Linear statistics restricted to the largest eigenvalues of random matrices

Aurélien Grabsch

joint work with Satya Majumdar and Christophe Texier

Random matrix M, eigenvalues $\lambda_1, \ldots, \lambda_N$
j.p.d.f. $P(\lambda_1, \ldots, \lambda_N)$ known $\rightarrow$ distribution of $L = \sum_{n=1}^{N} f(\lambda_n)$?

Linear statistics of eigenvalues

Random matrix M, eigenvalues $\lambda_1, \dots, \lambda_N$
j.p.d.f. $P(\lambda_1, \dots, \lambda_N)$ known $\rightarrow$ distribution of $L = \sum_{n=1}^{N} f(\lambda_n)$?

Applications:

- quantum transport
- random bipartite quantum states
- statistical mechanics
- number of eigenvalues in a given interval (ex: $f(x) = \theta(x - w)$)
- ...

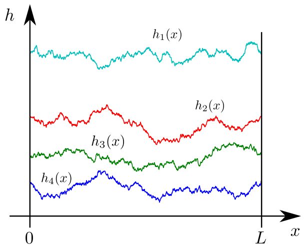
Truncated linear statistics

Distribution of

$$\tilde{L} = \sum_{n=1}^{N_1} f(\lambda_n), \quad \lambda_1 > \lambda_2 > \ldots > \lambda_N \quad \text{and} \quad N_1 < N,$$

in the limit $N \to \infty$, with $\kappa = N_1/N$ fixed.

Non intersecting Brownian interfaces with periodic boundary conditions, in the presence of a wall (Fisher '84).



Nadal and Majumdar '09: $\lambda_n = bh_n^2$, when $L \to \infty$:

$$P(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{n=1}^{N} \lambda_n^{\alpha - 1/2} e^{-\lambda_n}$$

Center of mass of the N_1 highest interfaces: $G = \frac{1}{N_1} \sum_{n=1}^{N_1} h_n$.

Coulomb gas

Rescaling $\lambda_n = N\, x_n$, introduce the density

$$\rho(x) = \frac{1}{N} \sum_{n=1}^{N} \delta(x - x_n).$$

$$P(\lambda_1, \dots, \lambda_N)\, \mathrm{d}\lambda_1 \dots \mathrm{d}\lambda_N \to e^{-N^2 \mathcal{E}[\rho]}\, \mathcal{D}\rho,$$

with the energy

$$\mathcal{E}[\rho] = -\int \rho(x)\rho(y) \ln|x - y|\, \mathrm{d}x\mathrm{d}y + \int x\, \rho(x)\mathrm{d}x.$$

Coulomb gas

Rescaled truncated linear statistics, with $N_1/N = \kappa$ fixed:

$$s = \frac{\kappa\, G}{\sqrt{N}} = \int_c \rho(x)\sqrt{x}\,\mathrm{d}x,$$

where c is a lower bound defined by

$$\int_c \rho = \kappa.$$

Rescaled truncated linear statistics, with $N_1/N = \kappa$ fixed:

$$s = \frac{\kappa\, G}{\sqrt{N}} = \int_c \rho(x)\sqrt{x}\,\mathrm{d}x,$$

where c is a lower bound defined by

$$\int_c \rho = \kappa.$$

$$P_{N,\kappa}(s) = \frac{\int \mathrm{d}c \int \mathcal{D}\rho\, e^{-N^2\mathcal{E}[\rho]}\, \delta\Big(\int_c \rho(x)\mathrm{d}x - \kappa\Big)\, \delta\Big(\int^c \rho(x)\mathrm{d}x - (1-\kappa)\Big)\, \delta\Big(\int_c \sqrt{x}\rho(x)\mathrm{d}x - s\Big)}{\int \mathrm{d}c \int \mathcal{D}\rho\, e^{-N^2\mathcal{E}[\rho]}\, \delta\Big(\int_c \rho(x)\mathrm{d}x - \kappa\Big)\, \delta\Big(\int^c \rho(x)\mathrm{d}x - (1-\kappa)\Big)}$$

Coulomb gas

$N \to \infty$: integrals dominated by the density $\rho_\star(x;s)$ that minimizes the energy under the constraints:

$$\kappa = \int_c \rho_\star(x;s)\,\mathrm{d}x,$$

$$1-\kappa = \int^c \rho_\star(x;s)\,\mathrm{d}x,$$

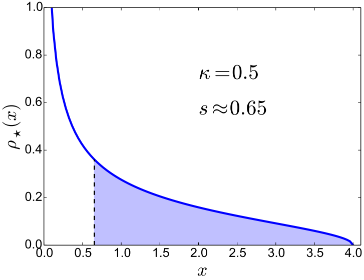
$$s = \int_c \rho_\star(x;s)\sqrt{x}\,\mathrm{d}x.$$

Then,

$$P_{N,\kappa}(s) \propto \exp\left[-N^2\left(\mathcal{E}[\rho_\star(x;s)] - \mathcal{E}[\rho_0^\star(x)]\right)\right]$$

Optimal configuration: Marčenko-Pastur distribution

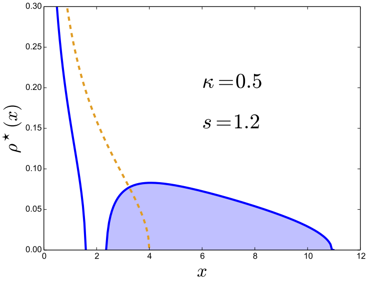
$$\rho_0^\star(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}.$$



$$s_0(\kappa) = \int_{c_0}^{4} \rho_0^\star(x)\sqrt{x}\,\mathrm{d}x, \quad \kappa = \int_{c_0}^{4} \rho_0^\star(x)\,\mathrm{d}x$$

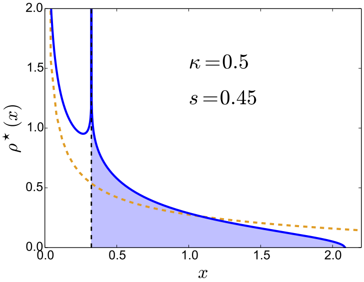
Case $s > s_0$: Phase 1

$$\rho(x) = \frac{\mu_1}{2\pi^2} \frac{\mathrm{sign}(x-c)}{\sqrt{d-b}} \sqrt{\frac{(c-x)(d-x)}{x(d-x)}} \Pi\left(\frac{d-c}{d-x}, \sqrt{\frac{d-c}{d-b}}\right),$$



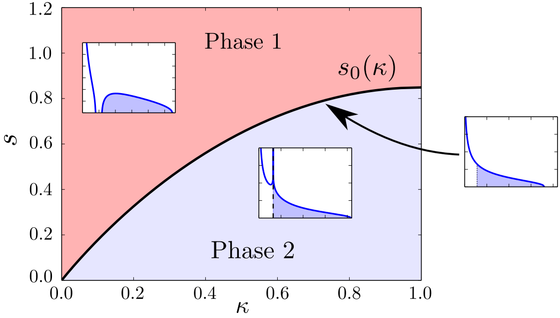
Case $s < s_0$: Phase 2

$$\rho(x) = \frac{1}{2\pi}\sqrt{\frac{d-x}{x}} + \frac{\mu_1}{4\pi^2\sqrt{x}} \ln \frac{\sqrt{d-c}+\sqrt{d-x}}{\left|\sqrt{d-c}-\sqrt{d-x}\right|}.$$



$\Rightarrow$ log-singularity in the density of eigenvalues

Phase diagram



$\mathcal{E}[\rho]$ and all its derivatives continuous at the transition
$\Rightarrow$ infinite order phase transition

Distribution of the linear statistics

- Gaussian peak around $s_0(\kappa)$,

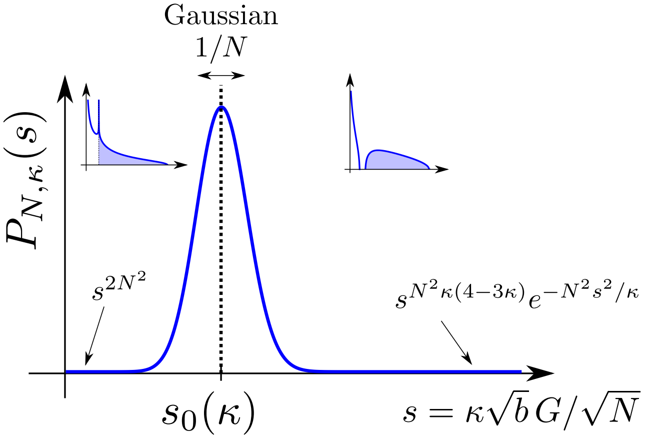
$$\mathrm{Var}(s) = \frac{1}{2\pi^2}\left(4 - c_0 + c_0 \ln \frac{c_0}{4}\right), \quad \kappa = \int_{c_0}^{4} \rho_0^\star .$$

- large deviations:

$$P_{N,\kappa}(s) \underset{s\to 0}{\sim} s^{2N^2},$$

$$P_{N,\kappa}(s) \underset{s\to\infty}{\sim} s^{N^2\kappa(4-3\kappa)} e^{-N^2 s^2/\kappa}.$$

Distribution of the linear statistics



Universal mechanism for truncated linear statistics: