Conformal Gravity

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Master Thesis in Theoretical Physics

September 2015

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Statutory Declaration

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly marked all material which has been quoted either literally or by content from the used sources.

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1 Motivation

Despite the great success of General Relativity in describing the physics of the Solar System and the dynamics of the Universe in total, there are still some puzzles left. When Albert Einstein introduced his theory of General Relativity in 1915 there were no measurements, which tested gravity on scales far beyond the distances of the Solar System. Therefore, as General Relativity was able to solve several problems like the gravitational bending of light or the precession of the perihelion of Mercury, it was quite natural to accept it as the standard theory for gravity. But nowadays there is data from far more distant objects, like other galaxies or galaxy clusters. On these distance scales there arise several problems. For example the measurement of rotational velocities of stars or gas in galaxies differs very strongly from the expectation. Thus, it is only possible to fit this behavior by invoking some unknown Dark Matter, which interacts with the baryonic matter only via gravity or via the weak force, but not electromagnetically or strongly.

Another issue is the cosmological constant problem. Since the measurements of Edwin Hubble in 1929 we know that the Universe is not static as it was first assumed by Einstein, but as we know today it expands accelerated. Since all types of matter known so far are attracting each other via gravity, there has to be some kind of matter or energy that pushes matter apart. This substance is called Dark Energy and its nature is unknown today, too. It would be appealing to identify this unknown Dark Energy with the zero-point energy of the matter fields of the Standard Model. But the calculated zero-point energy density of the Standard Model and the measured value, differ by 120 orders of magnitude. One could solve this problem simply by fine-tuning the comological constant, but this seems to be ridiculously unlikely.

Strongly coupled to this is the problem of combining gravity and quantum mechanics at high energies. This means that treating the gravitational field as a quantum field exposes that General Relativity is non-powercounting renormalizable. Therefore, it is not a reliable theory in the regime of high energies.

Hence, it is quite reasonable to ask whether instead of unknown matter or energy that cures all the problems of General Relativity, it is the theory of gravity itself, which has to be modified in such a way that there are modifications on big distance scales, but the behavior on Solar System scales is kept.

There are several approaches to modify gravity, like gravitational scalar fields, extra spatial dimensions, higher-order derivatives in the action or non-Christoffel connections, but non of them totally convincing.

In this thesis, we want to investigate an approach by Philip D. Mannheim [1, 2, 3], which belongs to the category of higher-order derivatives in the action. It is called "Conformal (Weyl) Gravity". This theory is able to tackle the mentioned problems of Dark Matter, Dark Energy and quantum gravity and thus it is worth investigating this theory in more detail.

We start with a presentation of the general idea of this theory in chapter 2. In chapter 3 we deal with the Dark Matter problem and its solution in

Conformal Gravity. After that in chapter 4 we have a brief look at the quantization mechanism of the theory and how it deals with the zero-point energy problem. In chapter 5, it is investigated, how the matter particles acquire mass via a dynamical Higgs mechanism. This mass generation mechanism is used in chapter 6 to study conformal cosmology. In chapter 7 we present an approach to study classical gravitational waves of gravitational bounded systems like binary systems of stars and black holes in Conformal Gravity. The first steps of solving the fourth-order wave equation are shown. The next step will be the calculation of the radiated power for such a system. But this is still work in progress. In the end we give an outlook of open problems and further work that has to be done.

2 Introduction to Conformal Weyl Gravity

This chapter is an introduction to the theory of Conformal Gravity. We start with the general formalism of General Relativity and introduce important quantities and conventions. After that we investigate, what changes in the formalism of Conformal Gravity and address some general problems and difficulties that may be solved within Conformal Gravity. This introduction follows mainly two publications by Mannheim et. al. [1, 2].

2.1 General Relativity

Let us first investigate which properties of GR make it so successful to describe the Solar System phenomenology. We know that Einstein wanted to generalize his theory of Special Relativity to a theory that does not only work for constant moving observers, but which is also compatible with accelerated observers. Furthermore, it was necessary to bring Newtonian Gravity, which was the standard theory of gravity before the invention of General Relatrivity, in line with the principle of relativity. The solution to both problems is given by a theory which is generally coordinate invariant and involves a metric $g_{\mu\nu}$ which can be identified with the gravitational field.

Thus, we introduce several important quantities that are necessary for a formalism of a metric theory. We adopt the sign conventions for the metric $(-+++)$, the Riemann tensor and the energy-momentum tensor from [1]. In addition, we set $c = \hbar = 1$.

The Christoffel symbols

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\rho g_{\nu\sigma} \right) \tag{1}$$

are central objects in metric-based theories. They appear in the geodesic equation which is the equation of motion for test particles in flat or curved space-time. The geodesic equation reads, after choosing some coordinate system $x^\mu(\tau)$ and an affine parameter τ, as follows

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = 0. \tag{2}$$

Moreover, there are tensors which carry the information about the curvature of space-time and are made up of the Christoffel symbols.
The components of the Riemann tensor are given by

$$R^{\lambda}_{\mu\nu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^\kappa} + \Gamma^{\lambda}_{\kappa\eta} \Gamma^{\eta}_{\mu\nu} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^\nu} - \Gamma^{\lambda}_{\nu\eta} \Gamma^{\eta}_{\mu\kappa}. \tag{3}$$

One can show that if and only if this tensor is zero, the space-time we are dealing with is flat. In that case, eq. (2) reduces to Newtonian's second law of motion describing a free particle.

By invoking the static, spherically symmetric Schwarzschild metric

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta)\, d\phi^2, \tag{4}$$

where

$$B(r) = A^{-1}(r) = 1 - 2\frac{\beta}{r}, \quad \beta = MG, \tag{5}$$

this theory predicts for example the bending of light in such a geometry. M represents the mass of a spherically symmetric object and G is the gravitational coupling constant. This behavior was confirmed by the observation of Arthur Eddington in 1919 (up to the accuracy of the experiment). Since then the validity of the above description of nature was established.

By this line of reasoning it is very likely that a valuable theory of gravity has to be a covariant metric theory.

Up to now, we did not specify the equation of motion for the gravitational potential. We only defined, what the solutions on Solar System distances look like. Therefore, let us turn to the equation of motion for the metric. This can be derived by a functional variation of an action for the Universe with respect to the metric $g_{\mu\nu}$

$$I_{UNIV} = I_{EH} + I_\Lambda + I_M = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R^\alpha_{\ \alpha} - 2\Lambda\right) + I_M, \tag{6}$$

where I_{EH} is the Einstein-Hilbert action, I_M the the matter part and I_Λ the action for the cosmological constant. The cosmological constant will be investigated in chapter 4, 5 and 6.

The equations of motion for the metric are called the Einstein equations and are given by the following expression

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^\alpha_{\ \alpha}\right) - \Lambda g^{\mu\nu} = -8\pi G T^{\mu\nu}_M, \tag{7}$$

where $R_{\mu\nu} = g_{\lambda\nu}R^\lambda_{\ \mu\nu\kappa}$ and $R^\alpha_{\ \alpha} = g^{\alpha\mu}R_{\mu\alpha}$ are the Ricci tensor and the Ricci scalar, respectively. Λ is the cosmological constant. The determinant of the metric is represented by $g = \det(g_{\mu\nu})$. $T^{\mu\nu}_M$ is the energy-momentum tensor of the matter field sector. It is defined in the following way

$$T^{\mu\nu}_M = \frac{2}{\sqrt{-g}}\frac{\delta I_M}{\delta g_{\mu\nu}}. \tag{8}$$

In vacuum we have $T^{\mu\nu}_M = 0$. This means that the Ricci tensor has to vanish and one finds that the solution to the Einstein equations exterior to a static, spherically symmetric source is the Schwarzschild solution (4) and (5).

It is commonly believed that a viable modification of gravity has to reproduce these Einstein equations in some limit or approximation. But as pointed out by Mannheim [1, 2] this is definitely not true. What a theory, which modifies gravity, really has to recover is the Solar System phenomenology to a level of accuracy that is required by the available data. This is because the range, where Newton's law is well established, runs from the millimeter scale up to 10^{15} cm, which is roughly the scale of the Solar System diameter. This means that a viable modification of gravity only has to reproduce the Schwarzschild solution for the exterior region of a static, spherically symmetric source. Of course, it also has to pass all the other test of General Relativity, like the

Mercury procession or to describe the evolution of the Universe so that it fits the current data. On top of that, there are measurements of the Hulse-Taylor binary pulsar (see chapter 7), which seems to have a shrinking orbital radius. A viable theory of gravity has to give an explanation for that. Therefore, some of these aspects are considered in this thesis.

2.2 Modifications of General Relativity

In this Chapter we introduce the "Conformal Weyl Gravity", which is still a coordinate invariant metric theory, but the Einstein-Hilbert action is substituted by a conformally invariant action

$$\begin{aligned} I_{UNIV} &= I_W + I_M = -\alpha_g \int d^4x \, (-g)^{1/2} \, C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} + I_M \\ &= -\alpha_g \int d^4x \, (-g)^{1/2} \left[R_{\lambda\mu\nu\kappa} R^{\lambda\mu\nu\kappa} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} \left(R^{\alpha}_{\ \alpha}\right)^2 \right] + I_M, \quad (9) \end{aligned}$$

where α_g is a dimensionless coupling constant and

$$\begin{aligned} C_{\lambda\mu\nu\kappa} &= R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^{\alpha}_{\ \alpha} \left[g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu} \right] \quad (10) \\ & \quad - \frac{1}{2} \left[g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu} \right] \end{aligned}$$

is the Weyl conformal tensor. It has the remarkable property that it is invariant under local stretching of the metric, which are called the Weyl transformation

$$g_{\mu\nu}(x) \to \Omega^2(x) \, g_{\mu\nu}(x) , \quad (11)$$

$$\begin{aligned} C^{\lambda}_{\ \mu\nu\kappa}(x) &\to C^{\lambda}_{\ \mu\nu\kappa}(x) , \quad (12) \\ C^{\lambda\mu\nu\kappa} C_{\lambda\mu\nu\kappa} &\to \Omega^{-4} C^{\lambda\mu\nu\kappa} C_{\lambda\mu\nu\kappa} . \quad (13) \end{aligned}$$

Note that a Weyl transformation on the metric is not the same thing as a usual conformal coordinate transformation

$$\begin{aligned} x &\to x' \quad (14) \\ g_{\mu\nu} &\to \Omega^2(x') \, g_{\mu\nu}(x') . \quad (15) \end{aligned}$$

Although the transformation laws look quite the same, their meaning is totally different. Conformal coordinate transformations are just special coordinate transformation, which does not change the physical scale. In covariant theories, physics is invariant under general coordinate transformations. In contrast, Weyl transformation are stretchings of the metric and therefore they change the physical scale. It cannot be associated with a general coordinate transformation, because it does not leave physical distances invariant.

Furthermore, the Weyl tensor is the traceless part of the Riemann tensor

$$g^{\mu\kappa} C^{\lambda}_{\ \mu\nu\kappa} = 0. \quad (16)$$

Now by using the Gauß-Bonnet theorem (Lanczos lagrangian)

$$L_L = (-g)^{1/2} \left[R_{\lambda\mu\nu\kappa} R^{\lambda\mu\nu\kappa} - 4 R^{\mu\nu} R_{\mu\nu} + \left(R^{\alpha}_{\ \alpha}\right)^2 \right], \tag{17}$$

which is a total derivative, we can rewrite the action in eq. (9) in the following way (see [2, 5])

$$I_{UNIV} = I_W + I_M = -2\alpha_g \int d^4x \, (-g)^{1/2} \left[R_{\mu\kappa} R^{\mu\kappa} - \frac{1}{3} \left(R^{\alpha}_{\ \alpha}\right)^2 \right] + I_M. \tag{18}$$

In Conformal Gravity the analog to the Einstein equations is called the Bach equations. It can be derived by a functional variation of the Weyl action I_W with respect to the metric

$$\frac{1}{\sqrt{-g}} \frac{\delta I_W}{\delta g_{\mu\nu}} \equiv -2\alpha_g W^{\mu\nu}. \tag{19}$$

Then by varying the matter term I_M we can add an energy-momentum tensor for the matter field sector

$$4\alpha_g W^{\mu\nu} = 4\alpha_g \left[2 C^{\mu\lambda\nu\kappa}_{\ \ \ \ \ \ ;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa} \right] = 4\alpha_g \left[W^{\mu\nu}_{(2)} - \frac{1}{3} W^{\mu\nu}_{(1)} \right] = T^{\mu\nu}_M, \tag{20}$$

where

$$W^{\mu\nu}_{(1)} = 2 g^{\mu\nu} \left(R^{\alpha}_{\ \alpha}\right)^{;\beta}_{\ \ ;\beta} - 2 \left(R^{\alpha}_{\ \alpha}\right)^{;\mu;\nu} - 2 R^{\alpha}_{\ \alpha} R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} \left(R^{\alpha}_{\ \alpha}\right)^2 \tag{21}$$

and

$$W^{\mu\nu}_{(2)} = \frac{1}{2} g^{\mu\nu} \left(R^{\alpha}_{\ \alpha}\right)^{;\beta}_{\ \ ;\beta} + R^{\mu\nu;\beta}_{\ \ \ \ \ ;\beta} - R^{\mu\beta;\nu}_{\ \ \ \ \ ;\beta} - R^{\nu\beta;\mu}_{\ \ \ \ \ ;\beta} - 2 R^{\mu\beta} R^{\nu}_{\ \beta} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta}. \tag{22}$$

We have to mention here that the rank-two gravitational tensor $W^{\mu\nu}$ inherits the properties to be traceless and covariantly conserved from the Weyl tensor.

Thus, one can see that this theory of gravity is able to reproduce a vacuum solution $R_{\mu\nu} = 0$ as in General Relativity. Further, it is possible that $W^{\mu\nu}$ vanishes not because of $R_{\mu\nu} = 0$, but by the cancellation of the different terms. Therefore, in Conformal Gravity there are also non-Schwarzschild solutions to the static, spherically symmetric geometry, see eq. (37).

Let us stop here for a moment and think about what this means. In principle, the requirement to the action to be a coordinate scalar does forbid you to add higher order terms to the Einstein-Hilbert action to modify gravity. Usually one can choose these terms to be negligible on Solar System distance scales and thus the derived equations of motion would reduce to the Einstein equations. Remarkable about conformal gravity is that the conformal invariance principle leads to the unique action in eq. (9). For this it is forbidden to add a cosmological constant term $-\int d^4x \, (-g)^{1/2} \, 2\Lambda$, where Λ is the cosmological constant [1, 2, 6], since the transformation $(-g)^{1/2} \to (-g')^{1/2} = \Omega^4(x) \, (-g)^{1/2}$. Furthermore, it terms like $\left(C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}\right)^2$ are also excluded. Hence, by demanding the conformal symmetry one found a mechanism to overcome the degeneracy in the gravitational action and the somehow well motivated, but still arbitrary choice of the Einstein-Hilbert action.

On top of that, Conformal Gravity addresses all the difficulties which appear in standard gravity, when one extrapolates the Einstein equations without a cosmological constant term beyond its weak classical non-Solar System limit. By extrapolation to scales of galaxies or galaxy clusters one ends up with the Dark Matter problem, see chapter 3. The cosmological constant or Dark Energy problem (chapter 6) appears by dealing with cosmological scales. Moreover, if one goes to the limit of strong classical gravity one runs into the singularity problem [1, 2].

Actually, the cosmological constant and zero-point problem are related to each other. In trying to find a theory of quantum gravity one recognizes that it is not clear if standard gravity is renormalizable and therefore the zero-point energy of gravity seems to have an infinite or at least very huge value. This does not mean that the value is really infinite, but it indicates that the classical theory of General Relativity is not valid in the quantum regime anymore. Additionally, when one quantizes the matter field particle physics part, there appear infinite zero-point terms. Moreover, for every quantum field theory we are able to make educated guesses of the magnitude of its zero-point energy. Generally, one assumes that there were several phase transitions during the evolution of the Universe. A phase transition means that above a certain temperature of the Universe a symmetry is unbroken and below it is broken. For the Weinberg-Salam electroweak theory the $SU(2) \times U(1)$ symmetry breaks at a temperature of about ~ 100 GeV. By this symmetry breaking the zero-point energy contribution to the cosmological constant readjust, because there is a potential energy difference between these phases. Therefore, one naturally expects a contribution to the net cosmological constant of

$$\rho_\Lambda^{EW} \sim (100 \text{ GeV})^4 . \tag{23}$$

Besides that, ones assumes a chiral symmetry breaking at a temperature of ~ 0.3 GeV which gives a zero-point energy contribution of

$$\rho_\Lambda^{QCD} \sim (0.3 \text{ GeV})^4 . \tag{24}$$

Moreover, it is usually assumed that ordinary quantum field theory fails at the Planck scale, because quantum effects of gravity become strong. Thus, this gives a contribution of

$$\rho_\Lambda^{Planck} \sim \left(10^{18} \text{ GeV}\right)^4 . \tag{25}$$

Furthermore, due to redshift measurements of supernovae type 1a one recognizes that the Universe is in a state of accelerated expansion. That means that one needs something that drives this expansion. Thus, there has to be some matter that is repulsive rather than attractive. Hence, standard cosmology has to assume that the composition of the Universe has to be roughly 70% Dark Energy, 25% Dark Matter and just 5% baryonic matter, where the Dark Matter and the Dark Energy are still unknown today. This Dark Energy is associated with a non-zero cosmological constant in eq. (7). By doing so, standard cosmology is really successful, since it encounters all crucial cosmological events like the big bang nucleosynthesis, the anisotropy of the cosmic microwave background and the strong lensing effect.

The net cosmological constant is given by a sum of the “bare” cosmological constant Λ and the zero-point energy contributions of the various fields. The zero-point energies of the fields contribute to the accelerated expansion, because after a renormalization that makes their magnitudes finite they provide the correct equation of state $p_{vac} = -\rho_{vac}$ for a repulsive contribution. Actually, it is not clear, whether the cosmological constant is really a constant or maybe slightly time dependent. But a time-independent cosmological constant is in good agreement with the data. By the measurement of the accelerated expansion of the Universe one finds a value for the observed energy density

$$\left|\rho_{\Lambda}^{(obs)}\right| \leq \left(10^{-12}\ \mathrm{GeV}\right)^{4}, \tag{26}$$

which is much smaller than the contributions from the particle physics matter fields. This is the well-known discrepancy of 120 orders of magnitude. One could solve this problem by assigning the “bare” cosmological constant a fine-tuned value such that the net cosmological constant yields the observed value in eq. (26). Nevertheless, one still tries to find a solution to this problem. For a detailed discussion of this issue, see [7].

As will be outlined in this work all the above problems or equivocalities can be solved by the Weyl invariance principle. We will investigate these issues in detail in chapter 4 and 6.

3 The Dark Matter Problem

This chapter deals with one of the biggest problems in standard gravity, namely the Dark Matter problem. We start to explain the Dark Matter problem in Newtonian gravity and after that we show how one can tackle this problem in Conformal Gravity.

The Dark Matter problem is the discrepancy between the expected and measured value of the rotational velocities of stars or hydrogen gas in galaxies. Hydrogen gas is distributed out to much further distances than the most stars. So it delivers data for the outer region of spiral galaxies and enables us to study the rotation curves to much further distances.

As we will see in this chapter, one would expect that the rotational velocities of these objects, which are far away from the galactic center, fall as $r^{-1/2}$ with the distance. This behavior is called the "Keplerian fall-off". But in contrast, all measurements show that the rotational velocities of outer objects do not decline, but become nearly constant, regardless of the distance from the center of the galaxy. In the following we will study the theoretical expectation and the discrepancy in detail.

But before doing so, let us mention that in chapter 4 it will be outlined that the Bach equations (20) are a priori purely quantum mechanical equations, opposed to the Einstein equations in General Relativity, which are treated classically. But Mannheim states in [1] that quite analogously to Quantum Electrodynamics it is possible to use these non-classical equations in a classical limit, i.e. we use matrix elements of the Bach equations with an indefinite high number of gravitational quanta. Therefore, we can treat the equations completely classical and can apply them to classically phenomena like the interior dynamics of galaxies.

3.1 Derivation and Solution of the Bach Equations

To investigate the rotation curves of galaxies we have to find the solutions of the Bach equations for a static, spherically symmetric source in Conformal Gravity. This is necessary, since as mentioned in section 2.2, there are additional non-Schwarzschild solutions.

The most general static, spherically symmetric line element is given by (see [1, 2, 6])

$$ds^2 = -b\left(\rho\right)dt^2 + a\left(\rho\right)d\rho^2 + \rho^2 d\Omega^2, \tag{27}$$

where $d\Omega^2 = d\theta^2 + \sin^2\left(\theta\right)d\phi^2$ is the infinitesimal surface element in spherical coordinates. By means of a general coordinate transformation

$$\rho\left(r\right) = p\left(r\right) \Rightarrow d\rho\left(r\right) = p'\left(r\right)dr, \quad B\left(r\right) = \frac{r^2 b\left(r\right)}{p^2\left(r\right)}, \quad A\left(r\right) = \frac{r^2 a\left(r\right)p'^2\left(r\right)}{p^2\left(r\right)} \tag{28}$$

(the prime denotes the derivative with respect to r) we can rewrite the the line element in the following way

$$ds^2 = \frac{p^2\left(r\right)}{r^2}\left[-B\left(r\right)dt^2 + A\left(r\right)dr^2 + r^2 d\Omega^2\right]. \tag{29}$$

Now we choose

$$-\frac{1}{p(r)} = \int_{const.}^{r} dr' \frac{1}{r'^2 \left(a(r')\, b(r')\right)^{1/2}} \tag{30}$$

and from that we find

$$p'^2(r) = \frac{p^4(r)}{r^4 a(r)\, b(r)}. \tag{31}$$

Inserting this in $A(r)$ and $B(r)$ in eq. (29) yields

$$ds^2 = \frac{p^2(r)}{r^2}\left[-B(r)\, dt^2 + B^{-1}(r)\, dr^2 + r^2 d\Omega\right] \tag{32}$$

just by replacing the two general functions $b(\rho)$ and $a(\rho)$ in eq. (27) by two different, but still general functions $p(r)$ and $B(r)$ in eq. (32). The resulting line element is now conformal to a line element with the property $g_{00} = -g_{rr}^{-1}$.

Thus, if we choose the prefactor $\frac{p^2(r)}{r^2}$ in eq. (32) to be the conformal factor $\Omega^2(x)$ of a Weyl transformation as in eq. (11), then both rank two tensors $W^{\mu\nu}$ and $T^{\mu\nu}$ transform as follows

$$W^{\mu\nu} \rightarrow \Omega^{-6}(x)\, W^{\mu\nu}, \tag{33}$$

$$T^{\mu\nu} \rightarrow \Omega^{-6}(x)\, T^{\mu\nu}. \tag{34}$$

As required the theory is invariant under a Weyl transformation. Therefore, it is enough to evaluate eq. (20) in the subsequent line element

$$ds^2 = -B(r)\, dt^2 + B^{-1}(r)\, dr^2 + r^2 d\Omega \tag{35}$$

without the conformal factor. Here we notice that all dynamics are included in this metric. Thereby, the conformal symmetry reduces the number of independent metric coefficients, although we are dealing with a theory of higher derivatives.

The tensor $W^{\mu\nu}$ has only one independent component, because all other components are related to it by the tracelessness and the covariant conservation of $W^{\mu\nu}\left[{W^0}_0 + {W^r}_r + 2{W^\theta}_\theta = 0,\ \left(\frac{B'}{2B} - \frac{1}{r}\right)\left({W^0}_0 - {W^r}_r\right) - \left(\frac{d}{dr} - \frac{4}{r}\right){W^r}_r = 0\right]$. In consequence, let us just analyze W^{rr}. By inserting (35) into the Bach equations (20) we find

$$\frac{W^{rr}}{B(r)} = \frac{B'B'''}{6} - \frac{B''^2}{12} - \frac{BB''' - B'B''}{3r} - \frac{BB'' + B'^2}{3r^2} + \frac{2BB'}{3r^3} - \frac{B^2}{3r^4} + \frac{1}{3r^4}. \tag{36}$$

First let us investigate a solution for an observer outside the static, spherically symmetric source of radius R, i.e. the source is embedded in a region with $T^{\mu\nu}(r > R) = 0$. The complete vacuum solution can be found, after several substitutions and integrations (see [8]), by $W^{rr} = 0$. It is given by

$$B(r > R) = 1 - \frac{\beta(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2. \tag{37}$$

The parameters γ, β and k are integration constants. Here we recognize a standard Newtonian $\frac{1}{r}$-term, but additionally a term which is linear in r and a term which is proportional to r^2. Let us mention here that in the limit of Solar System distances r and by assigning appropriate values to γ and β, we can reproduce the standard Schwarzschild solution. We will discuss this in detail below, see 3.2 and 3.3.

To obtain the non-vacuum solution it is useful to calculate the component

$$W^{00} = -\frac{B''''}{3} + \frac{B''^2}{12B} - \frac{B'''B'}{6B} - \frac{B'''}{r} - \frac{B''B'}{3rB} + \frac{B''}{3r^2} + \frac{B'^2}{3r^2B} - \frac{2B'}{3r^3} - \frac{1}{3r^4B} + \frac{B}{3r^4}. \quad (38)$$

Now we can combine eq. (36) and (38) to yield

$$\begin{aligned} \frac{3}{B(r)}\left(W^0{}_0 - W^r{}_r\right) &= B'''' + \frac{4B'''(r)}{r} = \frac{(rB)''''}{r} = \nabla^4 B(r) \\ &= \frac{3}{4\alpha_g B(r)}\left(T^0{}_0 - T^r{}_r\right) \equiv f(r), \end{aligned} \quad (39)$$

where ∇ is the radial part of the gradient in spherical coordinates.

This form of the coefficient (37) of the static, spherically symmetric line element is also present in some linearized limits of fourth-order theories, but here it is an exact result. We also note that in GR there is no such compact form of the Einstein equations. It does not reduce to a second-order Laplacian. Only in the Newtonian limit in linearized theory such a simple form can be achieved. However, this very simple form of eq. (39) in Conformal Gravity can be obtained only in the case of a static, spherically symmetric source. Already, when one considers a non-static, spherically symmetric case the equations are far more complicated, cf. [9].

We recognize eq. (39) to be a fourth-order Poisson equation. For a perfect fluid $T^0{}_0 - T^r{}_r$ reduces to $-(\epsilon - p)$. Therefore, for slowly moving sources, like in the case of dust ($p \approx 0$), this quantity is just the negative energy density $-\epsilon = -\rho$, where ρ is the mass density.

The solution to this fourth-order Poisson equation can be found by the methods of Green's function. Thus, we yield (see [6])

$$\begin{aligned} B(\vec{r}) &= -\frac{1}{8\pi}\int d^3\vec{r'} f(r') \left|\vec{r} - \vec{r'}\right| \quad (40) \\ &= -\frac{1}{12r}\int dr' f(r') r' \left[|r' + r|^3 - |r' - r|^3\right], \quad (41) \end{aligned}$$

where the second equality is due to the angular integration. We can evaluate the absolute value functions by separating between an exterior and interior solution

$$\begin{aligned} B(r > R) &= -\frac{r}{2}\int_0^R dr' r'^2 f(r') - \frac{1}{6r}\int_0^R dr' r'^4 f(r') + \omega - kr^2, \quad (42) \\ B(r < R) &= -\frac{r}{2}\int_0^r dr' r'^2 f(r') - \frac{1}{6r}\int_0^r dr' r'^4 f(r') \quad (43) \\ &\quad -\frac{1}{2}\int_r^R dr' r'^3 f(r') - \frac{r^2}{6}\int_r^R dr' r' f(r') + \omega - kr^2. \end{aligned}$$

One has to match the solutions of (37), (42) and (43) at $r = R$. Thus, we find

$$\begin{aligned} 1 - \frac{\beta\left(2-3\beta\gamma\right)}{R} - 3\beta\gamma + \gamma R - kR^2 &= -\frac{R}{2}\int_0^R dr' r'^2 f\left(r'\right) \\ &\quad -\frac{1}{6R}\int_0^R dr' r'^4 f\left(r'\right) + B_{hom}. \end{aligned} \tag{44}$$

Now we can identify $\gamma = -\frac{1}{2}\int_0^R dr' r'^2 f\left(r'\right)$ and $\beta\left(2-3\beta\gamma\right) = \frac{1}{6}\int_0^R dr' r'^4 f\left(r'\right)$ as the second and fourth gravitational source density moment. And from the general homogeneous solution, which has the form $B_{hom} = \alpha_1 + \alpha_2\frac{1}{r} + \alpha_3 r + \alpha_4 r^2$, we only have to keep a $\omega - kr^2$ term to fulfil the boundary condition. Therefore, we are able to reproduce eq. (37) in the $r > R$ region. Up to now, (42) and (43) are exact solutions and no approximation has been done so far.

In [10] Mannheim argues that because the term $\nabla^4\left(r^2\right)$ vanishes everywhere, it corresponds to $T^{\mu\nu}\left(r\right) = 0$ for all r. Therefore, the term $-kr^2$ is just a trivial vacuum solution and does not couple to the matter source. Hence, in the following Mannheim does not take this term into account. In subsection 3.3.2 we will see that once we allow for matter in the $r > R$ region, i.e. $T^{\mu\nu}\left(r > R\right) \neq 0$, a quadratic term in r can be generated.

3.2 Newtonian Limit

In this paragraph we want to analyze the Newtonian limit of the theory of Conformal Gravity and furthermore we compare it to the Newtonian theory. That it makes the same predictions on sub-Solar System scales is not trivial, since there is a problem with the Cavendish-type experiments [11]. And to yield the correct behavior for the gravitational potential, we have to change the structure of the gravitational sources from point-like to extended sources. Additionally, we will see that Conformal Gravity passes the Solar System tests of General Relativity [12].

In Newtonian Gravity one uses the second-order Poisson equation $\nabla^2\phi\left(r\right) = g\left(r\right)$ with its solution

$$\phi\left(r\right) = -\frac{1}{r}\int_0^r dr' r'^2 g\left(r'\right) - \int_r^\infty dr' r' g\left(r'\right), \; \frac{d\phi\left(r\right)}{dr} = \frac{1}{r^2}\int_0^r dr' r'^2 g\left(r'\right), \tag{45}$$

where $\phi\left(r\right)$ is the gravitational potential. The second equation of (45) is the gravitational force with the familiar $\frac{1}{r^2}$-term for static, spherically symmetric sources. It also shows that it does not matter whether the spherically symmetric source $g\left(r\right)$ is a delta function or extended to yield a Newtonian term. Even though a delta function is indeed sufficient to produce a Newtonian term, there is no experiment yet, which proves that the microscopical gravitational source has to be a delta function. Furthermore, as long as there is only a measurement of a test particle in the exterior region $r > R$, it is not possible to reconstruct $g\left(r\right)$. On top of that, in this equation we can see the local character of Newtonian gravity, since even if there is some material in the region $r' > r$, there is no contribution to the gravitational force from this material. This behavior is called the "shell theorem".

Although this behavior appears to be natural for us, it does not hold in all theories. Therefore, by defining $B(r) = 1 + 2\phi(r)$ and $h(r) = f(r)/2$ we can identify the gravitational potential of Conformal Gravity in the fourth-order Poisson equation

$$\nabla^4 \phi(r) = h(r). \tag{46}$$

The general solution to this equation is given analogously to eq. (43) by

$$\begin{aligned} \phi(r) &= -\frac{r}{2}\int_0^r dr' r'^2 h(r') - \frac{1}{6r}\int_0^r dr' r'^4 h(r') \\ &\quad -\frac{1}{2}\int_r^\infty dr' r'^3 h(r') - \frac{r^2}{6}\int_r^\infty dr' r' h(r'), \end{aligned} \tag{47}$$

$$\frac{d\phi(r)}{dr} = -\frac{1}{2}\int_0^r dr' r'^2 h(r') + \frac{1}{6r^2}\int_0^r dr' r'^4 h(r') - \frac{r}{3}\int_r^\infty dr' r' h(r'). \tag{48}$$

Here we see that the gravitational force in eq. (48) consists of the familiar Newtonian $1/r^2$-term (although the second-order Laplacian is not present in this theory), but additionally there is a constant term and a term which yields a contribution from the rest of the Universe that is proportional to r. The fact that the integral in the third term incorporates contributions from regions $r' > r$ shows that Conformal Gravity is an intrinsically non-local theory. As a consequence, a test particle in the orbit of a galaxy is also able to sample the global field due to the rest of the Universe. We will analyze this term in greater detail below.

Moreover, this is actually not the only choice of a Poisson equation, which produces a Newtonian term. In fact, it is possible to choose a Poisson equation of the form $\nabla^6 \phi(\vec{r}) = k(\vec{r})$, which yields a solution of the form $1/r + r + r^3$. Hence, for every higher even derivative we get an additional term which is a higher power of r. All these terms of higher power in r are negligible for small distances. So in principle, all these Poisson equations are possible candidates for a theory of gravity, since they reproduce the Solar System phenomenology.

Note here that the Newtonian term appears as the short distance limit ($r \ll 1/\gamma$) of the theory. The leading term is the large distance contribution of the $-\left|\vec{r} - \vec{r'}\right|/8\pi$ Green's function of the ∇^4-operator. The Newtonian term is an additional piece that is left over. Thus, it is not surprising that it is related to the internal structure of the gravitational source. This behavior exhibits a strong contrast to the usual treatment of fourth-order theories, since generally one tries to reduce the fourth-order equation to those of Einstein gravity on bigger distance scales, but in Conformal Gravity it explicitly differs from Einstein gravity on large scales.

3.2.1 Problem with Cavendish-type experiments

The Cavendish-type experiments are experiments to measure the gravitational attraction between masses at small distances and to determine Newton's constant
$G = (6,674 \pm 0,001) \times 10^{-11} kg\, m/s^2$ [13]. Therefore, let us have a look at a simplified apparatus used by Cavendish in 1798. The experiment consisted of

two small and two larger masses composed of lead. The smaller masses were fixed on the ends of a bar supported by a thin fiber. The bar was constructed as a torsion pendulum. Hence, it could rotate around a fixed axis. The bigger masses were fixed to the apparatus and could not move. Thus, one was able to measure the gravitational force between the masses by measuring the distance the bar has turned. With knowledge of the masses one is able to determine the value of Newton's constant G. In the following there were several experiments with much higher accuracy, but based on the same principle.

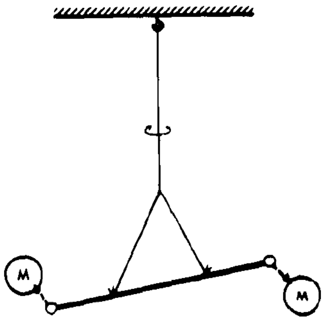

Figure 1: Simplified Cavendish-type experiment to measure the universal law of gravity and to determine the gravitational constant G. This figure is taken from [14].

As mentioned above, the gravitational force of a static, spherically symmetric source is given by eq. (48) in Conformal Gravity. If we assume that the source consists of some bulk matter distributed only in the interior of a volume with a radius R and $r > R$, then just the first two terms of (48) contribute to the force. But even if there is mass at a distance $r' > r$, the contribution to the gravitational force would be negligible, since the coefficients of the non-standard terms are very small. This is clarified and explained in subsection 3.3.2. Hence, if we go to the non-relativistic weak field limit ($T^r{}_r \approx 0$, $T^0{}_0 \approx \rho$, $B(r) \approx 1 + 2\phi(r)$, $\phi(r) \ll 1$), for distances $r \ll 1/\gamma$ the $1/r^2$-term is the dominant contribution to the force. That $T^0{}_0$ is approximately equal to the mass density only holds in the weak binding limit.The weak binding limit means that the binding energy of the system is negligible with respect to the mass energy. Thus, the force is given by the following expression

$$\frac{d\phi_{CG}(r > R)}{dr} = -\frac{1}{16r^2\alpha_g}\int_0^R dr' r'^4 \rho, \tag{49}$$

where ρ is the non-relativistic mass density and "CG" means Conformal Gravity.

In the Newtonian theory the formula for the gravitational force is

$$\frac{d\phi_{Newton}(r > R)}{dr} = -\frac{G}{r^2}\int_0^R 4\pi r'^2 \rho = -\frac{GM_{total}}{r^2}. \tag{50}$$

Thus, we see that in the Newtonian case the gravitational force only depends on the product of Newton's constant and the total mass GM_{total}. But in

Conformal Gravity there is a so-called fourth-moment of the mass density in eq. (49). This means that the gravitational force does not only depend on the total mass M_{total}, but also on the matter distribution.

Therefore, it has been predicted there to be a problem [11], because in principle it should be possible to rule Conformal Gravity out by these Cavendish-type experiments. To see this, consider two homogeneous, spherically symmetric objects with the same total mass, but different mass densities. In Newtonian Gravity the gravitational attraction should be the same, no matter which object is measured. But in Conformal Gravity the result should be different for different mass distributions. For example if the radius of the first object is two times bigger than the radius of the second one, then the gravitational force of the first one will be four times bigger than the second one.

3.2.2 Circumvention of the Cavendish-type Problem

In this subsection we present the solution to the problem raised in 3.2.1 developed by Mannheim and Kazanas [1, 11].

Experiments in the exterior region of some bulk matter yield that in the weak binding limit the coefficient of the gravitational force grows with the number of sources. Hence, the gravitational force of a macroscopic object is an extensive function of its microscopic constituents. Thus, it is sufficient to choose the mass density as the gravitational source. Hence, in the weak binding limit it is sufficient to just sum over the potentials of the individual nuclei of the bulk matter

$$\phi = \sum_{i=1}^{N} \frac{1}{\left|\vec{r} - \vec{r}_i^{\,\prime}\right|} \tag{51}$$

in Newtonian Gravity. To achieve this we can choose

$$g\left(r < R\right) = \sum_{i=1}^{N} mG \frac{\delta\left(r - r_i\right)}{r^2} \tag{52}$$

in eq. (45). This choice of a point-like source is sufficient to yield a potential

$$\phi = -\frac{NmG}{r}, \tag{53}$$

but, in fact, it is not mandatory to choose a point-like potential to yield a $\frac{1}{r}$-term.

Let us make an important remark here. Although we always talk about an interior solution for the regime $r < R$, this is not really an interior solution in the sense of microscopic sources. It is just the interior of some bulk material in a volume with the radius R. The gravitational potential of the microscopic sources (protons, neutrons or atoms) itself is unknown and does not have to be point-like. Everything we know is that the gravitational potential possesses a $1/r$-behavior and that it grows linearly with the number density in the weak binding limit. Therefore, every source, which yields these properties, is sufficient to be used as the gravitational source.

Although the Poisson equation also holds in the interior of the fundamental microscopic source, there is no experimental data, since gravity is not the dominating force in this regime. Therefore, we are just insensitive to the interior of the microscopic sources in the weak gravity and weak binding limit. Of course, in the interior of nuclei the explicit structure of the gravitational sources would be important and one has to take into account the two-body strong coupling nuclear forces between the protons. Here the weak binding limit would not be an accurate approximation, because the fourth-moment integral would not be proportional to the number density anymore. The same holds for compact macroscopic objects like neutron stars. For these sources it is not sufficient to assume the weak binding approximation and it is necessary to apply the complete equations (47) and (48). But actually also in standard gravity the strong binding limit is not tractable, since there is no reliable data of this regime yet.

Now with this argumentation, in Conformal Gravity we are free to choose a microscopic source which is extended in space and has the following form

$$h\left(r<R\right)=-\tilde{\gamma}\sum_{i=1}^{N}\frac{\delta\left(r-r_i\right)}{r^2}-\frac{3\tilde{\beta}}{2}\sum_{i=1}^{N}\left[\nabla^2-\frac{r^2}{12}\nabla^4\right]\left[\frac{\delta\left(r-r_i\right)}{r^2}\right]. \qquad (54)$$

This source is positive definite and by explicit calculation one recognizes that the first term of eq. (54) only couples to the second-moment integral of eq. (47) or (48), whereas the second term just couples to the fourth-moment integral. Therefore, the two coupling constants $\tilde{\gamma}$ and $\tilde{\beta}$ are indeed independent parameters. By explicit calculation we find $\beta\left(2-3\beta\gamma\right)=\tilde{\beta}$ and $\gamma=\tilde{\gamma}$. By inserting this potential in eq. (47) we yield

$$\phi\left(r>R\right)=-\frac{N\tilde{\beta}}{r}+\frac{N\tilde{\gamma}r}{2}. \qquad (55)$$

The meaning of this special source is that one chooses a specific interior distribution of the microscopic energy distribution to reproduce the Newtonian $1/r$-term and a term which is linear in r. Therefore, Mannheim was able to circumvent the problem regarding the Cavendish-type experiments. But let us mention that actually it is not clear if this gravitational source is a physical source that represents the interior of protons. It just shows that, in principle, you can find sources that are adequate to circumvent the Cavendish-type problem.

A remarkable fact is that within the theory of Conformal Gravity it is possible to make predictions about the microscopic structure of a macroscopic source by macroscopic measurements. This can be seen by assuming that the energy density h is not like in eq. (54), but continuous. In consequence, the gravitational potential is given by

$$\phi\left(r>R\right)=-\frac{NhR^5}{30r}-\frac{NhR^3r}{6}. \qquad (56)$$

Thus, the ratio of the two contributions is proportional to R^2, whereas the two parts of the gravitational potential in eq. (55), where we used (54) as the gravitational source, are independent of each other. Therefore, if Conformal Gravity appears to be the correct theory, one is able to state that the interior of a macroscopic source is not continuous but rather discrete. For the stellar potential (55) the ratio of the parameters is of the order $\left(\tilde{\beta}/\tilde{\gamma}\right)^{1/2} \sim 10^{23}$ cm, which can be seen in 3.3 in eq. (90). So in the continuous case $\left(\tilde{\beta}/\tilde{\gamma}\right)^{1/2}$ would be of the order of the radius of a star, which is much smaller than 10^{23} cm.

3.3 Galaxy Rotation Curves

As mentioned above, the additional term in the gravitational potential (47), which grows linearly with the distance, may be the reason for the fact that the rotational velocities of stars in galaxies do not undergo the so-called Keplerian fall-off. To get an impression of the problem with the velocities of objects rotating around a galaxy, let us make an estimate in the Newtonian limit of General Relativity. A simplified calculation for a test particle in a static, spherically symmetric potential of a point mass M, where we equate the gravitational force and the centripetal force yields

$$v_{rot} = \sqrt{\frac{GM}{r}}. \tag{57}$$

Here v_{rot} is the rotational velocity of the particle. Hence, we notice that the rotational velocity decreases like $r^{-1/2}$ with growing distance. This expected behavior can be seen in fig. 2. The figure shows the measured data points for the galaxy NGC 3198 and the theoretical curves for the Newtonian (dashed) and linear (dotted) potential. The sum of these contributions is depicted by the solid line. It is quite obvious that there is a large discrepancy between the measured data and the expected Keplerian fall-off for large distances. The velocities in the outer region seem to be systematically larger than the Keplerian expectation, thus the curve just flattens on larger distances.

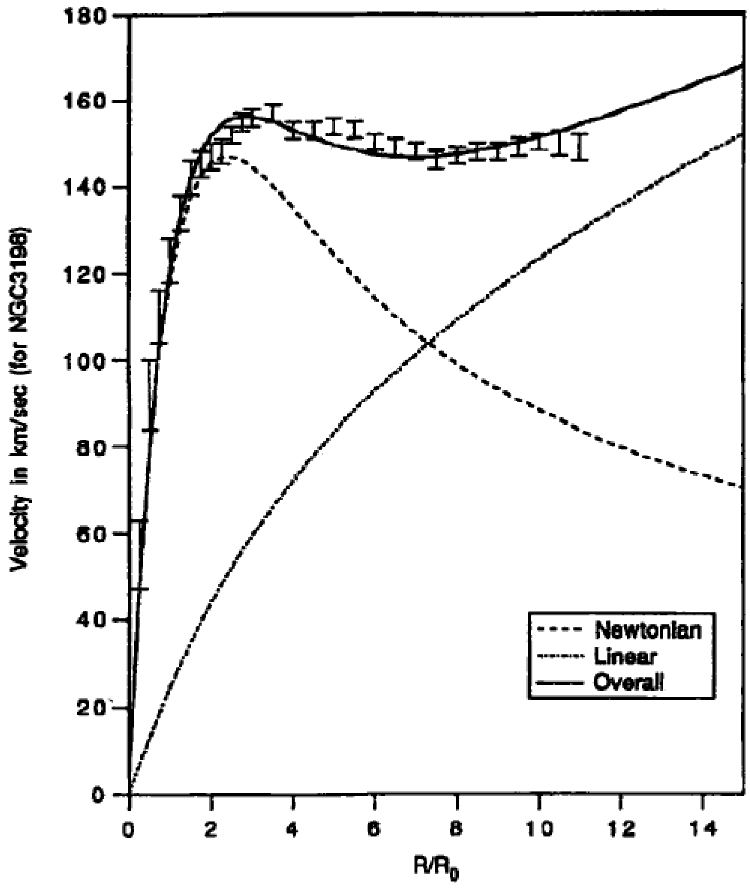

Figure 2: In this figure one can see how the rotational velocity of objects (stars, gas) changes with the distance for the galaxy NGC 3198. The velocity is given as a function of the distance R from the galactic center divided by a reference distance R_0. This is the distance at which the surface brightness of the galaxy decreased by the factor e^{-1}. It is called the exponential scale length and for this galaxy it is 2.72 kpc. The calculated rotational velocity curves are compared with Begeman's data [15, 16]. The dotted curve represents the linear potential term in Conformal Gravity, the dashed one is the Newtonian term and the solid line is the the sum of both terms. These curves are only related to luminous matter and no Dark Matter is assumed. This figure is taken from [6]

Hence, in the standard theory one cures this discrepancy by introducing some new kind of matter, which is not luminous, because it does not interact electromagnetically. It is called Dark Matter and consists of, up to now, hypothetical particles. There is a huge range of candidates which possess the appropriate properties for Dark Matter. But there are no such particles in the Standard Model of particle physics. Therefore, the cosmological data on Dark Matter require that the Standard Model has to be extended, when one assumes that General Relativity is the correct theory of gravity. Examples for Dark Matter particles are sterile neutrinos, axions and neutralinos. In fact, there is a huge zoo of proposed particles, but still there is no detection of any

Dark Matter particle yet. For further information about Dark Matter, see [17], chapter 9.

Let us first investigate how to cure the Keplerian fall-off problem in the standard theory. Therefore, we assume that the gravitational potential is proportional to the stellar surface brightness of a spiral galaxy

$$\Sigma(R) = \Sigma_0\, e^{-R/R_0} \tag{58}$$

$$L = 2\pi \int_0^\infty dR\, R\, \Sigma(R) = 2\pi\Sigma_0 R_0^2, \tag{59}$$

where Σ_0 is the surface brightness at the center of the galaxy, L is the total luminosity and R_0 is the scale length of the galaxy. It describes the distance, at which the surface brightness of the galaxy has decreased by the factor e^{-1}. Now we can define the galactic mass to light ratio Υ

$$\Upsilon L = \frac{M}{L} L = M = N^* M_\odot. \tag{60}$$

Here N^* represents the number of solar mass units of the galaxy. As discussed in [2, 8, 18, 19], by assuming that the surface number density follows the surface brightness density we get

$$\Theta(R) = \Theta_0 e^{-R/R_0} \tag{61}$$

for the surface number density. A surface integral over the surface number density yields

$$N^* = 2\pi\Theta_0 R_0^2. \tag{62}$$

The potential of the sun in General Relativity is given by $V^*(r) = -\frac{\beta^*}{r}$, when one identifies $\beta^* = GM_\odot$. Now we can integrate this potential over a thin exponential disk of stars of solar mass

$$V_\beta(R,z) = -\beta^* \int_0^\infty dR' \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' \frac{R'\rho(R',z')}{\left|\vec{R}-\vec{R}'\right|}, \tag{63}$$

where R, ϕ, z are cylindrical coordinates and $\rho(R,z)$ is the number volume density function. Now we expand the cylindrical coordinates Green's function in the Bessel function expansion

$$\frac{1}{\left|\vec{R}-\vec{R}'\right|} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk J_m(kR)\, J_m(kR')\, e^{-im(\phi-\phi')-k|z-z'|} \tag{64}$$

and rewrite eq. (63) to

$$-2\pi\beta^* \int_0^\infty dR' \int_0^\infty dk \int_{-\infty}^\infty dz' R'\rho(R',z')\, J_0(kR)\, J_0(kR')\, e^{-k|z-z'|}, \tag{65}$$

where we have used that all terms of the sum integrated over the ϕ' coordinate vanish, except for the $m = 0$ term. For an infinitesimal thin disk we can write

$\rho(R',z') = \Theta(R')\,\delta(z')$. Inserting this number volume density in eq. (65) yields

$$V_\beta(R) = -2\pi\beta^* \int_0^\infty dk \int_0^\infty dR' R' \Theta(R')\, J_0(kR)\, J_0(kR') \tag{66}$$

in the $z = 0$ plane. We use the following identities for Bessel functions

$$\int_0^\infty dR R J_0(kR)\, e^{-\alpha R} = \frac{\alpha}{(\alpha^2 + k^2)^{3/2}}, \tag{67}$$

$$\int_0^\infty dk \frac{J_0(kR)}{(\alpha^2 + k^2)^{3/2}} = \frac{R}{2\alpha}\left[I_0\left(\frac{R\alpha}{2}\right)K_1\left(\frac{R\alpha}{2}\right) - I_1\left(\frac{R\alpha}{2}\right)K_0\left(\frac{R\alpha}{2}\right)\right] \tag{68}$$

and by choosing $\alpha = 1/R_0$ we find

$$V_\beta(R) = -\pi\beta^*\Theta_0 R\left[I_0\left(\frac{R}{2R_0}\right)K_1\left(\frac{R}{2R_0}\right) - I_1\left(\frac{R}{2R_0}\right)K_0\left(\frac{R}{2R_0}\right)\right]. \tag{69}$$

Then by using further the Bessel function relations

$$I_0'(x) = I_1(x),\; I_1'(x) = I_0(x) - \frac{I_1(x)}{x},\; K_0'(x) = -K_1(x), \tag{70}$$

$$K_1'(x) = -K_0(x) - \frac{K_1(x)}{x}$$

and differentiating eq. (69) one finds

$$V_\beta'(R) = \frac{N^*\beta^* R}{2R_0^3}\left[I_0\left(\frac{R}{2R_0}\right)K_0\left(\frac{R}{2R_0}\right) - I_1\left(\frac{R}{2R_0}\right)K_1\left(\frac{R}{2R_0}\right)\right]. \tag{71}$$

Now for a test particle moving on a circular orbit around a massive body we can use the Virial theorem

$$v^2 = RV'. \tag{72}$$

Here v represents the rotational velocity of the particle and V is the potential of the central mass. By combining (71) and (72) we get the rotational velocity of the luminous matter

$$v_{lum}^2 = RV_\beta'(R) = \frac{N^*\beta^* R^2}{2R_0^3}\left[I_0\left(\frac{R}{2R_0}\right)K_0\left(\frac{R}{2R_0}\right) - I_1\left(\frac{R}{2R_0}\right)K_1\left(\frac{R}{2R_0}\right)\right]. \tag{73}$$

In the $R \gg R_0$ limit this expression simplifies to

$$v_{lum}^2 \to \frac{N^*\beta^*}{2R}\left(1 + \frac{9R_0^2}{2R^2}\right) \to \frac{N^*\beta^*}{2R}. \tag{74}$$

In fig. 2 we see that there is an initial increase of the rotational velocity from $R = 0$ to $R \approx 3R_0$ for the Newtonian term and after that it steadily falls. This behavior is represented by eq. (73) and this result fits nicely with our simplified calculation, cf. (57), where we assumed that the galaxy is a point mass.

For a more accurate calculation one has to take into account that the galactic disk has a finite thickness and that the HI gas has its own surface brightness. Furthermore, some galaxies have a central bulge. This more detailed calculation is given in [2, 19].

As one can see in fig. 2 the luminous matter fits the data in the inner region quite well, but in the outer region it yields much to low values for rotational velocities. To cure this discrepancy one can assume that there is some additional non-luminous matter distribution. Conventionally, one takes the spherical Dark Matter distribution $\sigma(r)$ to be an isothermal Newtonian sphere in hydrostatic equilibrium

$$\sigma(r) = \frac{\sigma_0}{(r^2 + r_0^2)}, \tag{75}$$

where r_0 denotes the radius of an inner core to prevent the distribution from diverging at $r = 0$. The integration of the Newtonian potential over this Dark Matter distribution results in a rotational velocity of the following form [2]

$$v_{dark} = 4\pi\beta^*\sigma_0 \left[1 - \frac{r_0}{R} \arctan\left(\frac{R}{r_0}\right)\right], \tag{76}$$

$$v_{dark} \overset{R \gg r_0}{\rightarrow} 4\pi\beta^*\sigma_0. \tag{77}$$

This yields an asymptotically flat curve, but nonetheless a simple addition of v_{lum} and v_{dark} does not fit the observed rotation curves automatically. Here Mannheim claims [2] that one has to adjust the two free parameters σ_0 and r_0 for every galaxy. But this seems to be natural, since each galaxy has a different mass, merger history and age. Therefore, the galaxies are in different phases of their evolution. Nevertheless one has to fine-tune the Dark Matter distribution to solve the problem of the galactic rotation curves.

3.3.1 Universality in the Data

In [19] it is pointed out that the data of rotational velocities of galaxies exhibit a universality. In this section we want to present this universal behavior of the rotational velocities and in section 3.3.2 we show how this universality matches to Conformal Gravity.

One can separate the measured galaxies in three categories. The first are galaxies of low luminosity, for which the rotational velocity generally still raises at the last data points. For intermediate to high luminosity galaxies the velocity seems to be flat and those for highest luminosity galaxies to be (slightly) falling. In [20] Mannheim investigated a subset of 11 galaxies for which the range of luminosity differs by a factor of about 1000, see tab. 1 in the Appendix. From fig. 6 in the Appendix the lack of flatness for the first four diagrams of galactic rotation curves is quite obvious, but it is commonly assumed that also these rotational velocities will flatten for larger distances. Further, there is one special feature for the galaxy DDO 154, for which the curve turns over and falls after some distance. The reason for this could be that this galaxy is the most gas dominated of the sample and random gas pressure could make a large contribution to the velocities in the turn over region. However, in [18]

Mannheim states that this fall of the rotation curve is not apparent in the more recent THINGS survey of the galaxy.

Mannheim points out that it could be more instructive to investigate the velocity discrepancy rather than the total velocity, i.e. to look at the excess of the actual velocity over the Newtonian contribution. This discrepancy clearly raises with the distance for the last detected data points, as can be seen in fig. 2.

Furthermore, for the flat rotation curve galaxies there is a regularity in the magnitude of the luminosity. One finds a phenomenological relation between the luminosity and the rotational velocity to the power of four. This relation is called the Tully-Fisher law [21]

$$L \propto v^4. \tag{78}$$

This means that if one knows the rotational velocity of a galaxy one can predict the Luminosity of that galaxy. Thus, by measuring the apparent brightness one can calculate the distance of the galaxy.

Another universality is given by the fact that the brighter spiral galaxies seem to have a common central surface brightness Σ_0^F. Freeman [22] was the first, who recognized this. But this common central surface brightness does not hold for all galaxies of lower luminosity.

In consequence, it is interesting to investigate a universality which is present for all three categories of galaxies. From tab. 1 we see that the quantity $\left(\frac{v^2}{c^2 R}\right)_{tot}$, which represents the centripetal acceleration at the last data point, varies just by a factor of 5. After subtracting the Newtonian contribution the quantity $\left(\frac{v^2}{c^2 R}\right)_{net}$ even only varies by a factor of 4, although the luminosity of the galaxies varies by a factor of 1000. Besides that, we recognize that the rotational velocity slightly increases with increasing mass.

Note that the last measured data point of the galaxy is only fixed by the instrumental limit and not by any dynamics of the galaxy. Therefore, it is completely arbitrary. The range of the distances of the last detected points runs roughly from 8 kpc to 40 kpc. Thus, because $\left(\frac{v^2}{c^2 R}\right)_{net}$ is roughly constant this immediately leads to the suggestion that v^2 grows universally and linearly with R. This means that the rotational velocity is not permitted to depend strongly on any galaxy specific quantity like N^*. This seems to be very speculative, since Mannheim just pics a small number of galaxies out of the presumably $\sim 10^{11}$ galaxies in the observable Universe. However, Conformal gravity is able to account for this hypothetically universal behavior. This is explained in subsection 3.3.2.

3.3.2 Fit with a Linear Potential Term

As we have pointed out before, the introduction of Dark Matter is not the only possibility to cure the problem with the mass discrepancy for galactic rotation curves. Mordehai Milgrom [22] suggested a theory in 1983 called "Modified Newtonian Dynamics" (MOND), which modifies Newton's second law for small accelerations. Therefore, in the regime of small accelerations Newton's second law behaves like $F = m\frac{a^2}{a_0}$ and this yields a constant rotation velocity for the stars in the outer region of a galaxy.

Besides that, Conformal Gravity provides a simple, but still viable solution to the fact that the rotation curves do not follow the expected Keplerian fall-off, but mostly flatten or just slightly increase in the outer region.

Remember eq. (73), which gives the rotational velocity just for luminous matter. We will repeat the same calculation, but now with an additional term linear in R [2, 18, 19]. The potential produced by the sun changes to

$$V^*_{CG}(R) = -\frac{\beta^*}{R} + \frac{\gamma^* R}{2}, \tag{79}$$

Hence, we are able to generalize the formalism to calculate the rotational velocity of stars in Conformal Gravity. Therefore, we set $\left|\vec{r} - \vec{r'}\right| = \frac{\left(\vec{r}-\vec{r'}\right)^2}{\left|\vec{r}-\vec{r'}\right|}$ and by using the second term of eq. (79) we immediately find

$$\begin{aligned} V_\gamma(R,z) &= \frac{\gamma^*}{2}\int_0^\infty dR' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} dz' R' \rho(R',z') \qquad (80) \\ &\quad \times \left(R^2 + R'^2 - 2RR'\cos(\phi') + (z-z')^2\right)^{1/2} \\ &= \pi\gamma^* \int_0^\infty dk \int_0^\infty dR' \int_{-\infty}^{\infty} dz' R' \rho(R',z') \\ &\quad \times \left[\left(R^2 + R'^2 + (z-z')^2\right) J_0(kR) J_0(kR') - 2RR' J_1(kR) J_1(kR')\right] e^{-k|z-z'|} \end{aligned}$$

Again, for an infinitesimally thin disk at $z = 0$ we find

$$\begin{aligned} V_\gamma(R) &= \pi\gamma^* \int_0^\infty dk \int_0^\infty dR' R' \Theta(R') \qquad (81) \\ &\quad \times \left[\left(R^2 + R'^2\right) J_0(kR) J_0(kR') - 2RR' J_1(kR) J_1(kR')\right]. \end{aligned}$$

With eq. (67) and eq. (68) and the relation $\int_0^\infty dR' R'^2 J_1(kR') e^{-\alpha R} = \frac{3\alpha k}{(\alpha^2+k^2)^{5/2}}$ we can write

$$\begin{aligned} V_\gamma(R) &= \pi\gamma^*\Theta_0 R R_0^2 \left[I_0\left(\frac{R}{2R_0}\right) K_1\left(\frac{R}{2R_0}\right) - I_1\left(\frac{R}{2R_0}\right) K_0\left(\frac{R}{2R_0}\right)\right] \qquad (82) \\ &\quad + \frac{\pi\gamma^*\Theta_0 R^2 R_0}{2}\left[I_0\left(\frac{R}{2R_0}\right) K_0\left(\frac{R}{2R_0}\right) - I_1\left(\frac{R}{2R_0}\right) K_1\left(\frac{R}{2R_0}\right)\right] \end{aligned}$$

and the by differentiation with respect to R and using eq. (70) we find

$$RV'_\gamma(R) = \frac{N^*\gamma^* R^2}{2R_0} I_1\left(\frac{R}{2R_0}\right) K_1\left(\frac{R}{2R_0}\right). \tag{83}$$

Combining this with eq. (73) the local contribution of the luminous matter reads

$$\begin{aligned} v_{lum}^2 &= \frac{N^*\beta^* R^2}{2R_0^3}\left[I_0\left(\frac{R}{2R_0}\right) K_0\left(\frac{R}{2R_0}\right) - I_1\left(\frac{R}{2R_0}\right) K_1\left(\frac{R}{2R_0}\right)\right] \qquad (84) \\ &\quad + \frac{N^*\gamma^* R^2}{2R_0} I_1\left(\frac{R}{2R_0}\right) K_1\left(\frac{R}{2R_0}\right) \end{aligned}$$

and in the limit for large R we get

$$v_{lum}^2 \overset{R \gg R_0}{\rightarrow} \frac{N^* \beta^*}{R} + \frac{N^* \gamma^* R}{2}. \tag{85}$$

We recognize that we produced two terms, which contribute to the rotational velocity that depend on the number of stars N^*. Hence, these contributions depend on the galaxy itself.

Now let us think about a major difference between Newtonian Gravity and Conformal Gravity. In Newtonian Gravity it is very familiar to us that just the matter interior of the volume given by a ball with the radius r of the observers position contributes to the gravitational potential. The rest of the Universe does not contribute. But this behavior is only valid for spherically symmetric $\frac{1}{r}$-potentials, thus it does not hold in Conformal Gravity anymore. As we mentioned before in 3.2, by this reason Newtonian Gravity is simply a local theory, whereas Conformal Gravity is a global theory. And, indeed, there are contributions to the rotational velocity from the rest of the Universe, i.e. contributions that are universal and do not depend on the morphology, size, luminosity or mass of the galaxy, just as was pointed out in 3.3.1.

Hence, as the Universe consists of a homogeneous and isotropic background and large scale inhomogeneities like galaxy clusters and superclusters we have two global contributions to the rotational velocity.

For the background part of the Universe we usually assume a Robertson-Walker (RW) geometry

$$ds^2 = -dt^2 + \frac{a^2(\tau)}{\left(1 + \frac{K\rho^2}{4}\right)^2} \left(d\rho^2 + \rho^2 d\Omega^2\right). \tag{86}$$

This is a Robertson-Walker line element in isotropic coordinates. It is homogeneous and isotropic and thus conformal to a flat geometry. Hence, after a appropriate coordinate transformation it is also a solution to the static, spherically symmetric equation (39). The Weyl tensor (10) and $W^{\mu\nu}$ (19) vanish in this geometry. Consequently, also the matter energy-momentum tensor $T_M^{\mu\nu}$ in eq. (20) has to vanish. Therefore, the background does not contribute to the inhomogeneous Bach equation (39). But there is a contribution to the homogeneous equation $\nabla^4 B(r) = 0$, when $W^{\mu\nu}$ vanishes non-trivially. Hence, this forces $T^{\mu\nu}$ to vanish non-trivially, i.e. by a positive contribution of the matter sources and a negative contribution of the gravitational part. This is indeed possible if the 3-curvature K is negative. In chapter 6 we will see that this is feasible in Conformal Gravity. Thus, in the following we want to look at this term in detail.

The cosmological Hubble flow is the motion of astronomical objects due to the expansion of the Universe. It is described in comoving coordinates which are given by eq. (86). But the galactic rotational velocities are measured in a coordinate system which is fixed to the center of the galaxy, i.e. the galaxy is at rest. Therefore, we have to find the transformation between these two coordinate systems. Explicitly it is given by

$$r = \frac{\rho}{\left(1 - \frac{\gamma_0 r}{4}\right)^2}, \quad \tau = \int dt R(t). \tag{87}$$

Therefore, the RW line element changes in the following way

$$\begin{aligned} &\Omega^2(\tau,\rho)\left[-d\tau^2 + \frac{R^2(\tau)}{\left[1 - \frac{\gamma_0^2\rho^2}{16}\right]^2}\left(d\rho^2 + \rho^2 d\Omega^2\right)\right] \\ &= \frac{\left(\frac{1+\frac{\gamma_0\rho}{4}}{1-\frac{\gamma_0\rho}{4}}\right)^2}{R^2(\tau)}\left[-d\tau^2 + \frac{R^2(\tau)}{\left[1 - \frac{\gamma_0^2\rho^2}{16}\right]^2}\left(d\rho^2 + \rho^2 d\Omega^2\right)\right] \\ &= -\left(1 + \gamma_0 r\right) dt^2 + \frac{dr^2}{\left(1 + \gamma_0 r\right)} + r^2 d\Omega^2, \end{aligned} \tag{88}$$

where $\Omega^2(\tau,\rho) = \frac{1}{R^2(\tau)}\left(\frac{1+\frac{\gamma_0\rho}{4}}{1-\frac{\gamma_0\rho}{4}}\right)^2$ is the conformal factor, but $d\Omega^2$ is the angular part of the line element of the spherical coordinates. Thus, after a conformal transformation a RW geometry written in a static coordinate system is equivalent to a static line element with a universal linear term $\gamma_0 r$. Since the spatial 3-curvature of these coordinates $K = -\frac{\gamma_0^2}{4}$ is negative, γ_0 has to be real. This is necessary for the $\frac{\gamma_0 R}{2}$ term to be able to serve as a potential. Hence, in analogy to the calculation for the linear term of eq. (79) and because the rotational velocities are non-relativistic we can simply add this contribution to eq. (85) and yield

$$v^2 \overset{R \gg R_0}{\rightarrow} v_{lum}^2 + \frac{\gamma_0 R}{2} = \frac{N^*\beta^*}{R} + \frac{N^*\gamma^* R}{2} + \frac{\gamma_0 R}{2}. \tag{89}$$

With this formula the data of 11 galaxies were fitted [20]. The first two terms are galaxy dependent, because they contain N^*. But the third term is universal and linear. This is the term, as Mannheim argues, which represents the universality we explained in 3.3.1. It is totally luminosity independent and therefore on a very different footing than the other two terms. The fit yields for the parameters [1]

$$\beta^* = 1.48 \times 10^5 \text{cm}, \quad \gamma^* = 5.42 \times 10^{-41} cm^{-1}, \quad \gamma_0 = 3.06 \times 10^{-30} cm^{-1}. \tag{90}$$

Here one also gets a quantitative answer to the question whether the Solar System is influenced by this linear modifications to the rotational velocities. We recognize that the parameter γ^* is small and therefore $N^*\gamma^* \sim 10^{-30}\ cm^{-1}$ is still a very small quantity. Only on the length scales of galaxies ($R \sim 10$ kpc) the linear contributions become of the order of the Newtonian contribution. Thus, on Solar System distances ($R \ll 10$ kpc) the linear term is negligible.

Additionally the value for γ_0 indicates that it is of cosmological magnitude, since it is of the order of magnitude of the inverse Hubble radius. So if the theory turns out to be correct, then it also provides a possibility to measure the spatial 3-curvature of the Universe. On top of that, in Conformal Gravity there is only one free parameter per galaxy, namely the optical disk mass to light ratio Υ or equivalently the total amount of stars or gas in the galaxy, N^*, in solar mass units. Hence, no fine-tuning of parameters is necessary in Conformal Gravity.

Now let us turn to the mentioned universality in the data. The local contribution $\frac{N^*\gamma^*}{2}$ varies enormously with the luminosity, i.e with N^*. But still the global contribution dominates $\left(\frac{v^2}{c^2R}\right)_{net}$ and it only slightly depends on the galactic mass or luminosity. Except for the largest galaxies, for which N^* is of the order of the critical mass $N_{crit} = 5,65 \times 10^{10}$. Here the local contribution is of the order of the global one, see Appendix tab. 1.

With this theory we are also able to explain, why the low luminosity galaxies do not flatten. For these galaxies N^* is much lower than N_{crit} and also Σ_0 is so low that the global γ_0-term immediately wins against the local contributions and the rotation curves raise from the beginning. For the largest galaxies where N^* is greater than N_{crit} the local contribution is temporarily larger than the global contribution, but after that it slightly falls and in the end the linear contribution wins, see tab. 1, NGC 2841. Note that the data of this galaxy goes to much larger distances.

As mentioned above, there is still another contribution from the rest of the Universe. In [19, 24] Mannheim states that inhomogeneities contribute to the r^2-term in eq. (48). These inhomogeneities are typically clusters and superclusters of galaxies. Their distance scale ranges from 1 Mpc up to about 100 Mpc. Mannheim argues that the integral of the r^2-term is constant, if we assume that the integral is evaluated between fixed endpoints. And furthermore, Mannheim assumes that the lower integration limit begins at some $r_{cluster}$ that is larger than the radius of the galaxy and independent of it. Therefore, on scales $r < r_{cluster}$ and by introducing the constant $\kappa = \frac{1}{3}\int_{r_{clust}}^{\infty} dr' r' h(r')$ one finds the following approximate formula in the weak gravity and asymptotic limit

$$v_{TOT}^2 \overset{R \gg R_0}{\to} \frac{N^*\beta^*}{R} + \frac{N^*\gamma^* R}{2} + \frac{\gamma_0 R}{2} - \kappa R^2. \tag{91}$$

It is important to mention here that such a form can be achieved by starting with a metric term $B(r) = -\kappa r^2$. This term has the de Sitter-like form, but it can't be associated with a de Sitter geometry, because the inhomogeneities are not distributed maximally 4-symmetrically. Anyway, a test particle would be influenced by that r^2-term and therefore the result is the same as if it had been embedded in a de Sitter geometry. But this whole argumentation is at least questionable, because there is no spherical symmetry on these distance scales.

At the time when eq. (89) was fitted to the set of 11 galaxies the data did not go to so far distances. Therefore, the fits of the galactic rotation curves without the r^2-term were perfectly in line with the data. But in the following, when there was data available to further distances, it seems that the rotation curves were not raising anymore, but staying more or less flat, see fig. 7. Hence, only with the additional r^2-term one was able to reproduce this feature of the rotation curves. With a new data set of 111 galaxies 21 galaxies were found, in which the expectation of eq. (89) totally overruns the data points for large distances [10, 19]. Consequently, these 21 galaxies were fitted with the repulsive term $-\kappa R^2$. By doing so the value of the parameter $\kappa = 9,54 \times 10^{-54}\ cm^{-2} \approx (100\ \text{Mpc})^{-2}$ was fixed and flat galactic rotation curves for larger distances could be reproduced. The fitting of the other 90 galaxies was unaffected by this new term, see [19, 24]. Because the quadratic term eventually will dominate, eq. (91) will become negative at some distance. But v_{TOT}^2 cannot be negative. Therefore, beyond the distance $R \sim (N^*\gamma^* - \gamma_0)/2\kappa \sim 100$ kpc for $N^* = \gamma_0/\gamma^* = 5,56 \times 10^{10}$ there cannot be any bound circular orbits. By this fact, a galaxy terminates naturally at this distance scale.

Despite this effective fitting mechanism for galactic rotation curves, still there are at least three arguments for the existence of Dark Matter. These are the cosmic growth of structures (see chapter 7 in [25]), Dark Matter in galaxy clusters measured by strong lensing and the Bullet clusters [26]. It seems that these indications for Dark Matter are hard to explain for Conformal Gravity. Nevertheless, Conformal Gravity does not completely exclude the existence of Dark Matter. It is still possible to have some kind of Dark Matter in Conformal Gravity, which could lead to a solution to these issues.

In this chapter we applied the macroscopic equations of the theory of Conformal Gravity to solve the Dark Matter problem. Now in the next chapter we want to turn to the microscopic side of the theory to investigate the zero point energy problem and the quantization of gravity.

4 Quantization

This chapter deals with the quantization in the theory of Conformal Gravity. We will mainly follow the discussion in [1, 3, 27, 28].

The reason to seek a viable theory of quantum gravity is that there are situations in the Universe where gravity and quantum effects are equally important. For example such a situation was the beginning of the Universe. We actually think, in the standard Big Bang models, that the Universe began in a space-time singularity. This means there was a boundary of space-time. The energy density of the Universe was very large and the matter was so strongly compressed that quantum-mechanical effects should play a significant role. Another example are Black Holes, which also represent a space-time singularity. The crucial point is that the standard theory of gravity, namely General Relativity, does not include the laws of quantum mechanics and is treated purely classical. It is commonly assumed that the Einstein equations break down when quantum effects become important. So to describe the above mentioned situations, one has to come up with a theory of quantum gravity.

There are several approaches to quantum gravity like String Theory or Loop Quantum Gravity, but there is no consensus of which of these theories, if any, is the correct theory for describing quantum gravitational effects, yet.

Thus, let us see how Conformal Gravity treats this problem. The action of Conformal Gravity is scale-free on the fundamental level. This means that initially there are no length or mass scales in the theory. How mass is produced in the theory of Conformal Gravity is the topic of chapter 5 and in this chapter we will focus on how the zero-point problem and the problem with the cosmological constant can be cured in Conformal Gravity. So since there is no length scale, we can state that on this fundamental level the only allowed geometry for macroscopic Conformal Gravity is the flat Minkowski space. Mannheim argues this in [1] as follows. If there are no length scales, there is nothing to define curvature, because one always has to compare curvature to a certain length. So if we take $W^{\mu\nu}$ to be a classical function, eq. (37) is an exterior vacuum solution to $W^{\mu\nu} = 0$, but there is no reason for taking the dimensionful parameters β, γ or k to be non-zero. Therefore, we will see that there has to be some symmetry breaking in order to yield a solution which has less symmetry than the differential equation itself. Now, if all of that symmetry breaking is produced quantum-mechanically, then in the classical case space-time has to be flat. Thus, mass in the Schwarzschild solution that is produced quantum-mechanically generates also curvature by a quantum-mechanical process. Explicitly, the commutation relations produce the length scales in the theory. This can be seen by the fact that a commutation relation of the form

$$\left[\phi\left(\vec{x},t\right), \pi\left(\vec{x'},t\right)\right] = i\hbar\delta^3\left(\vec{x}-\vec{x'}\right) \tag{92}$$

produces a delta function, which is a non-linear relation and is thus not scale-invariant. $\phi\left(\vec{x},t\right)$ denotes some quantum field and $\pi\left(\vec{x},t\right)$ is its momentum conjugate.

When one quantizes General Relativity usually one expands the metric in a power series of the gravitational coupling constant G, but in Conformal Gravity one has to expand the metric in a power series of Planck's constant $\hbar$, because by the above argumentation quantum mechanics is an indicator for the deviation from flat space-time. Nevertheless, one cannot use the standard canonical quantization mechanism. The reason for this is that the gravitational field is the space-time metric $g^{\mu\nu}$ itself. So for the matter fields one can define an energy-momentum tensor $T_M^{\mu\nu}$ by varying the matter action I_M with respect to the metric. Besides that, one yields the field equations for a matter field ϕ, when one varies the matter action with respect to the associated field. Furthermore, the field equations are constrained by the stationarity condition of the Hamilton's principle. This says that one yields the field equation when the deviation of the action with respect to the field is zero, i.e. $\frac{\delta I_M}{\delta\phi} = 0$. Then the field equation is constrained, but the matter energy-momentum tensor is not. Now for gravity the situation is different. For the field equation of gravity we also have to vary with respect to the metric, since the metric is the gravitational field. But as there is the constraint for the field equation from $\frac{\delta I_{GRAV}}{\delta g_{\mu\nu}} = 0$, then also the energy-momentum tensor is constrained, i.e. $\frac{\delta I_{GRAV}}{\delta g_{\mu\nu}} \equiv T^{\mu\nu}_{GRAV} = 0$. So in the absence of a coupling between matter and gravity, gravity is still coupled to itself. This does not result in a problem as long as we treat gravity purely classical. But when one wants to quantize gravity one gets into trouble. To clarify this, consider the Einstein equations (7). To be an operator identity both sides have to be either classical or quantum mechanical. On the one hand gravity is not well defined quantum mechanically, because radiative corrections are possibly not renormalizable, but on the other hand the matter side consists of quantum fields. Therefore, one assumes a semi-classical form of the Einstein equations

$$\frac{1}{8\pi G} G^{\mu\nu}_{CL} = - \langle Q | \, T^{\mu\nu}_{M} \, | Q \rangle \,, \tag{93}$$

where $G^{\mu\nu}_{CL} = \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^{\alpha}{}_{\alpha} \right)_{CL}$ denotes the classical Einstein tensor and $\langle Q | \, T^{\mu\nu}_{M} \, | Q \rangle$ is a c-number matrix element in the quantum state $|Q\rangle$. So basically one replaces the matter energy-momentum tensor $T^{\mu\nu}_{M}$ by its expectation value. But still the expectation value contains products of fields at the same point. Thus, its matrix elements are not finite and one has to subtract the infinities by hand. In the standard theory it is not clear why this is correct, because actually gravity couples to the full energy-momentum tensor and not just to energy differences. So conventionally, one uses the following form of the Einstein equations

$$\frac{1}{8\pi G} G^{\mu\nu}_{CL} = - \left(\langle Q | \, T^{\mu\nu}_{M} \, | Q \rangle - \langle \Omega | \, T^{\mu\nu}_{M} \, | \Omega \rangle_{DIV} \right) = - \langle Q | \, T^{\mu\nu}_{M} \, | Q \rangle_{FIN} \,. \tag{94}$$

$|\Omega\rangle$ denotes the matter field vacuum state. So for example to describe a black body like the cosmic microwave background one uses an ensemble average over an appropriate set of positive-energy Fock space states. These are eigenstates to the Hamilton operator

$$\sum \omega \left(a^{\dagger} \left(\vec{k} \right) a \left(\vec{k} \right) + \frac{1}{2} \right) \tag{95}$$

and one simply neglects the $\sum \omega/2$ zero-point contributions.

So now that we have an idea of the problems with quantum gravity in General Relativity, we continue with Conformal Gravity and its solution to these problems. As we have explained above, it is not possible to quantize gravity alone. Therefore, if we impose stationarity not just on $T^{\mu\nu}_{GRAV}$ alone, but on the energy-momentum tensor of the whole Universe

$$T^{\mu\nu}_{UNIV} = T^{\mu\nu}_{GRAV} + T^{\mu\nu}_{M} = 0, \tag{96}$$

we can cancel the zero-point energy contributions consistently. This is because Conformal Gravity is renormalizable, since the gravitational coupling constant α_g in (9) is dimensionless. In contrast, for General Relativity it is unclear if it could be a finite theory. When gravity is quantized it is now possible to have a non-zero $T^{\mu\nu}_{GRAV}$, because it is coupled to the matter energy-momentum tensor $T^{\mu\nu}_{M}$. But to achieve this cancellation of the vacuum contributions, the commutation relations must have a form that depends on the commutation relations of the matter fields. Therefore, gravity is quantized through the coupling to matter and additionally, the gravitational commutation relations are also fixed by the coupling to matter.

In Conformal Gravity we can use a boson-fermion cancellation. In [1] Mannheim assumes assumes that this cancellation takes place due to the fact that bosons and fermions contribute with different signs to the vacuum energy. Thus, since gravitational quanta are bosonic one can achieve a cancellation between the gravitational and the matter field parts. For such a cancellation one needs three things: a symmetry, fermions in the matter field sector and a theory which is renormalizable. All of these conditions are fulfilled in Conformal Gravity and hence, in the following it will be explained how the cancellation takes place.

The following expansion of (96) follows the argumentation in [1]. Already in flat space-time there are zero-point and cosmological constant contributions, and since gravity is absent in flat space-time it cannot be responsible for it. Thus, the way to solve this problem is to put gravity on an equal footing with the matter field sector. This is done by the fact that the Bach equations are purely quantum mechanical equations what means that gravity and the matter field sector are treated only microscopically. So as was pointed out above, we expand the metric in a power series of $\hbar$. The lowest order quantum-mechanical contribution to $T_M^{\mu\nu}$ is a zero-point contribution of order $\hbar$. Thus, to cancel this contribution also the lowest order of the gravitational part has to be $\hbar$. Since the zero-point contribution is due to products of fields at the same point, the order $\hbar$ contribution to the gravitational vacuum part has to consist of a product of two gravitational fields. Hence, it is given by varying the part of the gravitational action in eq. (9), which is second-order in the metric perturbation $h^{\mu\nu}$ with $|h^{\mu\nu}| \ll 1$,

$$I_W\left(2\right) = -\frac{\alpha_g}{2}\int d^4x \,\Box\, h_{\mu\nu}\,\Box\, h^{\mu\nu}, \tag{97}$$

i.e. by

$$-4\alpha_g W^{\mu\nu}\left(2\right) \equiv T_{GRAV}^{\mu\nu}. \tag{98}$$

One can evaluate $T_M^{\mu\nu}$ in flat background, because it is already of order $\hbar$ and a non-flat metric $g^{\mu\nu}$ would produce a matter energy-momentum tensor of higher order in $\hbar$. So an energy-momentum tensor in flat background will generate curvature of order $\hbar$. This means that the $\hbar^2$ term in $T_M^{\mu\nu}$ is curvature dependent, because it contains the non-flat metric. Additionally, the zero-point cancellation in eq. (96) will occur to all orders in $\hbar$. Hence, one can decompose the gravitational and matter contribution into finite and divergent parts, which lead to

$$(T_{GRAV}^{\mu\nu})_{DIV} + (T_M^{\mu\nu})_{DIV} \quad = \quad 0, \tag{99}$$

and

$$(T_{GRAV}^{\mu\nu})_{FIN} + (T_M^{\mu\nu})_{FIN} = 0. \tag{100}$$

So there is no need for a renormalization of the individual terms, since the terms regulate each other and thus their sum is finite. In consequence, we are allowed to work with eq. (100) alone, because all infinities have been removed. This has a decisive consequence for the gravitational fluctuations $h^{\mu\nu}$ in Conformal Gravity. Since $W^{\mu\nu}\left(2\right)$ is of order $\hbar$ and contains products of two gravitational fields, the gravitational field has to be of order $\hbar^{1/2}$. This means that the Bach equations to lowest order read

$$-4\alpha_g W^{\mu\nu}(1) = 0, \tag{101}$$

since $W^{\mu\nu}(1)$ is of order $\hbar^{1/2}$, but the lowest order of $T_M^{\mu\nu}$ is of order $\hbar$. Hence, this provides Conformal Gravity with a wave equation, which is strictly homogeneous. This is totally different to standard gravity, where one expands in a power-series of the gravitational coupling constant G. There the first-order gravitational fluctuation is produced by a first-order matter source. Eq. (101) does not permit $h^{\mu\nu}$ to be non-zero, but because of the second-order equation

$$-4\alpha_g W^{\mu\nu}(2) + T_M^{\mu\nu} = 0 \tag{102}$$

the fluctuation has to be non-trivial.

Lastly, let us note that the operator identity $\langle\Omega| T_{GRAV}^{\mu\nu} |\Omega\rangle + \langle\Omega| T_M^{\mu\nu} |\Omega\rangle = 0$ does not only hold for the matter vacuum state $|\Omega\rangle$, but also after symmetry breaking in some spontaneously broken vacuum $|\Omega_M\rangle$

$$\langle\Omega_M| T_{GRAV}^{\mu\nu} |\Omega_M\rangle + \langle\Omega_M| T_M^{\mu\nu} |\Omega_M\rangle = 0. \tag{103}$$

This means that in the case of mass generation and an induced cosmological constant, all the various zero-point contributions have to readjust in such a way that the cancellation still occurs after the spontaneous symmetry breaking. Thus, Mannheim states in [1, 2, 3] that one has to treat the cosmological constant in conjunction with the zero-point energies. Hence, this provides a possibility to solve the cosmological constant problem, because there is a constraint for the cosmological constant by this requirement as we we will see in chapter 5 and 6.

The explicit cancellation of these zero-point contributions and the cosmological constant is done in [1, 3].

5 Mass Generation

This chapter deals with the generation of mass in Conformal Gravity. Initially there is no bare mass term on the level of the matter action I_M. The reason for this is that a mass term would introduce a mass scale to the theory, but a conformal invariant theory is not permitted to have a mass scale on the level of the action. This can also be seen by the fact that an energy-momentum tensor, which results from a conformal invariant action, has to be traceless and such a mass term would destroy the traceless property of the energy-momentum tensor. Mathematically this means that the action would not be invariant under a conformal transformation (11). But since we know from experiment that Standard Model particles indeed have a mass, we have to find a mechanism which is responsible for the particle physics matter to acquire some mass. That one is not allowed to start with a mass term on the level of the action is not a problem, since even in the Standard Model of particle physics masses are produced via the Higgs mechanism. The discussion in this chapter follows [1] and chapter 3 in [2].

5.1 The Energy-Momentum Tensor of Test Particles and Perfect Fluids

At the time when General Relativity was introduced there was no Quantum Field Theory established. The matter fields were assumed to consist of hard "billiard balls" like purely mechanical kinematic particles, which are provided with energy and momentum. However, today it is clear from the Standard Model of particle physics ($SU(3) \times SU(2) \times U(1)$ theory of strong, weak and electromagnetic interactions) that matter as the source of gravity has to be treated as fields and no longer as hard "billiard balls". But the kinematical energy-momentum tensor is still used as the source of gravity, because it leads to geodesic motion, which is well confirmed by experiment. This is a shortcoming we will overcome below in this chapter and we will justify, why it is sufficient to work with the energy-momentum tensor of test particles or perfect fluids.

So let us start with the energy-momentum tensor for test particles which is given by

$$\frac{2}{\sqrt{-g}}\frac{\delta I_T}{\delta g_{\mu\nu}} = T^{\mu\nu} = \frac{m}{\sqrt{-g}}\int d\tau \delta^4\left(x - y\left(\tau\right)\right)\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau}, \tag{104}$$

Now by using the covariant energy-momentum conservation $T^{\mu\nu}{}_{;\nu} = 0$ one recovers the geodesic equation (2).

For the energy-momentum tensor of the form of a perfect fluid

$$T^{\mu\nu}_{Fluid} = \left[\left(\rho + p\right)u^\mu u^\nu + g^{\mu\nu}p\right], \tag{105}$$

where ρ is the relativistic mass density and p is the pressure. u^μ is the four velocity, which obeys $u^\mu u_\mu = -1$ with our sign convention for the metric.

The covariant conservation for a perfect fluid can be written in the following form

$$[(\rho + p)\, u^\mu u^\nu + g^{\mu\nu} p]_{;\nu} = [(\rho + p)\, u^\nu]_{;\nu}\, u^\mu + (\rho + p)\, u^\mu_{;\nu} u^\nu + p^{;\mu} = 0. \tag{106}$$

The semicolon represents a covariant derivation, whereas the comma denotes a partial derivative. With $u^\mu u_{\mu;\nu} = 0$ and by multiplying eq. (106) with u_μ we can conclude

$$-\,[(\rho + p)\, u^\nu]_{;\nu} + p^{;\mu} u_\mu = -\,(\rho + p)\, u^\mu_{;\nu} u^\nu u_\mu = 0. \tag{107}$$

So if we insert eq. (107) in eq. (106) we find

$$(\rho + p)\, u^\mu_{;\nu} u^\nu + p_{;\nu}\, [g^{\mu\nu} + u^\mu u^\nu] = 0. \tag{108}$$

Now by dividing eq. (108) by $(\rho + p)$ and rewriting the covariant derivative we yield

$$\frac{D^2 x^\mu}{D\tau^2} = -\,[g^{\mu\nu} + u^\mu u^\nu]\, \frac{p_{;\nu}}{(\rho + p)}. \tag{109}$$

Here we see that we reproduce the geodesic equation when the right-hand side of (109) is negligible. E.g. this situation is achieved for non-interacting, pressureless dust.

It is obvious now why one naturally assumes that the sources of gravity should be described by test particles or a perfect fluid, because both lead to geodesic motion.

Nevertheless, it is still possible to add a term $T^{\mu\nu}_{extra}$ to the energy-momentum tensor, which is covariantly conserved independently. Thus, it would not change the equation of motion (109). Therefore, one can add a term $T^{\mu\nu}_{extra} = G^{\mu\nu}$ to the total energy-momentum tensor, because the Einstein tensor is covariantly conserved on its own and non-existent in flat space-time. Besides that, the same holds for the metric tensor, which is covariantly conserved, $\nabla_\mu g^{\mu\nu} = 0$, too. Hence, we can include a term $g^{\mu\nu}\Lambda$, where Λ is a constant, leaving eq. (109) unaffected. Such a term would lead to an unobservable shift in the zero of energy in flat space-time.

In conclusion we can say that the energy-momentum tensor of a perfect fluid is sufficient to yield geodesic motion, but it is not at all necessary. We will show this in more detail in the next section 5.2 and use this in chapter 6 to cure the cosmological constant problem.

5.2 Dynamical Mass Generation

Now, let us investigate why it is sufficient to work with test particles and perfect fluids, although we know that the matter part consists of fields rather than particles.

Therefore, we analyze the structure of the energy-momentum tensor for elementary particles whose masses are produced dynamically, i.e. they acquire mass by spontaneous symmetry breaking.

We consider a massless spin one-half matter field fermion $\psi(x)$, which gets its mass via a real spin-zero Higgs scalar boson field $S(x)$. So we start with an action which does not contain any kinematic mass scale. Furthermore, to maintain the tracelessness of the energy-momentum tensor, the scalar field is coupled conformally to the curvature. Hence, we can write down a curved-space matter action

$$I_M = -\int d^4x\sqrt{-g}\left[\frac{S^{,\mu}S_{,\mu}}{2} - \frac{S^2 R^{\mu}{}_{\mu}}{12} + \lambda S^4 + i\bar{\psi}\gamma^{\mu}(x)\left[\partial_{\mu} + \Gamma_{\mu}(x)\right]\psi - hS\bar{\psi}\psi\right], \quad (110)$$

where h and λ are dimensionless coupling constants and $\Gamma_{\mu}(x)$ is the fermion spin connection. This action I_M is the most general action, which is not only local coordinate invariant, but also invariant under local Weyl transformations of the form $S(x) \to e^{-\alpha(x)}S(x)$, $\psi(x) \to e^{-3\alpha(x)/2}\psi(x)$ and $g_{\mu\nu}(x) \to e^{2\alpha(x)}g_{\mu\nu}(x)$. A crucial difference between test particles and fields is that for fields the geodesic motion is not achievable just by the covariant energy conservation. Rather we have to use some equation of motion for the field itself. This means that there is additional dynamical information contained in the equation of motion of the matter fields. Therefore, by varying eq. (110) with respect to $S(x)$ and $\psi(x)$ we find the field equations

$$S^{,\mu}{}_{;\mu} + \frac{1}{6}SR^{\mu}{}_{\mu} - 4\lambda S^3 + h\bar{\psi}\psi = 0, \quad (111)$$

$$i\gamma^{\mu}(x)\left[\partial_{\mu} + \Gamma_{\mu}(x)\right]\psi - hS\psi = 0. \quad (112)$$

The energy-momentum tensor can be obtained by variation of eq. (110) with respect to the metric

$$\begin{aligned} T^M_{\mu\nu} &= i\bar{\psi}\gamma_{\mu}(x)\left[\partial_{\nu} + \Gamma_{\nu}(x)\right]\psi + \frac{2S_{,\mu}S_{,\nu}}{3} - \frac{g_{\mu\nu}S^{,\alpha}S_{,\alpha}}{6} - \frac{SS_{,\mu;\nu}}{3} + \frac{g_{\mu\nu}SS^{,\alpha}{}_{;\alpha}}{3} \quad (113)\\ &\quad -\frac{1}{6}S^2\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\alpha}{}_{\alpha}\right] - g_{\mu\nu}\left[\lambda S^4 + i\bar{\psi}\gamma^{\alpha}(x)\left[\partial_{\alpha} + \Gamma_{\alpha}(x)\right]\psi - hS\bar{\psi}\psi\right]. \end{aligned}$$

Now, we multiply eq. (112) with $\bar{\psi}$ and eq. (111) with $g_{\alpha\beta}S$ and isolate the $-4g_{\alpha\beta}\lambda S^4$ term to bring the energy-momentum tensor in the following form

$$\begin{aligned} T^M_{\mu\nu} &= i\bar{\psi}\gamma_{\mu}(x)\left[\partial_{\nu} + \Gamma_{\nu}(x)\right]\psi + \frac{2S_{,\mu}S_{,\nu}}{3} - \frac{g_{\mu\nu}S^{,\alpha}S_{,\alpha}}{6} - \frac{SS_{,\mu,\nu}}{3} \quad (114)\\ &\quad +\frac{g_{\mu\nu}SS^{,\alpha}{}_{,\alpha}}{12} - \frac{1}{6}S^2\left[R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^{\alpha}{}_{\alpha}\right] - \frac{g_{\mu\nu}hS\bar{\psi}\psi}{4}. \end{aligned}$$

One can explicitly calculate the tracelessness $\left(T^{\mu}_{M\,\mu} = 0\right)$ and the covariant conservation $\left(T^{\mu\nu}_{M\ ;\nu} = 0\right)$ of this energy-momentum tensor by employing the field equations of motion again. Therefore, we see that the structure of this energy momentum tensor is extremely different from that of a Newtonian test particle.

Now that we have found the correct energy-momentum tensor, we introduce a mass for the particles by spontaneous symmetry breaking. This happens quite analogously to the Higgs mechanism, but there is one decisive difference. Namely, there is no non-conformal invariant tachyonic Higgs mass term $-\mu S^2$

in the energy-momentum tensor. Nevertheless, we recognize the Ricci scalar term in the matter action (110). This term can play the role of the negative Higgs mass term and it provides a minimum of the scalar field potential away from the origin at some value S_0. In [29] Mannheim argues that the Higgs field $S(x)$ would acquire a space-time dependent vacuum expectation value, but after a conformal transformation of the form $S(x) \to \Omega^{-1}(x) S(x)$ it is possible to make the scalar field constant, i.e. $S(x) = S_0$. So the fermion acquires the dynamical induced mass $m = hS_0$. Hence, conformal symmetry is broken by the space-time geometry, i.e. the curvature, itself. However, this seems to be questionable, because Mannheim does no present this conformal transformation explicitly. Furthermore, he states that a conformal transformation which is needed to bring the metric to the form of eq. (35) is generally not the same like the one that makes the Higgs field constant. Thus, one has to work with a space-time dependent Higgs field $S(x)$. But Mannheim states that one is still able to reproduce the Solar System phenomenology within the experimental bounds, although one cannot exactly reproduce the geodesic equation (see [29]).

Nevertheless, let us do the calculation in this specific gauge for the sake of simplicity and to see how the mass generation mechanism works in principle in Conformal Gravity. So after choosing the scalar field to be constant the matter field energy-momentum tensor in the form of eq. (113) simplifies to

$$\begin{aligned} T^M_{\mu\nu} &= i\bar{\psi}\gamma_\mu(x)\left[\partial_\nu + \Gamma_\nu(x)\right]\psi - \frac{S_0^2}{6}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\alpha_\alpha\right] - g_{\mu\nu}\lambda S_0^4 \quad (115) \\ &= i\bar{\psi}\gamma_\mu(x)\left[\partial_\nu + \Gamma_\nu(x)\right]\psi - \frac{S_0^2}{6}\left[R_{\mu\nu} - \frac{R^\alpha_\alpha}{2}g_{\mu\nu}\right] - \frac{hS_0\bar{\psi}\psi}{4}g_{\mu\nu} \quad (116) \\ &= T^{kin}_{\mu\nu} + T_{\mu\nu}(S_0). \end{aligned}$$

Here we have subdivided the energy-momentum tensor in two parts. The first part is the kinematic fermionic part $T^{kin}_{\mu\nu}$. $T_{\mu\nu}(S_0)$ describes the Higgs field and its coupling to the curvature, i.e. especially the back reaction of the Higgs field to the space-time geometry. The flat space-time limit of this energy-momentum tensor reads

$$T^M_{\mu\nu} = i\bar{\psi}\gamma_\mu\partial_\nu\psi - \frac{1}{4}\eta_{\mu\nu}hS_0\bar{\psi}\psi \quad (117)$$

and for the fermionic field equation we get

$$i\gamma^\mu\partial_\mu\psi - hS_0\psi = 0. \quad (118)$$

Furthermore, let us make another remark to eq. (115). As we have said before, the energy-momentum conservation holds for the whole energy-momentum tensor, but additionally, it also holds for both parts of eq. (115) independently. This can be seen immediately by exploiting the property of the Einstein tensor $\left(G^{\mu\nu}{}_{;\nu} = 0\right)$ and the metric tensor $g^{\mu\nu}{}_{;\mu} = 0$ for the $T_{\mu\nu}(S_0)$ part. Therefore, also the $T^{kin}_{\mu\nu}$ part has to be conserved. This is the crucial point now. In principle, it would be possible that the fermionic part and the

Higgs field part interchange some energy, but by this independent energy conservation of both pieces, we see that this, indeed, does not happen. This is quiet remarkable, since neither of the fermionic or Higgs field part is traceless just by itself, but only their sum.

Thus, we notice that as far as we just invoke the fermionic part of the theory and the conformal gauge $S(x) = S_0$, we are able to reproduce the same equations as if we just had started with a purely kinematic fermionic field with a non-conformal invariant mechanical mass term. This means that the covariant energy conservation of $T^{kin}_{\mu\nu}$ leads to the same geodesic motion of particles on average in a gravitational field, where the $T^{kin}_{\mu\nu}$ part is taken to be the entire gravitational source.

Eq. (118) for the flat space-time limit describes a free fermion with mass $m = hS_0$. Hence, it has one-particle plane wave eigenstates $|k\rangle$ of the four momentum $k^\mu = \left(E_k, \vec{k}\right)$, where $E_k = \left(k^2 + m^2\right)^{1/2}$. Let us choose a spin up Dirac spinor with positive energy propagating in the z-direction. This yields the following matrix elements for the energy-momentum tensor

$$\int d^3x T^M_{\mu\nu} = \begin{pmatrix} E_k & 0 & 0 & -k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & 0 & \frac{k^2}{E_k} \end{pmatrix} + \begin{pmatrix} -\frac{m^2}{4E_k} & 0 & 0 & 0 \\ 0 & \frac{m^2}{4E_k} & 0 & 0 \\ 0 & 0 & \frac{m^2}{4E_k} & 0 \\ 0 & 0 & 0 & \frac{m^2}{4E_k} \end{pmatrix} \tag{119}$$

and the trace

$$\int d^3x T^{\mu}_{M\ \mu} = \left[-E_k + \frac{k^2}{E_k} + \frac{m^2}{E_k}\right] = 0. \tag{120}$$

The first matrix in (119) corresponds to the fermionic part and is given by $\langle k| \int d^3x T^{kin}_{00} |k\rangle = E_k$ and $\langle k| \int d^3x T^{kin}_{33} |k\rangle = \frac{k^2}{E_k}$. But besides that, it contains a part from the spontaneous symmetry breaking, i.e. a dynamic part $\langle k| \int d^3x T_{00}(S_0) |k\rangle = -\frac{m^2}{4E_k}$, $\langle k| \int d^3x T_{jj}(S_0) |k\rangle = \frac{m^2}{4E_k}$ for $j = 1, 2, 3$. In total we get $\langle k| \int d^3x T^M_{00} |k\rangle = E_k - \frac{m^2}{4E_k}$ and $\langle k| \int d^3x T^i_{M\ i} |k\rangle = \frac{k^2}{E_k} + \frac{3m^2}{4E_k} = -\langle k| \int d^3x T^0_{M\ 0} |k\rangle$. Here we sum over i. One recognizes that the energy of the one particle state is decreased by the contribution due to the Higgs field $-\frac{m^2}{4E_k}$, but the pressure is increased by $\frac{m^2}{4E_k}$. These vacuum contributions appear, when the mass is induced dynamically. They are responsible for the tracelessness of the energy-momentum tensor.

This energy-momentum tensor does not have the form of a perfect fluid yet. Therefore, Mannheim argues in [2, 29] that we need to average incoherently over the directions of $\vec{k}$, i.e. take a plane wave moving in every spatial direction (positive and negative direction), which all have the same $\left|\vec{k}\right| = k$ and E_k. Adding up their individual contributions to the energy-momentum tensor $T^{\mu\nu}$

yields

$$\begin{aligned} \int d^3x T_{00}^{kin} &= 6E_k, &(121)\\ \int d^3x T_{11}^{kin} &= \int d^3x T_{22}^{kin} = \int d^3x T_{33}^{kin} = \frac{2k^2}{E_k}, &(122)\\ \int d^3x T_{kin\ \mu}^{\mu} &= -\frac{6m^2}{E_k}. &(123)\end{aligned}$$

This energy-momentum tensor has exactly the form of a perfect fluid

$$T_{\mu\nu}^M = (\rho + p)\, u_\mu u_\nu + p\eta_{\mu\nu} + \Lambda\eta_{\mu\nu}, \tag{124}$$

when we identify $\rho = \frac{6E_k}{V}$, $p = \frac{2k^2}{E_k V}$ and $\Lambda = \frac{3m^2}{2VE_k}$, where V is a 3-volume. The traceless property of $T_{\mu\nu}^M$ is guaranteed by

$$3p - \rho + 4\Lambda = 0. \tag{125}$$

In fact, we mentioned this before, but as it is so important, let us restate here again that this is an explicit example, where we added to a kinematic energy-momentum tensor an additional tensor, which is conserved covariantly itself. So the covariant conservation of the total $T_{\mu\nu}^M$ is not affected by this. Another very important point here is that the tracelessness condition (125) constrains Λ to exactly the value of $\frac{\rho-3p}{4}$. We will use this property later, when we discuss the problem of Dark Energy in chapter 6.

The whole procedure generalizes to curved space straightforwardly due to the fact that the field equation for the fermions (112) has the same form as the flat space-time equation (118), one just has to add a term for the fermion spin connection $\Gamma_\mu(x)$. Moreover, in the energy-momentum tensor there appears an additional term, which represents the back reaction of the Higgs field to the geometry, but this term does not influence the fermionic part at all. This means that in the conformal gauge $S(x) = S_0$ we recover the standard kinematically massive perfect fluid conservation condition (108) and for a static, spherically symmetric fluid in hydrostatic equilibrium we find

$$\frac{\partial p}{\partial r} + (\rho + p)\frac{\partial}{\partial r}\ln(-g_{00})^{1/2} = 0, \tag{126}$$

which reduces to the standard Euler equation in the weak gravity limit.

Note that actually it is mandatory to work with wave packages rather than with plane waves. But for simplicity one can do the calculations with plane waves, since when it works for a plane wave, then it also works for a wave package.

5.3 Comparison with a Kinematic Fermion Mass Theory

To complete the mechanism of dynamical mass generation let us compare it with a kinematic fermion mass theory. The action is given by

$$I_M = -\int d^4x\, (-g)^{1/2}\left[i\bar{\psi}\gamma^\mu(x)\left[\partial_\mu + \Gamma_\mu(x)\right]\psi - m\bar{\psi}\psi\right]. \tag{127}$$

Hence, the equation of motion in the flat space-time limit reads

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0 \tag{128}$$

and the flat space-time energy-momentum tensor has the following form

$$T_{\mu\nu} = i\bar{\psi}\gamma_{\mu}\partial_{\nu}\psi. \tag{129}$$

The trace of (129) is non-zero and given by

$$T^{\mu}_{\ \mu} = m\bar{\psi}\psi. \tag{130}$$

The same incoherent averaging over plane waves (or wave packages) yields

$$T_{\mu\nu} = (\rho + p)\, u_{\mu}u_{\nu} + p\eta_{\mu\nu},\ \rho = \frac{6E_k}{V},\ p = \frac{2k^2}{VE_k}. \tag{131}$$

But opposed to Conformal Gravity, here the tracelessness condition does not hold and hence we have $T^{\mu}_{\ \mu} = 3p - \rho \neq 0$. So by noticing that both eq. (124) and eq. (131) lead to geodesic motion, we can conclude that the equivalence principle, in fact, does not require to use eq. (104) or (105) as the gravitational source. The crucial difference between both energy-momentum tensors is that eq. (124) does not just contain the fermionic part, but there is also some energy in the Higgs field. In flat space-time, where only energy differences are observable, this Higgs field energy can be neglected in the Standard Model, because it ignores gravity anyway. Thus, the neglect of this Higgs field energy does not lead to any problems. But for gravitational interactions the situation is totally different. Here the energy-momentum tensor couples directly to the curvature of space-time. Hence, the zero of energy is observable. So in conclusion we can say that the form of the geodesic equation is not sensitive to the Higgs mechanism, but the gravitational field does couple to the Higgs energy. Therefore, there is an indirect effect as can be seen in the Einstein equations (7) for standard gravity. These equations fix the background geometry in which the test particles move. So for example this means that the motion of a planet around a star is not sensitive to how the planet got its mass (when we treat the planet as a test particle), but it could depend on how the star got its mass. The reason for this is that in eq. (42) and (43) the integral of the $1/r$- term depends on the details of the energy-momentum tensor of the gravitational source.

Another issue, which is raised here, is that the right-hand side of eq. (7) is conformal invariant and traceless as long as one accepts that the energy-momentum tensor consists of fields. But the left-hand side is not conformal invariant and traceless. Hence, in standard gravity it is quite natural to use the energy-momentum tensor of eq. (105) as it fits to the left-hand side of eq. (7). But as we have seen that for geodesic motion it is not necessary to use the Einstein equations, one could look at it the other way round and let the symmetry of the matter fields imply the structure of the gravitational side. By this reasoning Mannheim argues in [29] that it would make sense to replace the Einstein equations by a conformally invariant theory, for example Conformal Gravity.

6 Cosmology in Conformal Gravity

This chapter is the direct continuation of the last chapter, where we showed how mass can be generated in a conformal invariant theory by a dynamical Higgs mechanism. Now we want to apply the energy-momentum tensor (115) to cosmology. But first, let us say a few words on the need for a different type of cosmology. As mentioned above, there is one of the biggest problems in physics today, namely the cosmological constant problem. Therefore, let us have a look on the Friedmann equation

$$\dot{a}^2(t) + k = \dot{a}^2(t)\left(\Omega_M(t) + \Omega_\Lambda(t)\right), \tag{132}$$

where $\Omega_M(t) = \frac{8\pi G\rho_M(t)}{3H^2(t)}$ and $\Omega_\Lambda(t) = \frac{8\pi G\Lambda}{3H^2(t)}$ are the time-dependent density parameters for matter and Dark Energy. Here the spatial 3-curvature $k = -1, 0, 1$ corresponds to an open, flat or closed Universe. The Friedmann equation can be derived by inserting the isotropic Robertson-Walker line element

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{133}$$

into the Einstein equations (7). A fit to the current high redshift supernovae Hubble plot data indicates that the Universe is in a phase of acceleration rather than deceleration. Moreover, the Dark Energy density parameter $\Omega_\Lambda(t_0)$, where t_0 is the age of the Universe, has to be of the order of 1. In detail it has to be in the range $\left(\frac{1}{2}, \frac{3}{2}\right)$, see fig. 4.6 in [17]. This means it has to be of the same order as the normal matter contribution. Therefore, one has to find a mechanism which quenches this contribution from its very large particle physics expectation by about 120 orders of magnitude. Even if such a mechanism were to be found, this raises another problem. The so-called cosmic coincidence problem for the standard model of cosmology. Because of the ratio $\frac{\Omega_\Lambda}{\Omega_M} \propto a^3$ the initial conditions of a Universe with an initial big bang singularity, where $\dot{a}(t=0)$ was extremely large, these two parameters had to be fine-tuned to an enormously accuracy. Given these problems, Mannheim suggests in [2] that there could be something wrong with the whole standard picture.

Furthermore, it is actually not clear that it is the particle physics part that has to be changed (there are some possible solutions like a slowly varying scalar field serving as Dark Energy or additional spatial dimensions). It is possible that the large value of Λ coming from particle physics is correct and the gravitational side has to be changed. But nevertheless, we have to say that it is still possible that the values of the cosmological constant and the zero-point contributions are just as predicted in the standard picture. Then all of them have to cancel in such a way that they result exactly in the measured value.

In this chapter we present within the theory of Conformal Gravity a possibility to cure these problems by not trying to quench the cosmological constant Λ itself, but its contribution to the gravitational effect.

The decisive point for conformal cosmology is that the Weyl tensor in (10), and hence also $W^{\mu\nu}$ (see eq. (20)), vanishes in such a geometry. Note that you can always perform a coordinate transformation on eq. (133) such that you end up with a conformally flat metric, i.e. it can be written in the form of a Minkowski metric. Any conformal transformation we need later on, would not destroy this flatness. Therefore, the Bach-equations (20) for conformal cosmology read

$$T_M^{\mu\nu} = 0. \tag{134}$$

Hence, by inserting the energy-momentum tensor (115) in this equation one yields

$$\frac{1}{6} S_0^2 \left[R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_\alpha^\alpha \right] = T_{\mu\nu}^{kin} - g_{\mu\nu} \lambda S_0^4, \tag{135}$$

where $T_{\mu\nu}^{kin} = (\rho_M + p_M) u_\mu u_\nu + p_M g_{\mu\nu}$ is the energy-momentum tensor of a perfect fluid. Here we recognize the standard form of a cosmological evolution equation, namely the form of the Einstein equations (7) with an additional term on the right-hand side

$$-\frac{1}{8\pi G_{eff}} \left[R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_\alpha^\alpha \right] = T_{\mu\nu}^{kin} - g_{\mu\nu} \lambda S_0^4, \tag{136}$$

but with an effective gravitational constant

$$G_{eff} = -\frac{3}{4\pi S_0^2}. \tag{137}$$

So we see that the gravitational coupling constant has a negative value and hence it is repulsive rather than attractive as in standard cosmology. Moreover, it becomes smaller with a larger Higgs field S_0. Thus, let us define the conformal analogs of the standard density parameters Ω_M and Ω_Λ as

$$\bar{\Omega}_M(t) = \frac{8\pi G_{eff} \rho_M(t)}{3H^2(t)}, \tag{138}$$

$$\bar{\Omega}_\Lambda(t) = \frac{8\pi G_{eff} \Lambda}{3H^2(t)}, \tag{139}$$

where $\Lambda = \lambda S_0^4$ is the cosmological constant. The matter density is given by $\rho_M = \frac{A}{a^n}$ with A being a constant and $3 \leq n \leq 4$ for the matter and radiation domination era. In consequence, by inserting the Robertson-Walker metric (133) into eq. (136) this yields

$$\dot{a}^2(t) + k = \dot{a}^2(t) \left(\bar{\Omega}_M(t) + \bar{\Omega}_\Lambda(t) \right), \tag{140}$$

where $\bar{\Omega}_M(t) + \bar{\Omega}_\Lambda(t) + \bar{\Omega}_k(t) = 1$. The curvature density parameter is defined as $\bar{\Omega}_k(t) = -\frac{k}{\dot{a}^2}$. The calculation is precisely analog to the derivation of the standard Friedmann equation (see [17], chapter 3) and thus eq. (140) is the conformal analog to the Friedmann equation of standard cosmology. The only difference is the replacement of Newton's Constant G by the the effective coupling constant G_{eff}. That the form of the equation is the same as in standard cosmology is already clear from the fact that the conformal invariance forces the matter field action to be of second order. Therefore, in a geometry where the left-hand side of the Bach equations vanishes the resulting equation of motion has to be of second order, too. This means that there cannot be a preference due to cosmology for General Relativity or Conformal Gravity on a basis of simplicity, since both are as simple as the other one.

Furthermore, let us define the conformal analog to the deceleration parameter

$$q(t) \equiv \frac{\ddot{a}(t)}{a(t) H^2(t)} = \frac{1}{2}\left(1 + \frac{3p_M}{\rho_M}\right)\bar{\Omega}_M(t) - \bar{\Omega}_\Lambda(t). \tag{141}$$

The crucial point for solving the cosmological constant problem is that the effective coupling constant G_{eff} decreases with a larger Higgs field S_0. Therefore, a larger contribution from the particle physics vacuum energy does not lead to a larger gravitational contribution. This means that in Conformal Gravity the $\bar{\Omega}_\Lambda(t)$ quenches itself to a smaller contribution to the cosmic evolution, opposed to standard cosmology, where the Ω_Λ has a fixed value, since G has a preassigned value. Hence, in Conformal Gravity a huge Λ from the particle physics side does not lead to a discrepancy of 120 orders of magnitude like in standard cosmology.

Let us comment on an issue the reader maybe already has recognized. Namely, that the theory of Conformal Gravity possesses two different gravitational coupling constants. In chapter 3 we developed the solution for the static, spherically symmetric source of the inhomogeneous Bach eq. (20). This solution is controlled by the parameter β in (35) and (37), which is determined by the conformal coupling constant α_g. In geometries where the Weyl tensor vanishes, the solution is independent of the β or α_g parameter. Therefore, G_{eff} is totally independent of the local $G = \frac{\beta}{M}$. Hence, we see that an attractive local G can coexist with a global repulsive G_{eff} in the same theory.

In [30] it is shown that the curvature parameter k has to take a negative value, because otherwise there would be no non-trivial solution to the equation $T^{\mu\nu}_{kin} = 0$, which holds above the temperature T_{EW}, where the electroweak symmetry breaking happens and S_0 is zero. In consequence of this, $\bar{\Omega}_k(t_0)$ has a positive value. Furthermore, Λ has to be negative, too (see [2]), because it is induced in the electroweak or Planck scale phase transition and therefore the free energy is lowered. Hence, $\bar{\Omega}_\Lambda(t_0)$ is a positive quantity, whereas $\bar{\Omega}_M(t_0)$ is negative. So in Conformal Gravity the signs of all parameters are fixed and in [2] one finds the following solutions to eq. (140)

$$a^2(t) = -\frac{k(\beta - 1)}{2\alpha} - \frac{k\beta \sinh^2\left(\alpha^{1/2} t\right)}{\alpha}, \tag{142}$$

where $\alpha = -2\lambda S_0^2 = \frac{8\pi G_{eff}\Lambda}{3}$, $\beta = \left(1 - \frac{16A\lambda}{k^2}\right)^{1/2}$. The solution to the conformal density parameter for Dark Energy in the limit $t \gg 0$ is given by

$$\bar{\Omega}_\Lambda(t \gg 0) = \tanh^2\left(\alpha^{1/2} t\right). \tag{143}$$

Hence, it has to lie between zero and one. Thus, no matter how big Λ might be, $\bar{\Omega}_\Lambda(t_0)$ is automatically bounded from above. In (142) we see that

$\dot{a}(t=0)$ is finite and that the conformal cosmology is singularity-free, because G_{eff} is repulsive, rather than attractive. The Universe has a minimum size $R_{min} = -\frac{k(\beta-1)}{2\alpha}$ from which it starts to expand. Another consequence is that the deceleration parameter $q(t)$ is always negative regardless at which cosmological epoch you look. Therefore, the Universe in conformal cosmology always expands accelerated. Furthermore, there is no initial singularity, but the temperature is still very high, because the product $a(t)T$ is constant for a blackbody.

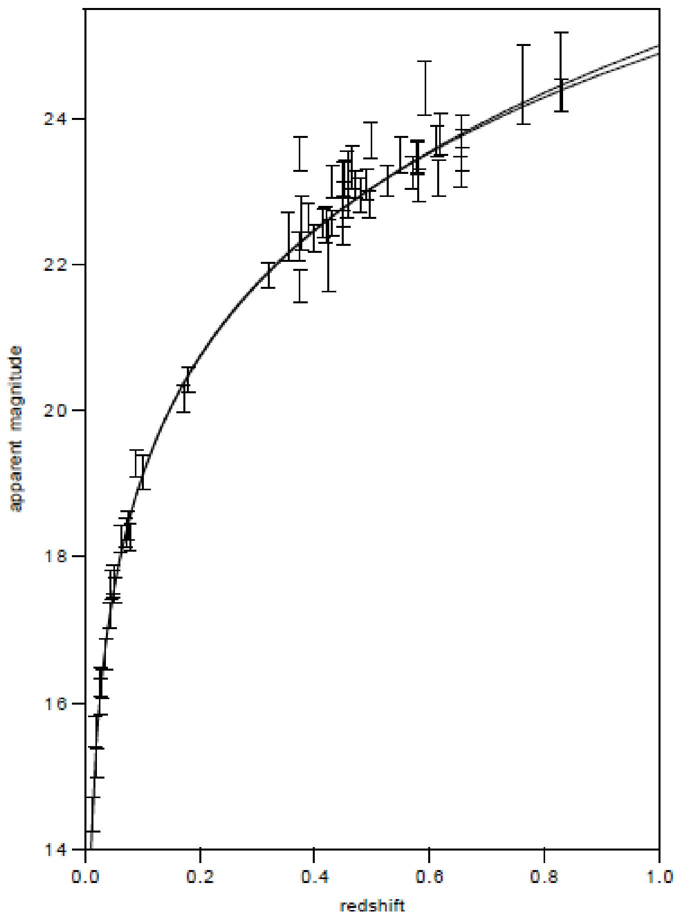

Figure 3: The figure shows the current high redshift supernovae Hubble plot data. The upper curve is the conformal gravity fit with $q_0 = -0.37$. The lower curve is the fit of the standard model with $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$. This figure is taken from [2].

Now, with this mechanism one is able to fit the current supernovae Hubble plot data by yielding the following values for the parameters: $\bar{\Omega}_M(t_0) = \mathcal{O}(10^{-60})$ $\bar{\Omega}_k(t_0) = 0,63$ and $\bar{\Omega}_\Lambda(t_0) = 0,37$. Here we see a decisive difference to the standard model, because the matter contribution is nearly zero and there is a significant contribution due to the spatial curvature. That this is a possible and realistic result needs some clarification. Therefore, we remark that there is some freedom in fitting the current supernovae data and the standard model parameters $(\Omega_M = 0,3, \Omega_\Lambda = 0,7)$ are not the only choice giving a low χ^2 parameter fitting value, see fig. 3. This figure shows the current high redshift supernovae Hubble plot data. The upper curve is the conformal gravity fit with $q_0 = -0.37$, where q_0 is the deceleration parameter today. Thus, with data for higher redshift sources it should be possible to decide which theory is correct, because standard cosmology predicts a decelerating expansion in the matter dominated phase, whereas Conformal Cosmology is always accelerating.

6.1 The wrong Sign of the linear Potential Term

Now we want to investigate a problem that was raised by Yoon in [11]. He argued that the sign of the first term of eq. (47) has the wrong sign. This is apparent by the following argumentation. The integral

$$\int_0^r hr'^2 dr' \tag{144}$$

is basically the total mass up to a positive coefficient. Hence, in (48) you can see that the linear term and the Newtonian term have different signs. But the linear term should be used to fit the radial velocity curves of galaxies in such a way that it is not necessary to put any unknown Dark Matter in the Halo of the galaxy. As one can see in fig. 2 we would need additional attraction and not less attraction to be compatible with the data. So Yoon claims Mannheim's Conformal Gravity program to be problematic, unless there is some argument that α_g in eq. (9) can be negative.

But in [31] Mannheim gives a direct answer to this critique of Yoon. There he states that it is not just the matter fields that carry energy and momentum, but also the Higgs field $S(x)$. So the matter source has the following form [32]

$$f(r) = \frac{1}{4\alpha_g B}\left[-3\left(\rho_M + p_M\right)\right] + BSS'' - 2BS'^2. \tag{145}$$

For the trace of the energy-momentum tensor one yields

$$\begin{aligned} T^{\mu}{}_{\mu}(tot) &= 3p(r) - \rho(r)\frac{S^2}{6r^2}\left[r^2B'' + 4rB' + 2B - 2\right] \\ &\quad -4\lambda S^4 + \frac{S}{r}\left[rBS'' + 2BS' + rB'S'\right] = 0. \end{aligned} \tag{146}$$

So Mannheim states that in the case $\rho(r) \gg p(r)$ the matter source is still not dominated by the matter energy density alone, since there could be a huge contribution due to the Higgs field.

7 Wave Equation and its Solution

In this thesis we presented several problems of the theory of General Relativity and how Conformal Gravity may be able to solve these problems. But a viable theory of gravity has to achieve more than the presented problems about Dark Matter, Dark Energy and Quantum Gravity. In subsection 2.1 we already mentioned the measurements of binary systems like the Hulse-Taylor binary. It was discovered by Russell Alan Hulse and Joseph Hooton Taylor of the University of Massachusetts Amherst. It is a gravitational bound system composed of the pulsar (PSR B1913+16) and another neutron star (PSR J1915+1606) orbiting around a common center of mass. By their observations Hulse and Taylor have shown that the orbit of the binary system is contracting. This could possibly be the result of energy emission of the system. General Relativity predicts such a energy emission due to gravitational waves. In fig. 4 one can see the observed decay of the periastron (data points) and the prediction of General Relativity. This prediction seems to be perfectly in line with the observation.

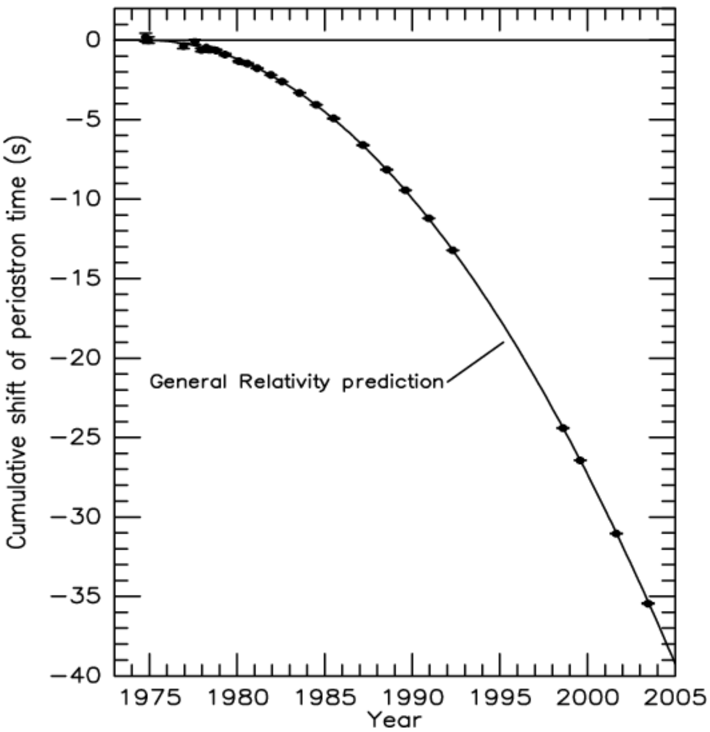

Figure 4: Orbital decay of PSR B1913+16. The data points show the observed decay of the periastron in time. The solid line is the prediction of General Relativity. This figure is taken from [33].

Thus, also Conformal Gravity has to contain a mechanism that explains the contraction of the orbit of the binary system. Therefore, in this chapter we want to present the first steps to analyze classical gravitational waves and the radiated power of such a system in Conformal Gravity.

7.1 Linearization of the Bach Equation

To analyze classical gravitational waves in Conformal Gravity one can proceed quite analogously to the mechanism in General Relativity.

Therefore, we can expand the metric around flat space-time

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{147}$$

with $|h_{\mu\nu}| \ll 1$. $h_{\mu\nu}$ is called the metric perturbation which represents the gravitational wave. Now we have to insert (147) in the Bach equations (20) and keep just the terms linear in $h_{\mu\nu}$. Hence, we give a list of linearized quantities here,

$$R^{\mu}_{\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho h^{\mu}_{\sigma} + \partial^\mu\partial_\sigma h_{\nu\rho} - \partial^\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h^{\mu}_{\rho}\right), \tag{148}$$

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\mu h^{\rho}_{\nu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\nu\partial_\rho h^{\rho}_{\mu}\right), \tag{149}$$

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h, \tag{150}$$

with $h = \eta^{\mu\nu}h_{\mu\nu}$ denoting the trace of the metric perturbation. By inserting this linearized quantities we yield the following expressions

$$\frac{1}{3}W^{(1)}_{\mu\nu} = \frac{2}{3}\eta_{\mu\nu}\partial_\rho\partial_\sigma \Box h^{\rho\sigma} - \frac{2}{3}\eta_{\mu\nu}\Box\Box h - \frac{2}{3}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma} + \frac{2}{3}\partial_\mu\partial_\nu\Box h, \tag{151}$$

$$\begin{aligned} W^{(2)}_{\mu\nu} = {} & \frac{1}{2}\eta_{\mu\nu}\partial_\rho\partial_\sigma \Box h^{\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}\Box\Box h - \frac{1}{2}\Box\Box h_{\mu\nu} + \frac{1}{2}\partial_\mu\partial_\nu\Box h \\ & -\partial_\mu\partial_\nu\partial_\sigma\partial_\rho h^{\rho\sigma} + \frac{1}{2}\partial^\nu\partial_\sigma\Box h^{\sigma}_{\mu} + \frac{1}{2}\partial^\mu\partial_\sigma\Box h^{\sigma}_{\nu}. \end{aligned} \tag{152}$$

Finally, this gives

$$\begin{aligned} W_{\mu\nu} = W^{(2)}_{\mu\nu} - \frac{1}{3}W^{(1)}_{\mu\nu} = {} & -\frac{1}{6}\eta_{\mu\nu}\partial_\rho\partial_\sigma\Box h^{\rho\sigma} + \frac{1}{6}\eta_{\mu\nu}\Box\Box h - \frac{1}{3}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma h^{\rho\sigma} \\ & -\frac{1}{6}\partial_\mu\partial_\nu\Box h - \frac{1}{2}\Box\Box h_{\mu\nu} + \frac{1}{2}\partial_\nu\partial_\rho\Box h^{\rho}_{\mu} + \frac{1}{2}\partial_\mu\partial_\rho\Box h^{\rho}_{\nu}. \end{aligned} \tag{153}$$

Now we introduce a new quantity which is the analog to the trace reversed metric perturbation in the standard theory: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}\eta_{\mu\nu}h$. By rewriting eq. (20) in terms of $\bar{h}_{\mu\nu}$ we yield

$$\begin{aligned} \frac{1}{4\alpha_g}T_{\mu\nu} = {} & -\frac{1}{6}\eta_{\mu\nu}\partial_\rho\partial_\sigma\Box\bar{h}^{\rho\sigma} - \frac{1}{3}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} - \frac{1}{2}\partial_\mu\partial_\rho\Box\bar{h}^{\rho}_{\nu} \\ & -\frac{1}{2}\Box\Box\bar{h}_{\mu\nu} + \frac{1}{2}\Box\partial_\nu\partial_\rho\bar{h}^{\rho}_{\mu} + \frac{1}{2}\partial_\mu\partial_\rho\Box\bar{h}^{\rho}_{\nu}. \end{aligned} \tag{154}$$

Here we can see that only the traceless part $\bar{h}_{\mu\nu}$ of the metric perturbation contributes to the wave equation. As $\bar{h}_{\mu\nu}$ is a symmetric tensor and traceless it has only 9 degrees of freedom. Further, by choosing the Lorenz gauge $\partial_\nu\bar{h}^{\mu\nu} = 0$ we are left with only five degrees of freedom. The result is a fourth-order wave equation

$$-\frac{1}{2}\Box\Box\bar{h}_{\mu\nu} = \frac{1}{4\alpha_g}T_{\mu\nu}. \tag{155}$$

In the vacuum case we see that the equation basically consists of the $\Box^2$-operator applied to the metric perturbation $\bar{h}_{\mu\nu}$. So the solutions to the second-order wave equation

$$\bar{h}_{\mu\nu} = C_{\mu\nu} e^{\pm i\left(\vec{k}\vec{x} - \omega t\right)} \tag{156}$$

trivially satisfy eq. (155), with $\omega = \pm \left|\vec{k}\right|$ and $C_{\mu\nu} k^{\mu} = 0$, $C^{\mu}_{\ \mu} = 0$.

But a fourth-order equation has additional solutions. These solutions are of the following type

$$C_{\mu\nu} \left(n \cdot x\right) e^{\pm i\left(\vec{k}\vec{x} - \omega t\right)}, \tag{157}$$

where n is a unit vector that obeys $n^{\mu} n_{\mu} = -1$. This seems to be a problem, because this solution grows linearly in time, which could lead to energy conservation being violated. But although the vacuum solution in Conformal Gravity shows a quite different behavior this does not automatically lead to rule out the theory of Conformal Gravity. So first let us have a look at gravitational waves in General Relativity.

The sources which lead to gravitational waves are far away from our gravitational wave observatories. We know that gravitational waves decrease with the distance to the source like r^{-1}. But as spatial diameter of the detectors on earth are relatively small the amplitude of the wave does not change significantly over this distance. Therefore, we can approximate the gravitational waves as a plane wave $h^{\mu\nu}_{plane}\left(x\right) = C^{\mu\nu} e^{ikx}$ in General Relativity.

In Conformal Gravity the vacuum solution actually grows in time or space. But this does not mean that this behavior has to be observable, since it is not a priori clear that these modes are actually excited by any source that radiates gravitational waves. Furthermore, even if the radiated waves were really growing in time or space, it is still not clear that we are able to observe this, since it could be a coordinate artefact. Thus, it could be possible that by choosing appropriate coordinates these growing solutions do not appear.

Another point is that a wave has to be produced at some specific point of space-time and only after being excited the wave can start to grow. But our detectors will always measure the same amplitude of the gravitational wave. A possibility to measure differences to the standard gravitational waves is to measure gravitational waves of systems which radiate gravitational waves with the same amplitude, but with different distances to the earth or to by a sequence of detectors along the path of the wave.

Consequently, to learn more about the behavior of gravitational waves in Conformal Gravity we study the inhomogeneous wave equation in the next section.

7.2 Inhomogeneous Wave Equation

In this paragraph we want to investigate the solution to the inhomogeneous wave equation (155).

Of course the solutions to the inhomogeneous second-order wave equation are solutions to the inhomogeneous fourth-order wave equation. Thus, the standard gravitational wave solutions are also present in Conformal Gravity.

But again we can find additional solutions. Let us make a simplified calculation to get a basic understanding of the form of these additional solutions. Hence, we apply the d'Alembert operator to a radial propagating wave which is constant in amplitude

$$\Box e^{i(t-r)} = \left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right) e^{i(t-r)} = -\frac{2i}{r} e^{i(t-r)}. \tag{158}$$

The result is a radially propagating wave with an amplitude that decreases like r^{-1}. So we know that this is a solution to the second-order Poisson equation. Thus, we can apply the d'Alembert operator to eq. (158) and we see that a constant spherical wave is a solution to the fourth-order Poisson equation. Let us note here that this is the solution to a very specific energy-momentum tensor $T_M^{\mu\nu}$ and it is not clear whether this represents a physical system or not.

Now we continue with the general solution to (155) by making use of the method of Green's function. Let us write

$$\bar{h}_{\mu\nu}(x) = -\frac{1}{2\alpha_g} \int d^4x' G(x-x') T_{\mu\nu}(x'), \tag{159}$$

where x is the position 4-vector. The defining equation for the Green's function is

$$\Box \Box G(t-t', \vec{x}-\vec{x}') = \delta(t-t') \delta(\vec{x}-\vec{x}'), \tag{160}$$

where we can set $t' = \vec{x}' = 0$. We can reintroduce arbitrary t', $\vec{x}'$ in the end without loss of generality. We start with a Fourier transformation on the left hand side of eq. (160) and apply the squared d'Alembert operator to the integrand

$$\Box \Box \frac{1}{(2\pi)^4} \int \int d\omega d^3k e^{\left(-i\left(\omega t - \vec{k}\vec{x}\right)\right)} \tilde{G}\left(\omega, \vec{k}\right) = \delta(t) \delta(\vec{x}), \tag{161}$$

$$\frac{1}{(2\pi)^4} \int \int d\omega d^3k \left(\omega^4 + \vec{k}^4 - 2\omega^2 \vec{k}^2\right) e^{\left(-i\left(\omega t - \vec{k}\vec{x}\right)\right)} \tilde{G}\left(\omega, \vec{k}\right) = \delta(t) \delta(\vec{x}). \tag{162}$$

From (162) we can read off the Fourier transformation of the Green's function

$$\tilde{G}\left(\omega, \vec{k}\right) = \frac{1}{\omega^4 + \vec{k}^4 - 2\omega^2 \vec{k}^2}. \tag{163}$$

Thus, we have to solve the following integral

$$G(t, \vec{x}) = \frac{1}{(2\pi)^4} \int \int d\omega d^3k \frac{e^{\left(-i\left(\omega t - \vec{k}\vec{x}\right)\right)}}{\omega^4 + \vec{k}^4 - 2\omega^2 \vec{k}^2}. \tag{164}$$

At first let us execute the angular integration in spherical coordinates. This yields

$$G(t, \vec{x}) = \frac{2}{(2\pi)^3 |\vec{x}|} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} dk \frac{k \sin(k|\vec{x}|)}{\omega^4 + \vec{k}^4 - 2\omega^2 \vec{k}^2} e^{(-i\omega t)}. \tag{165}$$

By recognizing that the integrand is an even function in k we can integrate from $-\infty$ to ∞ and divide by 2

$$G(t,\vec{x}) = \frac{1}{(2\pi)^3 |\vec{x}|} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \frac{k \sin(k|\vec{x}|)}{\omega^4 + \vec{k}^4 - 2\omega^2 \vec{k}^2} e^{(-i\omega t)}. \tag{166}$$

The denominator is singular at $\omega = \pm k$. Thus, we can solve this integral by a complex contour integral. So we integrate over a closed curve which includes both poles. Hence, by choosing the contour as seen in fig. 5(b) we get

$$\oint_{C_R} F(\omega)\, d\omega = -2\pi i \sum Res \tag{167}$$

with $F(\omega) = \frac{k\sin(k|\vec{x}|)}{\omega^4 + \vec{k}^4 - 2\omega^2 \vec{k}^2} e^{(-i\omega t)}$.

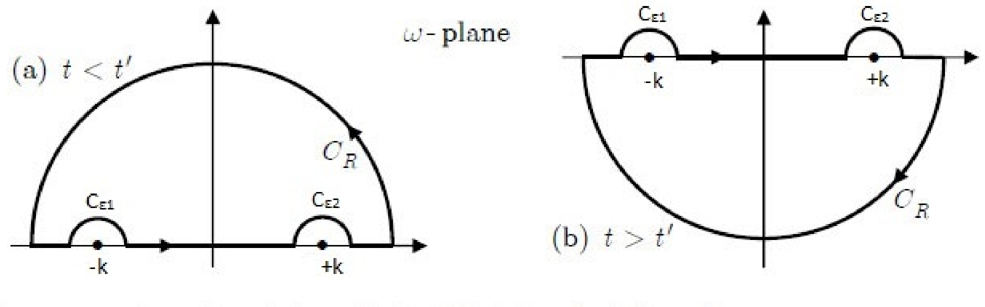

Figure 5: Contour in the complex ω-plane. (a) corresponds to a path in the upper plane and (b) to a path in the lower plane.

The next step is to evaluate the integral by the methods of calculus of residues. At first we investigate the integral over the big half circle. We continue ω to a complex variable, $w = R\exp(i\phi)$, and use that $t > 0$ and $\sin(\phi) < 0$ for $\phi \,\epsilon\, [\pi, 2\pi]$ for the chosen contour. So we can write the integral as

$$\underset{R\to\infty}{lim} \left\{ \int_{2\pi}^{\pi} Rie^{(i\phi)} F(\phi)\, d\phi \right\} \tag{168}$$

$$= \underset{R\to\infty}{lim} \left\{ \int_{2\pi}^{\pi} iRe^{\left(-i\left(tRe^{(i\phi)} - \phi\right)\right)} \frac{k\sin(k|\vec{x}|)}{R^4 e^{(4i\phi)} + \vec{k}^4 - 2R^2 e^{(2i\phi)} \vec{k}^2} d\phi \right\} \tag{169}$$

$$= \underset{R\to\infty}{lim} \left\{ \int_{2\pi}^{\pi} iRe^{(i\phi)} e^{(-itR\cos(\phi) + tR\sin(\phi))} \frac{k\sin(k|\vec{x}|)}{R^4 e^{(4i\phi)} + \vec{k}^4 - 2R^2 e^{(2i\phi)} \vec{k}^2} d\phi \right\} \tag{170}$$

For $t > 0$ and $\sin(\phi) < 0$ for $\phi \,\epsilon\, [\pi, 2\pi]$ this integral does not contribute. In the following we look at the two integrals over the small semi-circles C_{ϵ_1} and C_{ϵ_2}. Therefore, we have to find the residues of these functions. So we rewrite $F(\omega)$ in the following way

$$F(\omega) = \frac{k\sin(k|\vec{x}|)}{\omega^4 + \vec{k}^4 - 2\omega^2 \vec{k}^2} e^{(-i\omega t)} = \frac{k\sin(k|\vec{x}|)}{(k-\omega)^2 (k+\omega)^2} e^{(-i\omega t)}. \tag{171}$$

Now we define the functions $N(\omega) = \frac{\exp(-i\omega t)}{(k+\omega)^2}$ and $D(\omega) = \frac{1}{(k-\omega)^2}$. We expand $N(\omega)$ at $\omega = k$, and hence we get

$$N(\omega) = \frac{1}{4k^2} e^{(-ikt)} - \left(\frac{1}{4k^3} e^{(-ikt)} + \frac{it}{4k^2} e^{(-ikt)} \right) (\omega - k) + \mathcal{O}(\omega - k)^2 \tag{172}$$

and

$$N(\omega)\,D(\omega) = \frac{\frac{e^{(-ikt)}}{4k^2} - \left(\frac{e^{(-ikt)}}{4k^3} + \frac{ite^{(-ikt)}}{4k^2}\right)(\omega - k) + \mathcal{O}(\omega - k)^2}{(\omega - k)^2} \tag{173}$$

So the residuum is

$$Res_{+k} = \left(\frac{1}{4k^3}e^{(-ikt)} + \frac{it}{4k^2}e^{(-ikt)}\right) = \frac{e^{(-ikt)}(1 + ikt)}{4k^3}. \tag{174}$$

In analogy to this calculation we get

$$Res_{-k} = \frac{e^{(ikt)}(-1 + ikt)}{4k^3}. \tag{175}$$

If we combine the remaining integrals, they give a Cauchy principal value integral. So this is the integral we want to evaluate

$$\begin{aligned} \lim_{R\to\infty} \int_{-R}^{R} d\omega F(\omega) &= -\frac{1}{2}\pi i \left\{ \frac{e^{(ikt)}(-1 + ikt)\sin(k|\vec{x}|)}{k^2} \right. \\ &\quad \left. + \frac{e^{(-ikt)}(1 + ikt)\sin(k|\vec{x}|)}{k^2} \right\} \Theta(t). \end{aligned} \tag{176}$$

Then we insert this result in (166) and get the following remaining integral

$$\begin{aligned} G(t,\vec{x}) &= \frac{\Theta(t)}{32\pi^2|\vec{x}|} \int_{-\infty}^{\infty} dk \left\{ \frac{(1 - ikt)}{k^2} \left[e^{(ik(t+|\vec{x}|))} - e^{(ik(t-|\vec{x}|))} \right] \right. \\ &\quad \left. - \frac{(1 + ikt)}{k^2} \left[e^{(-ik(t-|\vec{x}|))} - e^{(-ik(t+|\vec{x}|))} \right] \right\}. \end{aligned} \tag{177}$$

The next step is to execute the k-integral. Therefore, we bring the integral in the following form and notice that this integral is non-singular

$$\begin{aligned} G(t,\vec{x}) &= \frac{\Theta(t)}{16\pi^2|\vec{x}|} \int_{-\infty}^{\infty} dk \frac{1}{k^2} \left[-\cos(k(|\vec{x}| - t)) + kt\sin(k(|\vec{x}| - t)) \right. \\ &\quad \left. + \cos(k(|\vec{x}| + t)) + kt\sin(k(|\vec{x}| + t)) \right]. \end{aligned} \tag{178}$$

This can be integrated and yields

$$G(t,\vec{x}) = -\frac{\Theta(t)}{16\pi} \frac{(|\vec{x}|(|t - |\vec{x}|| - |t + |\vec{x}||) + t(|t - |\vec{x}|| + |t + |\vec{x}||))}{t^2 - \vec{x}^2}. \tag{179}$$

Now this equation can be solved for the cases $t > |\vec{x}|$, $t < |\vec{x}|$, $t = |\vec{x}|$. In the first case we get

$$G(t,\vec{x}) = -\frac{1}{8\pi}\Theta(t - t')\,\Theta\left[(t - t') - \left|\vec{x} - \vec{x}'\right|\right]. \tag{180}$$

In the second case the whole term gives just zero. The limit $t \to |\vec{x}|$ does not exist, because the limits from the left and the right do not coincide. Thus, after reintroducing $t \to t - t'$ and $|\vec{x}| \to \left|\vec{x} - \vec{x'}\right|$, we get for the Green function

$$G\left(t - t', \vec{x} - \vec{x}'\right) = -\frac{1}{8\pi}\Theta\left(t - t'\right)\Theta\left[\left(t - t'\right) - \left|\vec{x} - \vec{x'}\right|\right]. \qquad (181)$$

Now we can insert the Green function into eq. (159) and yield

$$\bar{h}_{\mu\nu}\left(x\right) = \frac{1}{16\pi\alpha_g}\int d^4x'\Theta\left(t - t'\right)\Theta\left[\left(t - t'\right) - \left|\vec{x} - \vec{x'}\right|\right]T_{\mu\nu}\left(x'\right). \qquad (182)$$

It is quite remarkable that there is a constant Green's function for the past light cone and everywhere else it is zero. One would intuitively expect that there has to be some retardation in the solution in order to have a wave solution, which travels at the speed of light. Furthermore, it seems that there could be a problem with the energy conservation too, because the wave does not decrease with growing distance. But in Conformal Gravity one does not have the same energy-momentum tensor as in General Relativity. Hence, to understand more about these issues one has to investigate the energy content of such a wave and how it couples to matter.

8 Summary and Outlook

In this thesis we presented an overview of the theory of Conformal Gravity. In chapter 2 we started with the fundamental principles of the standard formalism for gravity given by General Relativity. Thus, we figured out which properties of General Relativity are necessary for a viable theory gravity and should be kept. Taking this into account we introduced Conformal Gravity with its underlying Weyl symmetry. We found the equation of motion for the gravitational field, the Bach equations, which replace the well-known Einstein equations. After that we turned to the Dark Matter problem in chapter 3. General Relativity has to invoke some kind of unknown Dark Matter to make correct predictions about the velocities of stars and gas in galactic rotation curves. Conformal Gravity can solve this problem without Dark Matter. An additional attractive term in the non-relativistic weak field limit makes Dark Matter redundant. In chapter 4 we turned to the microscopic description of Conformal Gravity. We pointed out that Conformal Gravity and especially the Bach equations have to be treated quantum-mechanically on a fundamental level. Additionally, we had a brief look on how the gravitational field is quantized in Conformal Gravity and how this could solve the zero-point energy problem that occurs in the standard formalism. Remarkable about Conformal Gravity is that the gravitational field is not quantized by its own, but by the coupling to the particle matter fields. Another important issue is the generation of masses in chapter 5. In principle mass terms are forbidden on the level of the lagrange density and thus, quite analogously to the Standard Model of particle physics, masses have to be generated by dynamical Higgs mechanism. Chapter 6 gives a short introduction to the non-perturbative cosmology in Conformal Gravity. Here we saw that one ends up with the same form for the evolution equation for the Universe, the Friedmann equation, but quite remarkably the gravitational coupling constant acts repulsive rather than attractive. Nevertheless, with conformal cosmology one is able to fit the current high redshift supernovae Hubble plot data. In chapter 7 we dealt with classical gravitational waves in Conformal Gravity. Here we calculated the Green's function for the wave equation in Conformal Gravity to first order in perturbation theory. Noteworthy, the modes we have found do not show the usual properties of waves like a decreasing amplitude with a growing distance. Furthermore, these modes do not necessarily travel with the speed of light.

By this summary we see that Conformal Gravity tackles the main problems of the standard theory of gravity and possibly is able to solve them. But still there are several concerns. One of the biggest concerns was that usually theories of higher than second order derivatives are believed to consist ghosts. That means that there are states of negative energy or negative norm. Thus, Conformal Gravity was supposed to be a non-unitary theory. But in several publications [3, 34, 35] Mannheim and Bender showed that a hermitian Hamilton operator is sufficient to have real eigenstates, but in fact also PT-symmetric Hamilton operators yield real eigenstates. In consequence, they were able to show that the evolution of quantum states in Conformal Gravity is unitary.

But still there are a lot of open questions and tasks in proving that Conformal Gravity is a viable theory of gravity. As was already mentioned in chapter 3 it is not clear that Conformal Gravity is able to explain phenomena like the Bullet cluster or the gravitational lensing of galaxy clusters. Especially, it could be important to include the cosmological $-\kappa R$ term in (91). Besides that, one has to investigate whether Conformal Gravity is able to describe the anisotropies of the cosmic microwave background as exact as the standard theory of cosmology. Furthermore, there is not very much progress in developing the theory of cosmological perturbations δ_M and structure growth, yet. A priori in Conformal Gravity it seems that it is quite hard to explain how cosmological inhomogeneities can grow up to the non-linear regime of $\delta_M \sim 1$ fast enough without any Dark Matter. First steps towards addressing these problems are given in [36, 37].

Extraordinarily, conformal cosmology provides a test for Conformal Gravity by redshift measurements above $z \sim 1$. If these measurements find a decelerating expansion of the Universe, this immediately falsifies the theory of Conformal Gravity, because in Conformal Gravity the Universe is accelerating in all epochs.

Lastly, also the decreasing orbits of binary pulsars bears data to test Conformal Gravity. In chapter 7 we presented first steps for calculating gravitational waves in Conformal Gravity. Although this is still work in progress, one recognizes that these gravitational waves have a different structure compared to them in General Relativity. Therefore, the next step is to investigate the radiated energy by these waves to see whether energy conservation is violated or not. On top of that, the Advanced Ligo experiment announced to be able to measure gravitational waves radiated by binary systems in the near future. Thus, it would be possible to decide which theory provides the correct form for gravitational waves.

As a conclusion we can say that Conformal Gravity holds some remarkable properties. It is able to explain several striking problems of theoretical physics in an excellent way. But nevertheless, Conformal Gravity has some weaknesses that have to be investigated in more detail. And furthermore, there is a huge bunch of unsolved tasks, especially for cosmology. To sum up, one can state that it is indeed reasonable as well necessary to put further investigation in the theory of Conformal Gravity as long as there is no detection of any kind of Dark Matter and Dark Energy which would solve the problems of General Relativity.

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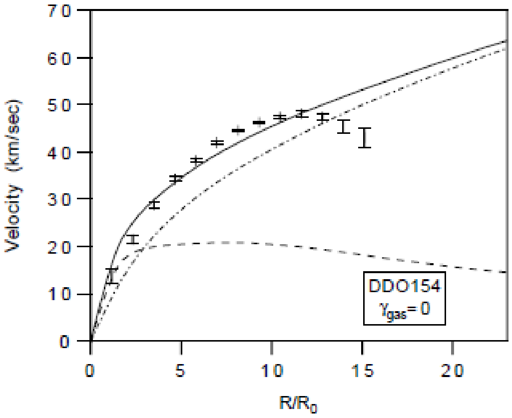
A Appendix



Velocity (km/sec)
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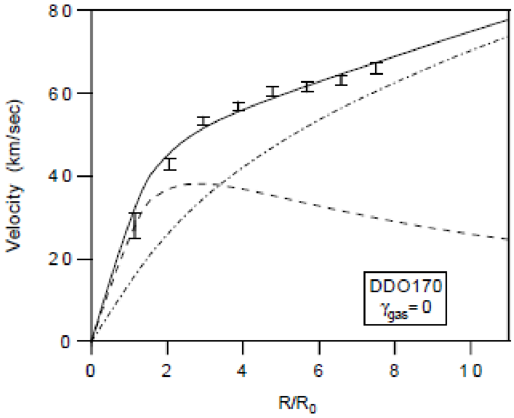
60
50
40
30
20
10
0
0
5
10
15
20
R/R0
DDO154
γgas = 0



Velocity (km/sec)
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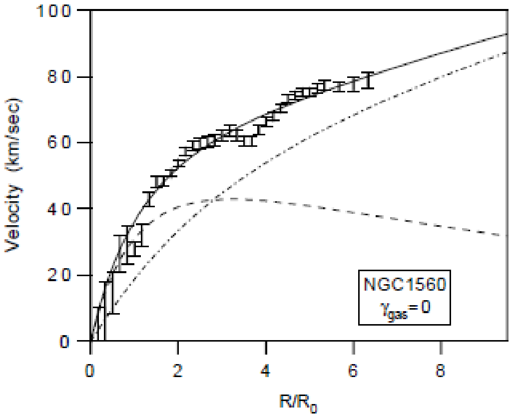
60
40
20
0
0
2
4
6
8
10
R/R0
DDO170
γgas = 0



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40
20
0
0
2
4
6
8
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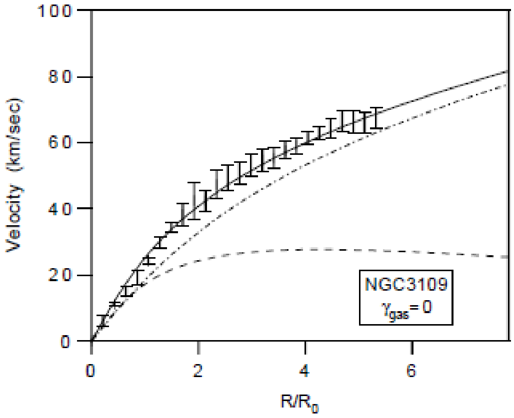
NGC1560
γgas = 0



Velocity (km/sec)
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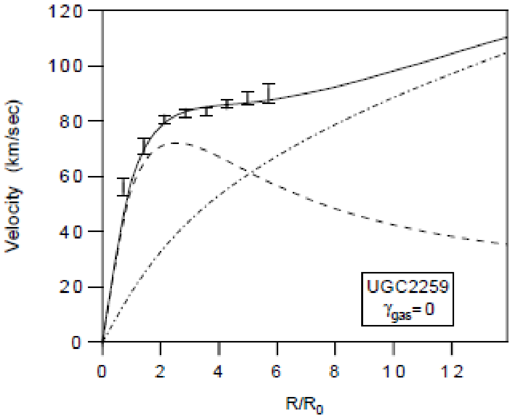
60
40
20
0
0
2
4
6
R/R0
NGC3109
γgas = 0



Velocity (km/sec)
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100
80
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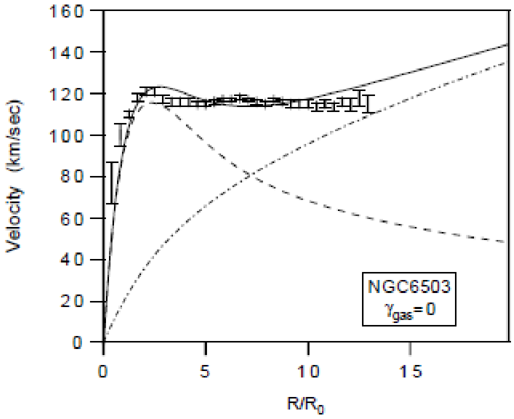
40
20
0
0
2
4
6
8
10
12
R/R0
UGC2259
γgas = 0



Velocity (km/sec)
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120
100
80
60
40
20
0
0
5
10
15
R/R0
NGC6503
γgas = 0

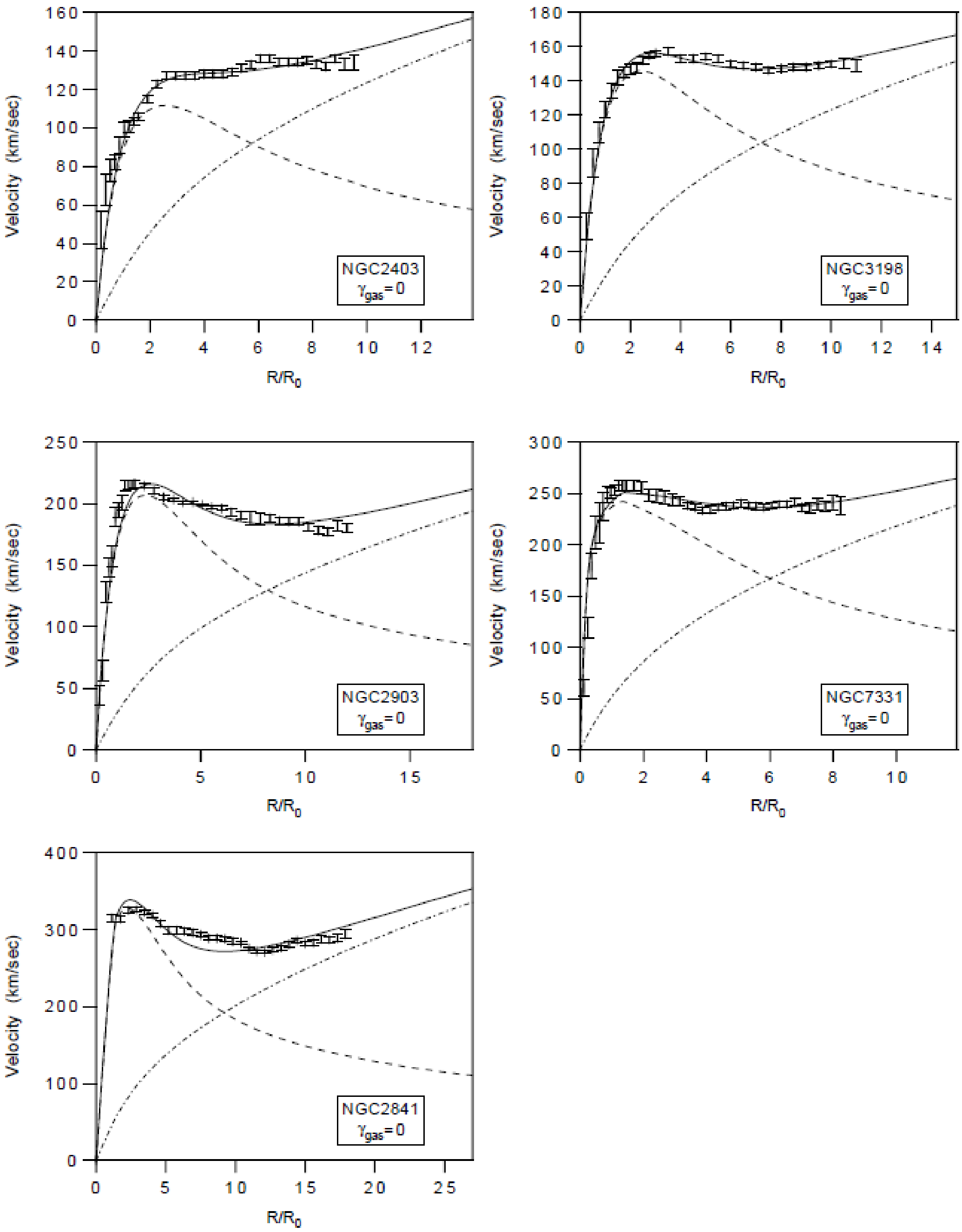

Figure 6: Galactic rotation curves of 11 galaxies. The dashed line is the Newtonian prediction. The dashed-dotted curve represents the additional linear term in Conformal Gravity and the solid line the sum of both contributions. These figures are taken from [4].

Galaxy	Distance (Mpc)	Luminosity ($10^9 L_{B\odot}$)	$(v^2/c^2R)_{tot}$ ($10^{-30}cm^{-1}$)	(M/L) $\left(M_\odot L_{B\odot}^{-1}\right)$	$(v^2/c^2R)_{net}$ ($10^{-30}cm^{-1}$)	$\gamma^* N^*/2$ ($10^{-30}cm^{-1}$)
DDO 154	3,80	0,05	1,15	0,71	1,49±,04	0,01
DDO 170	12,01	0,16	1,63	5,36	1,47±,07	0,04
NGC 1560	3,00	0,35	2,70	2,01	1,68±,13	0,08
NGC 3109	1,70	0,81	1,98	0,01	1,74±,19	0,03
UGC 2259	9,80	1,02	3,85	3,62	1,99±,26	0,15
NGC 6503	5,94	4,80	2,14	3,00	1,58±,15	0,46
NGC 2403	3,25	7,90	3,31	1,76	2,04±,17	0,66
NGC 3198	9,36	9,00	2,67	4,78	2,23±,13	0,97
NGC 2903	6,40	15,30	4,86	3,15	2,83±,19	1,80
NGC 7331	14,90	54,00	5,51	3,03	4,42±,50	3,39
NGC 2841	9,50	20,50	7,25	8,26	5,75±,30	4,76

Table 1: The table shows the measured data for the 11 galaxies analyzed in chapter 3.3.2. The data is taken from [17].

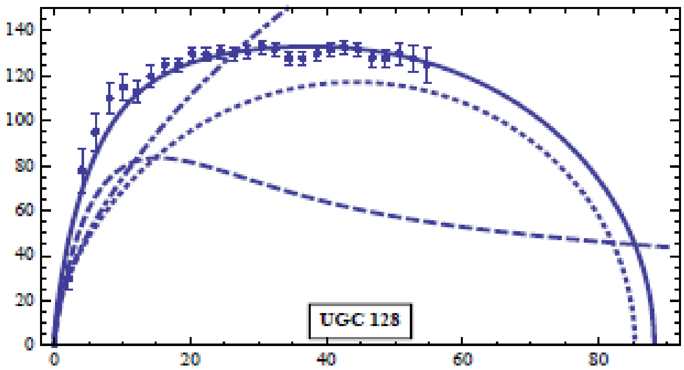

Figure 7: This figure shows the rotation curve of the galaxy UGC 128. One notices that the curve decreases again after the flat regime. This figure is taken from [16].