{1}{\pi r^2}\int_{B(z_0,r)} p^{(m)}(z)\,dA(z) + \frac{R_n}{\pi r^2}, \qquad (1)$$

  where for fixed $r > 0$
  $$\lim_{n\to\infty} R_n = 0$$
  and $B(z_0,r) := \{z : |z - z_0| \le r\}$.

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- $m = 1$: P. Bourgade, H.-T. Yau and J. Yin (PTRF, 2014, 3 parts); Tao and Vu (Annals of Prob., 2015), J. Alt, L. Erdös, T. Krüger (Ann. Appl. Probab., 2018)

  $m \geq 1$ Y. Nemish (EJP, 2017).

Local law for products

- Let $z_0 : ||z_0| - 1| \geq \tau > 0$ and $f(z)$ be a smooth non-negative function with compact support, such that $\|f\| \leq C, \|f'\| \leq n^C$ for some constant C independent of n. For any $a \in (0, 1/2)$ we define *smoothed indicator*

$$f_{z_0}(z) := n^{2a} f((z - z_0)n^a).$$

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- Theorem[Goetze, Naumov, T.] *Let* $m \geq 1$. *Assume* $(\mathbf{C0})$ *and* $\max_{j,k,q} \mathbb{E} |X_{jk}^{(q)}|^{4+\delta} < \infty$ *for some* $\delta > 0$.
Then for any $Q > 0$ *there exists* $c > 0$ *such that with probability at least* $1 - n^{-Q}$*:*

$$\left| \frac{1}{n} \sum_{j=1}^{n} f_{z_0}(\lambda_j) - \int_{\mathbb{C}} f_{z_0}(z) p^{(m)}(z) dA(z) \right| \leq \frac{q(n)}{n^{1-2a}} \|\Delta f\|_{L^1},$$

where $q(n) < c \log^4 n$.

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  where $q(n) < c \log^4 n$.
- Previous results with $q(n) \leq n^{\varepsilon}$ under condition:

$$\exists\, \theta > 0 : \max_{1 \leq q \leq m} \max_{1 \leq j,k \leq n} \mathbb{P}(|X_{jk}^{(q)}| \geq t) \leq \theta^{-1} e^{-t^{\theta}}.$$

  (or existence of sufficient number of moments)+ Tao, Vu 4-moment theorem.

Products of random matrices

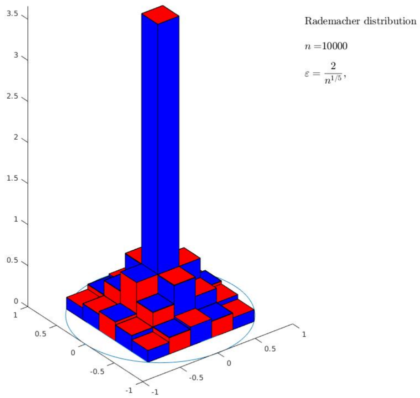

Figure: $n = 10000, a = 2/n^{\frac{1}{5}}$

Products of random matrices

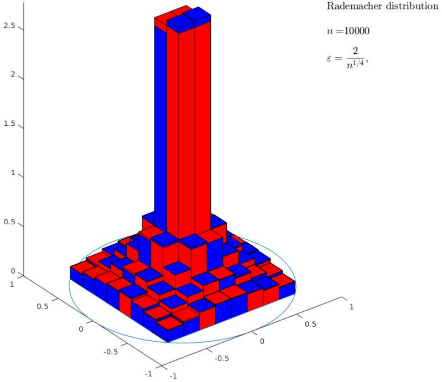

Figure: $n = 10000, a = 2/n^{\frac{1}{4}}$

Products of random matrices

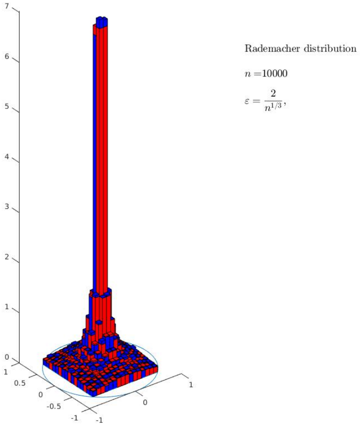

Figure: $n = 10000, a = 2/n^{\frac{1}{3}}$

Products of random matrices

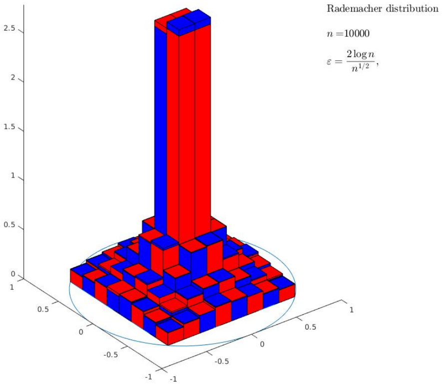

Figure: $n = 10000, a = 2/n^{\frac{1}{2}}$

Linearization

- Consider block-matrix

$$\mathbf{V} := \begin{bmatrix} \mathbf{O} & \mathbf{X}^{(k)} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{X}^{(2)} & \cdots & \mathbf{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{X}^{(n-1)} \\ \mathbf{X}^{(n)} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} \end{bmatrix}$$

Note that $\mathbf{V}^m$ is equal to diagonal matrix

$$\begin{matrix} \prod_{k=1}^m \mathbf{X}^{(1)} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \prod_{k=2}^m \mathbf{X}^{(k)}\mathbf{X}^{(1)} & \mathbf{O} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{O} & \cdots & \mathbf{X}^{(m-1)}\mathbf{X}^{(m)}\prod\limits_{k=1}^{m-2}\mathbf{X}^{(k)} & \mathbf{O} \\ \mathbf{X}^{(n)} & \mathbf{O} & \cdots & \mathbf{X}^{(m)}\prod\limits_{k=1}^{m-1}\mathbf{X}^{(k)} \end{matrix}$$

and has eigenvalues $\lambda_1, \ldots, \lambda_m$ with multiplicity m.

Logarithmic potential and Stieltjes transform

- Logarithmic potential of ν is given by

$$U_\nu(z) := -\int_{\mathbb{C}} \log|z-w|\nu(dw), \quad z \in \mathbb{C}.$$

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- Let F be an arbitrary d.f. on $\mathbb{R}$. The Stieltjes transform of F is given by

$$m_F(z) = \int_{\mathbb{R}} \frac{1}{x-z} dF(x), \quad z \in \mathbb{C}.$$

From logarithmic potential to Stieltjes transform

- Assume that

$$U_\nu(z) = -\int_{\mathbb{R}} \log|x| dF(z,x), \quad z \in \mathbb{C},$$

where $F(z,x), x \in \mathbb{R}$, is some d.f. for any z.

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- Let us define the following function

$$g_F(z,v) := \int\limits_{-\infty}^{\infty} \log|x - iv|\, dF(z,x), \quad U_\nu(z) = -g_F(z,0)$$

  By the Newton-Leibniz formula

$$g_F(z,M) - g_F(z,0) = \int\limits_0^M \operatorname{Im} m_F(z,iv)\, dv.$$

From logarithmic potential to Stieltjes transform

- Green's formula: For any compactly supported $f \in C^2(\mathbb{C})$

$$\int f(z)\nu(dz) = -\frac{1}{2\pi}\int \Delta f(z) U_\nu(z)\, dA(z),$$

From logarithmic potential to Stieltjes transform

- Green's formula: For any compactly supported $f \in C^2(\mathbb{C})$

$$\int f(z)\nu(dz) = -\frac{1}{2\pi}\int \Delta f(z) U_\nu(z)\, dA(z),$$

- Let μ_1 and μ_2 be arbitrary probability measures on $\mathbb{C}$ and $f \in C_0^2(\mathbb{C})$. Then

$$\int f(z) d(\mu_1 - \mu_2) = \frac{1}{2\pi}\int \Delta f(z) \int_0^M [\operatorname{Im} m_{F_2}(z, iv) - \operatorname{Im} m_{F_1}(z, iv)]\, dv\, dA(z)$$
$$- \frac{1}{2\pi}\int \Delta f(z) [g_{F_2}(z, M) - g_{F_1}(z, M)]\, dA(z).$$

Moreover,

$$g_{F_k}(z, M) = \log M + r_{M,k}, \quad |r_{M,k}| \le \frac{1}{M^2}\int_{-\infty}^{\infty} x^2\, dF_k(z, x), \quad k = 1, 2.$$

- Introduce the following matrix

$$\mathbf{V}(z) := \begin{bmatrix} \mathbf{O} & \mathbf{V} - z\mathbf{I} \\ \mathbf{V}^* - \bar{z}\mathbf{I} & \mathbf{O} \end{bmatrix},$$

and let $s_j(z) := s_j(\mathbf{X} - z\mathbf{I}), j = 1, \ldots, n$, be the singular values of $\mathbf{X} - z\mathbf{I}$.

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- Then $\pm s_j(z), j = 1, \ldots, nm$, are the eigenvalues of $\mathbf{V}(z)$. Let

$$F_n(z, x) := \frac{1}{2n} \sum_{j=1}^{n} I[s_j(z) \leq x] + \frac{1}{2n} \sum_{j=1}^{n} I[-s_j(z) \leq x].$$

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- Following Girko we use *hermitization trick* and write

$$U_{\mu_n}(z) = -\frac{1}{2n}\log|\det \mathbf{V}(z)| = -\int_{-\infty}^{\infty} \log|x|\, dF_n(z, x).$$

Moreover, there is d.f. $G(z,x)$:

$$U_{\mu^{(1)}}(z) = -\int\limits_{-\infty}^{\infty} \log|x| dG(z,x).$$

The Stieltjes transform of the distribution $G(w,z)$ we denote by $s(w,z)$. It is well-known that

$$s(z,w) = -\frac{w + s(z,w)}{(w + s(z,w))^2 - |z|^2}. \quad (2)$$

Local law for singular values of shifted matrices

- - The Stieltjes transform of F_n may be rewritten as follows

$$m_n(z,w) = \int_{-\infty}^{\infty} \frac{dF_n(z,\lambda)}{\lambda - w} = \frac{1}{2nm}\operatorname{Tr}(\mathbf{W}-w\mathbf{I})^{-1} =: \frac{1}{2nm}\operatorname{Tr}\mathbf{R}(w,z),$$

where $w = u + iv, v \geq 0$ (i.e. $w \in \mathbb{C}^+$) and

$$\mathbf{W} - w\mathbf{I} = \begin{bmatrix} -w\mathbf{I} & \mathbf{V}(z) \\ \mathbf{V}^*(z) & -w\mathbf{I} \end{bmatrix} \tag{3}$$

. We introduce the functions, for $\nu = 1, \ldots, 2m$,

$$m_n^{(\nu)}(w,z) = \frac{1}{n} \sum_{j=(\nu-1)m+1}^{\nu m} R_{jj}(w,z).$$

Note that

$$m_n(w,z) = \frac{1}{2m} \sum_{\nu=1}^{2m} m_n^{(\nu)}(w,z)$$

Local circular law

- Introduce the notations for $\nu = 1, \ldots, 2m$

$$\Lambda^{(\nu)} = m_n^{(\nu)}(w, z) - s(w, z),$$

and

$$\mathbf{\Lambda}_n = (\Lambda_n^{(1)}, \ldots, \Lambda^{(2m)})^T.$$

- Let $a = -s^{-2}(w, z)$ and $b = \frac{|z|^2}{|w+s(w,z)|^2}$. Introduce matrices

$$\mathbf{A}_{11} := \begin{bmatrix} a & 0 & 0 & \cdots & 0 & b \\ b & a & 0 & \cdots & 0 & 0 \\ 0 & b & a & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & b & a \end{bmatrix}$$

Local circular law

- and

$$\mathbf{A}_{12} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

- We define now matrix

$$\mathbf{A} := \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12} & \mathbf{A}_{11}^T \end{bmatrix}$$

Local circular law

- Using Shur complement formula, we may show that

$$\mathbf{A}\boldsymbol{\Lambda}_n = \mathbf{r}_n + \frac{1}{s(w,z)}\mathbf{T}_n$$

, where

$$\|\mathbf{r}_n\| \le C(|z|, w, s(w,z))\|\boldsymbol{\Lambda}_\mathbf{n}\|^{\mathbf{2}},$$

and vector $\mathbb{T}_n$ defined as follows

$$T_n^{(\nu)} := \frac{1}{n}\sum_{j=1}^{n} \varepsilon_{j_\nu} R_{j_\nu j_\nu}, \quad j_\nu := (\nu - 1)m + j.$$

The error term ε_{j_ν} are defined in standard way via linear and quadratic of q-th row and q-th column of matrices $\mathbf{X}^{(q)}$.

Local law for singular values of shifted matrices

- Let

$$v_0 := \frac{A \log n}{n}, \quad V_0 \gg 1$$

and define the following region in the complex plane

$$\mathcal{D}(z) := \{v_0 \leq v \leq V_0, u \in \operatorname{supp} G\},$$

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- Theorem. Let $m \ge 1$. Assume $(\mathbf{C0})$ and $\max_{j,k,q} \mathbb{E}\,|X_{jk}^{(q)}|^{4+\delta} < \infty$ for some $\delta > 0$.
Let $V_0 > 0$ be some constant. For any $Q > 0$ there exist positive constants A and C depending on V_0, δ, Q such that

$$\mathbb{P}\left(|m_n(z, u+iv) - s(z, u+iv)| \ge \frac{C \log n}{nv}\right) \ge 1 - n^{-Q}$$

for all $u + iv \in \mathcal{D}(z)$

Stein's approach to estimation of $\mathbb{E}\,\|\mathbf{T}_n\|^p$ for $p \sim \log n$

- Let $\mathfrak{M}_j$ be σ-subalgebras of $\mathfrak{M}$ and denote $\mathbb{E}_j(\cdot) := \mathbb{E}(\cdot\big|\mathfrak{M}^{(j)})$.

Stein's approach to estimation of $\mathbb{E}\,\|\mathbf{T}_n\|^p$ for $p \sim \log n$

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- Assume that $\xi_j, f_j, j = 1, \ldots, n$, are $\mathfrak{M}$-measurable r.v. and

$$\mathbb{E}_j(\xi_j) = 0. \tag{4}$$

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- We consider the following statistic:

$$T_n^* := \sum_{j=1}^{n} \xi_j f_j + \mathcal{R},$$

where $\mathcal{R}$ is some $\mathfrak{M}$ measurable function.

Stein's approach

- Let $\widehat{f}_j$ be an arbitrary $\mathfrak{M}^{(j)}$-measurable r.v. and

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- Lemma. For all $p \geq 2$ there exists some absolute constant C such that
$$\mathbb{E}\,|T_n^*|^p \leq C^p\Big(\mathcal{A}^p + p^{\frac{p}{2}}\mathcal{B}^{\frac{p}{2}} + p^p\mathcal{C} + p^p\mathcal{D} + \mathbb{E}\,|\mathcal{R}|^p\Big),$$

  where
$$\mathcal{A} := \mathbb{E}^{\frac{1}{p}}\left(\sum_{j=1}^n \mathbb{E}_j\,|\xi_j(f_j - \widehat{f}_j)|\right)^p,$$
$$\mathcal{B} := \mathbb{E}^{\frac{2}{p}}\left(\sum_{j=1}^n \mathbb{E}_j(|\xi_j(T_n^* - T_n^{(j)})|)|\widehat{f}_j|\right)^{\frac{p}{2}},$$
$$\mathcal{C} := \sum_{j=1}^n \mathbb{E}\,|\xi_j||T_n^* - \widetilde{T}_n^{(j)}|^{p-1}|\widehat{f}_j|,$$
$$\mathcal{D} := \sum_{j=1}^n \mathbb{E}\,|\xi_j||f - \widehat{f}_j||T_n^* - T_n^{(j)}|^{p-1}.$$

Stein's approach, Toy example

- Let $\mathbb{E}\, X_j^2 = 1$ and

$$T_n^* = \sum_{j=1}^{n} a_j X_j. \tag{5}$$

$\xi_j := X_j, f_j := a_j =: \widehat{f}_j, \mathcal{R} = 0$. It is easy to see that $T_n^{(j)} = \sum_{k \neq j} a_k X_k$.

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- $$\mathbb{E}\, |T_n^*|^p \leq C^p \Big(\mathcal{A}^p + p^{\frac{p}{2}} \mathcal{B}^{\frac{p}{2}} + p^p \mathcal{C} + p^p \mathcal{D} + \mathbb{E}\, |\mathcal{R}|^p \Big),$$

  where

$$\mathcal{A} = \mathbb{E}^{\frac{1}{p}} \left(\sum_{j=1}^{n} \mathbb{E}_j\, |\xi_j (f_j - \widehat{f}_j)| \right)^p = 0,$$

$$\mathcal{B} = \mathbb{E}^{\frac{2}{p}} \left(\sum_{j=1}^{n} \mathbb{E}_j (|\xi_j (T_n^* - T_n^{(j)})|) |\widehat{f}_j| \right)^{\frac{p}{2}} = \sum_{j=1}^{n} a_j^2,$$

$$\mathcal{C} = \sum_{j=1}^{n} \mathbb{E}\, |\xi_j| |T_n^* - T_n^{(j)}|^{p-1} |\widehat{f}_j| = \sum_{j=1}^{n} a_j^p\, \mathbb{E}\, |X_j|^p,$$

$$\mathcal{D} = \sum_{j=1}^{n} \mathbb{E}\, |\xi_j| |f - \widehat{f}_j| |T_n^* - T_n^{(j)}|^{p-1} = 0.$$

Stein's approach, for matrices

- $\xi_j := n^{-1}(\varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3}), f_j := \mathbf{R}_{jj}, \mathcal{R} = n^{-1} \sum_j \varepsilon_{j4} \mathbf{R}_{jj}$.

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- For all $p \geq 2$ there exists some absolute constant C such that

$$\mathbb{E}\,|T_n^*|^p \leq C^p\left(\mathcal{A}^p + p^{\frac{p}{2}}\mathcal{B}^{\frac{p}{2}} + p^p\mathcal{C} + p^p\mathcal{D} + \mathbb{E}\,|\mathcal{R}|^p\right),$$

where

$$\mathcal{A} = \mathbb{E}^{\frac{1}{p}}\left(\sum_{j=1}^n \mathbb{E}_j\,|\xi_j(f_j - \widehat{f}_j)|\right)^p \sim \frac{1}{(nv)},$$

$$\varepsilon_{j\alpha} \sim (nv)^{-1/2}, f_j - \widehat{f}_j \sim (nv)^{-1/2}$$

$$\mathcal{B} = \mathbb{E}^{\frac{2}{p}}\left(\sum_{j=1}^n \mathbb{E}_j(|\xi_j(T_n^* - T_n^{(j)})|)|\widehat{f}_j|\right)^{\frac{p}{2}} \sim \frac{1}{(nv)^2},$$

$$\varepsilon_{j\alpha} \sim (nv)^{-1/2}, T_n^* - T_n^{(j)} \sim (nv)^{-3/2}$$

$$\mathcal{C} = \sum_{j=1}^n \mathbb{E}\,|\xi_j||T_n^* - T_n^{(j)}|^{p-1}|\widehat{f}_j| \sim \frac{C^p}{(nv)^p},$$

$$\mathcal{D} = \sum_{j=1}^n \mathbb{E}\,|\xi_j||f - \widehat{f}_j||T_n^* - T_n^{(j)}|^{p-1} \sim \frac{C^p}{(nv)^p}.$$

Stein's approach, Toy example

- Let $\mathbb{E}\, X_j^2 = 1$ and

$$T_n^* = \sum_{j=1}^{n} a_j X_j. \tag{6}$$

$(\xi_j := X_j, f_j := a_j =: \widehat{f}_j)$. It is easy to see that $T_n^{(j)} = \sum_{k \neq j} a_k X_k$.

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- For all $p \geq 2$ there exists some absolute constant C such that

$$\mathbb{E}\, |T_n^*|^p \leq C^p p^{\frac{p}{2}} \left(\sum_{j=1}^{n} a_j^2 \right)^{\frac{p}{2}} + C^p p^p \sum_{j=1}^{n} a_j^p \, \mathbb{E}\, |X_j|^p \Big),$$

How to reduce the condition $\mathbb{E}\,|X_{jk}|^{4+\delta} < \infty$ with $\delta > 0$ to $\delta = 0$.

- Define

$$A_{jk} := \{|X_{jk}| < n^{\frac{1}{4}} \log n\},\; B_{jk} := A_{jk}^c = \{n^{\frac{1}{4}} \log n \le |X_{jk}| \le n^{\frac{1}{2}} R^{-1}\},$$

  and $p_n := \mathbb{P}(B_{11})$. Moreover $p_n \le \beta_4\, n^{-1} \log^{-4} n$.
- Define $\mathbf{L} := \mathbf{L}(\mathbf{X}) := [L_{jk}]_{j,k=1}^n$, where $L_{jk} := I[A_{jk}]$. Let ξ_{jk} and η_{jk}, $j,k = 1,\ldots,n$ be mutually independent and $\mathbb{P}(\xi_{jk} \in \cdot) = \mathbb{P}(X_{jk} \in \cdot \big| A_{jk})$ and $\mathbb{P}(\eta_{jk} \in \cdot) = \mathbb{P}(X_{jk} \in \cdot \big| B_{jk})$.
- Define $\mathbf{X}(\mathbf{L}) := [X_{jk}(L_{jk})]_{j,k=1}^n$, where

$$X_{jk}(L_{jk}) := \begin{cases} \xi_{jk}, & \text{if } L_{jk} = 1, \\ \eta_{jk}, & \text{if } L_{jk} = 0. \end{cases}$$

Local semicircle law, $z = 0$. 4 moments.

- Let $r := r_n := \log^3 n$. We say that $\mathbf{L}$ is r-admissible, if $\mathbf{L}$ can be represented as follows (up to the permutation of rows and columns)

$$\mathbf{L} = \begin{bmatrix} \mathbf{A}_1 & 1\ldots & 1\ldots & 1\ldots \\ 1\ldots & \mathbf{A}_2 & \ldots & 1\ldots \\ \ldots & \ldots & \ldots & \\ 1\ldots & 1\ldots & \mathbf{A}_L & 1\ldots \\ 1\ldots & 1\ldots & \ldots & 1\ldots \\ \ldots & \ldots & \ldots & \\ 1\ldots & 1\ldots & \ldots & 1\ldots \end{bmatrix},$$

  where $\mathbf{A}_\nu$ random Hermitian of size $r_\nu \leq r$, $\nu = 1, \ldots, L$. Here $r_1 + \ldots + r_L \leq \log^3 n \max(1, n^2 p_n)$. Moreover, the zero-entries of matrix $\mathbf{L}$ can only be inside of $\mathbf{A}_\nu$, and in each row (column) may contain at most r zero-entries.

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$$\mathbb{P}(\mathbf{L} \text{ is not } r\text{-admissible}) \leq n^{-c\log^2 n}, \tag{7}$$

  where $c > 0$

Thank you!