Precision Measurement of Angular Separation in Cosmology

Master thesis
attainment of the academic degree Master of Science
Faculty of Physics
University of Bielefeld

submitted by
Tatiana Esau

Supervisor and first assessor: Prof. Dr. D. J. Schwarz
Second assessor: Dr. D. Boriero

Bielefeld, April 10, 2014
Contents

0.1 Abstract ......................................................... 3

1 Introduction .................................................. 5
   1.1 Brief overview of cosmology ............................... 5
   1.2 Brief overview of astronomy ............................... 6
       1.2.1 Radio astronomy ........................................ 8
       1.2.2 Active Galactic Nuclei ............................... 9
   1.3 Interferometry ............................................ 13
       1.3.1 Very Large Array ...................................... 16
       1.3.2 Very Long Baseline Array ............................. 16
       1.3.3 RadioAstron ........................................... 17
       1.3.4 JIVE ................................................ 17

2 Cosmography ............................................... 19
   2.1 Redshift ................................................. 19
   2.2 Trigonometric parallax ................................. 20
   2.3 Luminosity distance .................................... 21
   2.4 Cosmic parallax ......................................... 22

3 The LTB and FLRW models .................................. 25
   3.1 Basics of general relativity ............................. 25
   3.2 A short introduction to LTB models ..................... 26
   3.3 A short introduction to FLRW models ................... 27

4 Cosmic parallax in the LTB metric ....................... 31
   4.1 The derivation of the cosmic parallax in the LTB metric . 31

5 Cosmic parallax in the FLRW metric ..................... 37
   5.1 Cosmic parallax in flat FLRW metric for comoving sources . 37
CONTENTS

5.2 Cosmic parallax in the ΛCDM model for gravitationally bound sources .......................... 41
5.3 Estimation of the secular parallax .............................................. 48

6 Estimation of cosmological parameters .................................. 53
   6.1 Likelihood estimation method ............................................ 53
   6.2 Fisher Information Matrix .................................................. 56
   6.3 Fisher matrix analysis of angular sizes
       in the ΛCDM model ....................................................... 57

7 Conclusion .................................................................................. 65

8 Appendix ...................................................................................... 67
0.1. Abstract

This thesis is handed in to obtain the degree "Master of Science" in physics with the emphasis laid on cosmology. The goal of this work is to investigate the application of the precise position measurement of astrophysical objects on the sky in cosmology. The thesis is structured as follows: chapters 1 to 3 briefly summarize theoretical foundations used in this work, chapters 4 to 6 illustrate the main ideas of this master thesis and the final chapter sums up the obtained results.
Chapter 1

Introduction

1.1 Brief overview of cosmology

The word "cosmology" is derived from Greek "cosmologia", and signifies the teaching of the world. This science studies the universe as a whole. The goal of cosmology is the mathematical description of the origin, development and the basic underlying structures of the universe. For thousands of years, humans had the desire to know more about the cosmos. Almost since the beginning of mankind, every religion and every society have been taking and still take the cosmology as the foundation of their doctrine. Only in the 20th century cosmology was recognized as a part of exact sciences, thanks to the development of Einstein's theory of relativity. In cosmology, different sciences as astronomy, astrophysics, mathematics, nuclear and elementary particle physics work together. Currently, cosmology is the most rapidly developing science. Even though a little more than one century ago, cosmology did not exist as a scientific discipline, nowadays, the Big Bang Theory is a generally accepted theory of the description of the universe. Two basic assumptions underlie the Big Bang Theory: General Relativity and the cosmological principle. The theory of relativity states that the laws of nature are universal which allows to describe the universe with the same laws that are valid close to the earth. The cosmological principle states that the universe is spatially homogeneous and isotropic on the large scale. By using the statement of the cosmological principle and General Relativity one obtains the Friedmann equations which describe the evolution of the spatially homogeneous and isotropic universe. To solve these equations, one starts
from the present state of the universe. Subsequently, one can go backwards in time to obtain a statement about the early universe. The solutions depend in particular on the measured values of the Hubble constant and the density parameters which describe the mass and the energy of the universe. With the help of astronomical observations, the cosmological parameters can be determined. There are three strong evidences in favor of the Big Bang Theory:

* Abundances of light chemical elements in the universe. After about the first three minutes of the existence of the universe, the formation of the four light elements $^2H$, $^3He$, $^4He$ and $^7Li$, was completed. The theoretical values are in excellent agreement with the measured ones, apart from $^7Li$.

* Cosmic Background Radiation (CMB). The cosmic microwave background radiation is a relic radiation from earlier stages of the hot Big Bang. CMB originates from the time about 300,000 years after the Big Bang, when radiation was decoupled from matter by the falling temperatures and the universe became transparent. In 1946, CMB was predicted theoretically and its temperature was estimated to be about 5K. After about twenty years, the CMB was measured by a coincidence. The measured temperature of the relic radiation amounts to 2.725 K.

* Expansion of the universe. In 1929 E.Hubble observed nearby galaxies and noticed that their spectra redshifted. He showed that the redshift linearly increases with the distance and plotted the distance of the galaxies against their speed. Thus he determined the Hubble constant $H_0$. However, he did not connect these with the expansion of the universe. The expansion of the universe was discovered by Lemaître who explained the redshift of the galaxies by the Doppler effect.

Thanks to projects like COBE, WMAP and Planck, the cosmological parameters, including the Hubble constant, could be measured very precisely [1, 2, 3, 4].

1.2 Brief overview of astronomy

Astronomy is one of the oldest sciences which studies the different objects in the universe using scientific methods. Although it is a very old science, it is
flourishing as never before. Due to the fast development of technologies in the last century, new disciplines with a fairly arbitrary division were created as a part of astronomy. The main instrument of astronomy is the study of electromagnetic radiation. Most of the direct information coming from the cosmos reaches us in this form. The astronomy experienced a major step forward in the Renaissance era which comes along with the discovery of the first telescope. However, despite this promising discovery, people were restricted to the visible region of the electromagnetic spectrum and had no way to do research offside our planet. Over several thousands of years, our knowledge about the universe was based on the observations in the visible range of light. The Earth’s atmosphere is transparent only in two wavelength ranges: the optical window is in the range from 290 nm to 1 µm and the radio window is in the range from about 5 cm to 20 m, see figure 1.1.

Figure 1.1: Permeability of the Earth’s atmosphere. The figure is taken from www.ipac.caltech.edu (NASA)
The discovery of the radio window has brought out a new field of the astronomy, the radio astronomy. Since the current research is no longer just ground-based, the entire spectrum of electromagnetic waves is available. The continuous development in the past several years of large ground-based telescopes and space telescopes has changed the understanding of the universe fundamentally. Thanks to the discovery and application of the maser and the supercooled detectors, it has become possible to increase the accuracy of radio telescopes [5, 6, 7].

1.2.1 Radio astronomy

In 1888 Heinrich Hertz discovered the electromagnetic radiation at radio frequencies which led to the observation of radio signals from the sky by Carl Jansky in 1931. This discovery opened up a new field of science the radio astronomy. Over several years, it turned out that this field is very promising. The measurements in the radio wave range are particularly suitable for the creation of high-resolution maps of distant galaxies. Many of the most distant galaxies have been discovered in this way. The first radio telescope was built during the second world war by an amateur astronomer Grote Reber. In 1946 his observations were confirmed by an astronomer team from England. Unfortunately, with their telescope it was not possible for them to determine the position of the observed objects exactly. Since the wavelength of the radio waves is much longer than the wavelength of the optical light, one needs to build much bigger antennas to get a sharp image. This problem could be solved by having several small telescopes connected together. Such an equipment was called a radio interferometer. The first interferometer was built in 1954 by British and Australian astronomers. At the end the 50's and the beginning of the 60's, astronomers made a catalog of radio sources over the whole sky. One of the most famous catalog was the Third Cambridge Catalogue, which can be abbreviated to "3C". Many objects in this catalog have not been regarded as anything unusual. Most of these objects were considered to be normal stars of our galaxy, until A.Sandage uncovered a special case in 1960. He discovered and investigated the "star" 3C48. The 3C48 was very bright, thus it was believed that the star could not be located outside of our galaxy, while its spectrum was completely unidentified. A little time later, several objects with the same properties like 3C48 were discovered and only in 1963 M. Schmidt has been able to explain the spectrum of the radio source 3C273. He noted that all the lines were shifted in the entire
1.2. BRIEF OVERVIEW OF ASTRONOMY

spectrum relatively to their usual position, which could be explained by the Doppler effect. The distance to 3C273 was calculated according to the Hubble law and it was found that the object was at a great distance from the Milky Way. However, astronomers have known that a normal star could not produce enough energy to appear bright to us at such a large distance. Hence, such sources are not stars, which has also been confirmed in later studies. Such sources are named quasi-stellar objects or quasars. Quasars were identified as the most distant objects. Meanwhile, quasars were detected up to redshift $z = 7$. They are among the most luminous objects in the universe.

Nowadays, it has been found that a quasar is a nucleus of an active massive galaxy, which is point-like in the optical images and emits a lot of energy in the other parts of spectrum. In the 1960s, using radio interferometry, many different radio objects were observed. Today they are known as radio galaxies [4, 6, 7].

1.2.2 Active Galactic Nuclei

By means of radio astronomy, it was observed that some galaxies have an active nucleus. An active galactic nucleus (short AGN) is a region in the center of a galaxy that emits strong radiation over the full electromagnetic spectrum. The most of active nuclei have a stronger luminosity than their entire host galaxy. A lot of these objects have been found at large $z$, which may indicate that the activity of the nucleus is a characteristic of the early life of the galaxy, and it makes the study of AGNs complicated. They are most common in galaxies with a large mass. In addition, the galaxy must be fused to the center, so the most of AGNs occur in elliptical galaxies. The hallmark of galaxies with an AGN is a changing luminosity, the period can vary from months to only a few hours. From this, one can conclude that the emission region is not larger than our solar system, which implies that the likely source of energy for AGNs is a super massive black hole in the center of the galaxy. There also almost always are large amounts of gas in the central regions of the galaxy which form an accretion disk. It is believed that the matter in an accretion disk swirls around before it falls into the black hole. This means that the gravitational potential energy of matter that falls into a black hole transforms into the kinetic energy. Collisions between particles convert their kinetic energy into thermal energy, which is the cause of the intense radiation of AGNs. Thus the reason of the strong radiation of the AGN is the hot gas in the accretion disk, which is placed around the black
The nucleus of such galaxies often emits relativistic jets. The jets consist of matter radiation and move away from the center of AGNs almost with the speed of light. Depending on the viewing angle, one can observe one or two jets directed in opposite directions. These jets extend far beyond the area where the stars of the galaxy are located. Sometime, the jets come up against the intergalactic gas and form so-called hotspots which can be observed very well.

The AGNs can be classified depending on the presence (or absence) of broad lines in the spectrum, depending on the band of the spectra which can be observed or on the intensity of the nuclear luminosity and also on the radio properties of the object. This means that one and the same object can be found in two or more classes of AGNs, depending on its physical characteristics. There are following classes of active galaxies:

**Seyfert galaxies** show broad emission lines from the galactic nucleus. The most host galaxies of Seyfert AGNs are either spiral or irregular galaxies. They were originally subdivided into Seyfert 1 (S1) and Seyfert 2 (S2) classes, where S1 has very broad emission lines and more likely emits low-energy X-rays, while S2 has only moderate lines.

**Low luminosity AGN.** In the spectrum of radiation of this class it lacks the "Big Blue Bump" which could be seen in the UV-range in many AGNs. Hence, it is common to think that these objects do not possess the inner region of the accretion disk which would produce the UV Bump.

**Low-ionization nuclear emission-line regions (LINERs).** This class shows only weak nuclear emission-line regions. The AGNs of this group are less luminous than a S2. It is questionable whether this group really belongs to AGNs.

**Radio-quiet quasars.** These can be regarded as more luminous versions of Seyfert nuclei. The distinction of these classes is arbitrary and is usually connected with a limit of optical magnitude. The quasars can be in different host galaxies such as spirals, irregulars or ellipticals. There is also some correlation between the mass of the host galaxy and the luminosity of the quasar. The greater the mass is, the more powerful the quasar is.

**Radio-loud quasars** have the same behavior as radio-quiet quasars. The difference is the emission of a jet. The spectrum shows broad and narrow emission lines and also strong X-rays and radio emission.

**Blazars (BL Lac objects and OVV quasars)** are quasars which have very weak emission lines. The luminosity of these AGNs can rapidly vary
during only a few days. Quasars which have the same variability but stronger emission lines are called optically violently variables (OVVs). Blazars are the most luminous objects in the universe.

**Fanaroff-Riley Type I (FRI) and Fanaroff-Riley Type II (FRII).**

These two groups were introduced by the astronomers Fanaroff and Riley to distinguish between radio galaxies. Fanaroff and Riley defined a ratio, $R_{FR}$, by the distance of the two brightest areas of the surfaces of the galaxy and the largest radius of the entire source. Thus, the sources are classified as follows: FR Type I, if $R_{FR}$ less than 0.5; FR Type II, if $R_{FR}$ larger than 0.5. An example of FRI and FRII is shown in figure 1.2.

![Example of Fanaroff-Riley type I galaxy (FRI) and Fanaroff-Riley type II galaxy (FRII), respectively.](http://ned.ipac.caltech.edu/level5/Cambridge)

Later, these astronomers discovered that there is a correlation between the surface brightness and spectral radio luminosity of the source. According to a frequency-dependent radio luminosity, radio resources belong either to one
or to the other FR class. In 1993 it was noted by Young that for a given radio luminosity, there exists an optical luminosity limit which divides two types of sources into FRI and FRII, see figure 1.3.

Figure 1.3: Radio luminosity is plotted against absolute visual magnitude of the host galaxy for the FRI sources (1) and FRII sources (2). The figure is taken from [8].

Figure 1.4 is a schematic representation of all described classes above. In this figure one can see that the belonging to one or the other class depends on the observing angle.
1.2. INTERFEROMETRY

Figure 1.4: Schematic representation of the classification of the AGN. The class of object which we see depends on the observing angle. The figure is taken from [26].

The big possibilities for the study of active galactic nuclei open up the discovery and further development of radio interferometry [2, 4, 5, 7, 9, 26].

1.3 Interferometry

The spatial resolution of a telescope is proportional to $\lambda/D$, where $\lambda$ is the wavelength of the observed light and $D$ is the diameter of the telescope. Thus it is clear that the resolution of a single radio antenna cannot be very
good. A method to improve the resolution of radio telescopes was developed in the 50’s. It has been found that the resolution can be increased by an interconnection of several telescopes. This method is called interferometry, since it is based on the known physical phenomenon, the interference of light. The further development of technology in the 60s finally made possible to realize an interferometer without a cable connection. The distance between telescopes can amount to several thousand kilometers. This method has become known as Very Long Baseline Interferometry. This observational technique is one of the most used principles in radio astronomy.

The easiest case to explain is by using the example with only two telescopes, see figure 1.5.

Figure 1.5: Schematic representation of the VLBI. The figure is taken from www.see.leeds.ac.uk with some small changes.

Since two telescopes cover only a small part of the hypothetical large telescope, they provide a very poor image compared to many telescopes. Nevertheless, useful information about the angular structure on the fine scale can be obtained. Consider a radio signal from a point-like object which reaches the telescopes which are placed at a distance \( B \) (\( B \) is called the baseline) from each other. If the direction of the source of cosmic radiation is perpendicular
to the base of the interferometer the signals from the two antennas yield the same phase signal. That means that one observes an interference maximum. The signal is amplified and registered. After some time, due to the rotation of the Earth, the source of the light changes its position relatively to the baseline of the interferometer.

Then the radio waves from the source will reach the antennas with a different phase. The path to radio antenna 1 is greater on the segment $P'$ and the magnitude of the resultant signal is reduced. If the segment $P'$ is equal to a half of the wavelength (or $3\lambda/2$, $5\lambda/2$ etc.), the radio waves are coming in with the opposite phase and the signals from antennas 1 and 2 completely cancel each other. So the power of the received signal will vary, increasing and decreasing, i.e., the radiation pattern of the radio interferometer is a series of petals, see figure 1.6.

To estimate the resolution of an interferometer, we choose the baseline equal to the multiple of the wavelength at which the measurement is performed. That means

$$B = n\lambda. \quad (1.1)$$

The waves of the observed object arrive at the antennas on different paths. So the path difference $P'$ for the constructive interference is given by the formula

$$P' = B \sin \Theta = m\lambda, \quad (1.2)$$

where $m \in \mathbb{N}$, $\Theta$ is the angle between the direction of incoming rays and a normal to the line on which the antennas are located. If we assume that $m = 1$ and we consider small angles $\Theta$ we obtain

$$\sin \Theta \approx \Theta \approx \frac{\lambda}{n\lambda} = \frac{1}{n} \text{ [rad]}. \quad (1.3)$$

This means the greater the baseline is, the better the resolution is. There currently exists a number of functioning VLBI systems. In the next sections we will discuss them briefly [6, 7, 9, 10].
CHAPTER 1. INTRODUCTION

1.3.1 Very Large Array

Two connected telescopes provide very poor images compared to a larger number of them. For this reason, a special arrangement of many telescopes is used. An example of such techniques is the Very Large Array (VLA) which is a radio astronomy observatory and is located in the desert of New Mexico, USA. The VLA contains 27 large independent antennas (each with the diameter of about 25 m) extending over about 40 km. The antennas of the VLA are placed Y-shaped. Due to its large area the VLA has a good sensitivity. The best possible angular resolution which can be reached with the VLA is about $\Theta \approx 0.05''$ at a wavelength of 7 mm [4, 6, 30].

1.3.2 Very Long Baseline Array

VLBA is an interferometer which consists of ten telescopes of which eight are located in the U.S., one in Hawai‘i and one in St. Croix, Virgin Islands. Each of the 10 antennas has a diameter of 25 m, and they span a baseline of about 8000 km which provides an angular resolution $\Theta \lesssim 1$ mas. The operating frequencies of the VLBA are ranging from 330 MHz to 43 GHz (from 0.9 m to 7 mm). The VLBA is controlled by the Science Operations Center of VLA in New Mexico. The signal is amplified, digitized and saved on a fast recorder for each antenna. These data are sent to the Science Operations Center in New Mexico and
are evaluated there [6, 7, 31].

1.3.3 RadioAstron

RadioAstron is a space telescope, whose operating principle is based on the VLBI method. It consists of an antenna which is moving on an elliptical orbit with an orbital period of 9.5 days, and several ground-based telescopes. The maximal distance from the Earth is 350,000 km.

The space telescope was launched in July 2011 and it is planned that the telescope remains active for at least five years. The goal of this mission is to study the AGNs, black holes and neutron stars, and also to collect information about the interstellar plasma. Due to such a great baseline, the resolution of RadioAstron is very high, $\Theta \approx 7\,\mu$as. The operating frequencies of the RadioAstron range from 18 GHz to 25 GHz (from 92 cm to 1.35 cm) [32, 33].

1.3.4 JIVE

The Joint Institute for VLBI in Europe (JIVE) is located in the Netherlands and is a member of the EVN (European VLBI Network). The EVN was formed in 1980 by a consortium of five radio astronomy institutes in Europe and performs observations of cosmic radio sources with a high angular resolution. In 1993 JIVE was established by the consortium for VLBI in Europe. The task of JIVE is to operate the EVN VLBI Data Processor and to develop it further. Currently, the JIVE and the consortium include 9 institutes with 14 radio telescopes in 8 European countries and also institute with telescopes in Russia, Ukraine, China, Italy, Poland, South Africa, Australia,
Chile, Finland, Germany, the Netherlands, Puerto Rico, Spain, Sweden and the UK [36, 37].
Chapter 2

Cosmography

The standard model of cosmology assumes that the universe can be described by the hot Big Bang. However, the Big Bang Theory does not provide an unique description of the present universe. First, the cosmological parameters have to be determined by the observations and after this, we can choose one or the other cosmological models. In this chapter, we will introduce important cosmological quantities and give a short explanation about how the distance of celestial objects can be measured.

2.1 Redshift

The cosmological observations show that almost everything in the universe moves away from us. The further an object is away, the faster it moves away from us. It was seen through spectral lines of light of distant objects which were shifted into the longer wavelength range. This effect is known as the redshift, which is denoted $z$ and is basically the Doppler effect. Thus, the redshift is given by

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}},$$

(2.1)

where $\lambda_{\text{em}}$ is the wavelength of the emitted light and $\lambda_{\text{obs}}$ is the wavelength of the observed light [1].
2.2 Trigonometric parallax

A possibility to determine the distance between an observer and a source is the trigonometric parallax. The term "parallax" is derived from the Greek and can be translated as "alteration". The parallax in astrophysics and in cosmology is an apparent change of the placement of a nearby object against the background of the much more distant objects when the observer moves from one end of the baseline to the other. That means that the objects which are closer to the observer appear to move relatively to the background objects. If the length of the baseline is known, the measurement of the parallax allows us to calculate the angular distance to the object.

Figure 2.1: Graphical representation of the trigonometric parallax that is induced by the motion of the Earth around the Sun (annual parallax). Figure 2.1 is taken from [6] with small changes.

An object is at a distance of one parsec, which is the abbreviation for "parallax arcsecond", if it has a parallax of one second of arc and the length of base is equal to 1 au. That means that the angular distance of an object

\[ 1 \text{ au} = 149597870700 \text{ m} \]
2.2. LUMINOSITY DISTANCE

is given by

\[ d_A = \frac{l/2}{\tan(\theta/2)} \approx \frac{l}{\theta}, \]  

(2.2)

where \( \theta \) is the parallax \(^2\) and \( l \) is the baseline.

There are many causes which can induce the trigonometric parallax. We consider two of these causes, the first one is the motion of the Earth around the Sun which leads to the annual parallax, and the second is the motion of the solar system around the galactic center, which generates the so-called secular parallax. Figure 2.1 shows the graphical representation of the trigonometric parallax which is caused by the motion of the Earth around the Sun.

For an object that is located at a distance of many Mpc, this method is not suitable because the parallax is immeasurably small. The usual method for determining the distance of such objects is to use the luminosity distance [1, 2, 6].

2.3 Luminosity distance

This method is based on the application of the so-called standard candles. Standard candles are the objects which are assumed to have the same absolute luminosity \( L \) (also the emitted energy flux per unit area and solid angle) everywhere in the universe. The standard candles include, for example, Cepheids and supernovae of type Ia. By comparing the absolute luminosity of standard candles with the measured radiation flux density \( S \) per time and per unit area there can be made conclusions to the distance of the source.

The luminosity distance \( d_L \) is an apparent distance of an object assuming that the light intensity decreases with the square of the distance, that means

\[ d_L^2 = \frac{L}{S}. \]  

(2.3)

By the expansion of the space the emitted photons lose energy \( \propto 1 + z \) and by the expansion of the universe the time between the arrival of photons increases \( \propto 1 + z \). Thus, the received energy flux is given by

\[ S = \frac{L}{d_{ph}^2(1 + z)^2}. \]  

(2.4)

\(^2\)In astronomy \( \theta \) has only small values and therefore, \( \tan \theta \approx \theta \)
where $d_{ph}$ is the physical distance to the observed source. Therefore, the luminosity distance is given by

$$d_L = d_{ph}(1 + z).$$

(2.5)

From the formula (2.5) it can be seen that for nearby objects ($z \ll 1$) it is $d_L \approx d_{ph}$ and that more distant objects appear to be further than they really are $d_L > d_{ph}$ [1, 2, 3].

### 2.4 Cosmic parallax

The next notion, which we will work closely with, is the "cosmic parallax". The cosmic parallax (also CP) is analogous to the classical trigonometric parallax, except that in this case, the parallax is induced by the expansion of the space.

![Figure 2.2: Overall scheme describing the possible cosmic parallax of two sources $a$ and $b$ in Lemaître-Tolman-Bondi (LTB) space-time, where $\theta := \theta_b - \theta_a$. Figure 2.2 is taken from [11].](image)
The cosmic parallax in addition to the redshift drift, are the parts of the real-time cosmology, which is a new field of cosmology. This field was created in the recent years thanks to improved astrometric and spectroscopic techniques. The real-time cosmology means that some cosmological quantities can be measured during about one human generation, that means during $10 - 15$ years.

The standard cosmology is built on two main assumptions, the general relativity and the cosmological principle. The General relativity is well proven, so there are no doubts about its truth. The cosmological principle, however, is an assumption which says that the universe is spatially homogeneous and isotropic. The evidence of isotropy is the cosmic microwave background, whereas the homogeneity cannot be verified directly. The study of the cosmic parallax provides an opportunity to investigate the homogeneity of space indirectly.

The redshift drift is the temporal variation of the redshift which is proportional to the local expansion rate. To observe this effect, high-precision spectroscopy is required. For example, in [35], the necessary precision to measure the redshift drift for the sources in the area $2 < z < 5$ was estimated. The result indicates that a 42m telescope is able to detect the redshift drift over a period of 20 years, using 4000 h of observing time. The CP is the temporal change of the angular separation between two sources. The observations of this effect relies on high-precision astrometry. Some examples of the high-precision astronomy were considered in the section 1.3.

To consider the general case of the cosmic parallax one uses models which are based on the Lemaitre-Tolman-Bondi (also LTB) metric. LTB models describe an anisotropic universe for every observer, except for the central one. In each anisotropic expansion the angular separation between two sources is not constant in time. That means that the cosmic parallax is induced by the differential expansion rate. The overall scheme describing the cosmic parallax is shown in figure 2.2. Details for LTB spacetime are presented in the next chapter [11, 12, 13, 18].
Chapter 3

The LTB and FLRW models

A metric is the basis of general relativity. By describing the distance between two space-time points a metric represents the geometry of the space-time. In this chapter the important foundations of the Lemaître-Tolman-Bondi (LTB) and Friedmann-Lemaître-Robertson-Walker (FLRW) space-time will be presented.

3.1 Basics of general relativity

The physical phenomenon of gravitation is described by the Einstein’s field equations. The fundamental idea of these equations is that the gravity influences the geometry of the space. The Einstein’s field equations can be written as

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{3.1} \]

where \( c \) is the speed of light, \( G \) is the gravitational constant, \( R_{\mu\nu} \) is the Ricci curvature tensor, \( R \) is the Ricci curvature scalar, \( T_{\mu\nu} \) is the energy-momentum tensor, \( g_{\mu\nu} \) is the metric and \( \Lambda \) is the cosmological constant. The definition of the Ricci scalar \( R \) is

\[ R = g^{\mu\nu} R_{\mu\nu}, \tag{3.2} \]

and of the Ricci curvature tensor \( R_{\mu\nu} \) is

\[ R_{\mu\nu} = g^\tau_{\mu}(\partial_\rho \Gamma^\rho_{\tau\nu} - \partial_\tau \Gamma^\rho_{\rho\nu} + \Gamma^\sigma_{\tau\nu} \Gamma^\rho_{\rho\sigma} - \Gamma^\rho_{\rho\sigma} \Gamma^\sigma_{\tau\nu}), \tag{3.3} \]
where $\Gamma^\theta_{\rho\nu}$ is a Christoffel symbol, which is given by

$$
\Gamma^\theta_{\rho\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu} \right),
$$

(3.4)

and the relation of the energy-momentum tensor for an ideal fluid is given by

$$
T_{\mu\nu} = (\rho + p/c^2) u_\mu u_\nu + p \delta_{\mu\nu},
$$

(3.5)

where $\rho$ is the mass density, $p$ is the pressure and $u_\mu$ is the 4-velocity of the fluid [3, 16, 34].

### 3.2 A short introduction to LTB models

The Lemaître-Tolman-Bondi metric is a spherically symmetric metric, which is the most general possibility to describe the space-time without the assumption of homogeneity, and it can be written as

$$
ds^2 = -c^2 dt^2 + \frac{[R'(t, r)]^2}{1 + \beta(r)} dr^2 + R^2(t, r) d\Omega^2,
$$

(3.6)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and $r, \theta, \phi$ are the spherical coordinates, $\beta(r)$ is a position dependent spatial curvature function, $R(r, t)$ is a function which depends on the time and the radial coordinates and primes correspond to $\partial_r$.

In this space-time two different Hubble parameters exist, the first one in the radial and the second one in the transverse direction, which are defined as follows:

$$
H_\parallel = \frac{\dot{R}'}{R'},
$$

(3.7)

$$
H_\perp = \frac{\dot{R}}{R},
$$

(3.8)

where dots correspond to $\partial_t$.

By inserting the LTB metric into the Einstein’s field equation one obtains the equation of motion, which, for a universe with dust and dark energy, is given as:
3.2. A SHORT INTRODUCTION TO FLRW MODELS

\[ H_\perp^2 + 2H_\perp H_\parallel - \frac{\beta(r)c^2}{R(t,r)^2} - \frac{\beta'c^2}{R(t,r)R(t,r)'} = 8\pi G\rho + \Lambda c^2, \quad (3.9) \]

and the acceleration equation which is given as:

\[ \frac{\ddot{R}(t,r)}{R(t,r)} + 2H_\perp - \frac{\beta(r)c^2}{R(t,r)^2} - 2H_\perp H_\parallel + \frac{\beta'c^2}{R(t,r)R(t,r)'} = -\frac{8\pi G(\rho + 3p/c^2) + 2\Lambda}{H_\parallel}. \]

(3.10)

The LTB universe appears anisotropic for any observer except the central one. If the homogeneity is assumed, LTB models must go into the FLRW models which describe an isotropic and homogeneous universe (see the chapter 3.3) [11, 12, 15, 16, 17, 22, 34].

3.3 A short introduction to FLRW models

The Friedmann-Lemaître-Robertson-Walker space-time describes a homogeneous and isotropic universe and is a special case of the LTB space-time. That means that, with the assumption of homogeneity of the space, the scale function \( R(t,r) \) is separable and defined for the FLRW universe as

\[ R(t,r) = a(t)r \quad (3.11) \]

and the curvature function for the FLRW universe is defined as

\[ \beta(r) = -kr^2. \quad (3.12) \]

Thus, the LTB metric goes into the FLRW metric and looks as

\[ ds^2 = -c^2dt^2 + a(t)^2 \left( \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right), \quad (3.13) \]

where \( a(t) \) is the scale factor, \( k \) is a curvature constant (\( k = +1, 0, -1 \) for the closed, flat and open universe, respectively) and \( r, \theta, \phi \) are the spherical coordinates.

If we derive the Hubble function using the definition (3.11) and (3.7), (3.8), we obtain that there is no difference between the radial and the transverse Hubble function in the FLRW space-time, i.e. \( H_\parallel = H_\perp \) and

\[ H(t) = \frac{\dot{a}(t)}{a(t)}, \quad (3.14) \]
By inserting the FLRW metric into the Einstein’s field equation (3.1) one obtains the Friedmann equation which is given by

$$\left( \frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi G \rho(t)}{3} - \frac{kc^2}{a(t)^2} + \frac{\Lambda c^2}{3} \quad (3.15)$$

and the acceleration equation is

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho(t) + 3p(t)/c^2) + \frac{\Lambda}{3}. \quad (3.16)$$

We introduce the dimensionless density parameters:

- of dark energy

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \quad (3.17)$$

- of matter (baryonic and dark matter)

$$\Omega_M = \frac{8\pi G \rho}{3H_0^2}, \quad (3.18)$$

- of spatial curvature

$$\Omega_k = -\frac{k}{a_0^2H_0^2}, \quad (3.19)$$

- of the radiation

$$\Omega_{rad} = -\frac{8\pi G \rho_{rad}}{3H_0^2}, \quad (3.20)$$

where $a_0$ is the scale factor today (it is chosen as $a_0 = 1$) and $H_0$ is the Hubble rate today. Therefore, the Friedmann equation can be rewritten as

$$H(z)^2 = H_0^2[\Omega_M(z + 1)^3 + \Omega_\Lambda + \Omega_k(1 + z)^2 + \Omega_{rad}(1 + z)^4]. \quad (3.21)$$
The Lambda-Cold-Dark-Matter (ΛCDM) model is currently the most discussed model and it is a special case of the FLRW model which has a curvature equal to 0 and $\Lambda > 0$. Since the curvature in the ΛCDM model is equal to zero and we live in matter dominated epoch, the Friedmann equation is given by

$$H(z)^2 = H_0^2 [\Omega_M (z + 1)^3 + \Omega_\Lambda],$$

where $z$ is the redshift, which can be calculated for the FLRW metric using the definition (2.1) and the fact that for light $ds^2 = 0$. So we obtain

$$z + 1 = \frac{a_0}{a(t)}.$$

Figure 3.1: CMB map of the Planck mission (with 3% of the sky replaced by a constrained Gaussian realization). The figure is taken from [29].
The values of the cosmological parameters describing our universe can be obtained, for example, from the Planck mission. The Planck mission provides a good estimation of the parameters because of its high sensitivity, its high angular resolution and its wide frequency range. The accuracy of the Planck mission is much better than of all previous missions. In the Table 1 there are results for the cosmological parameters obtained by Planck which were calculated with different assumptions.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Planck (CMB + lensing)</th>
<th></th>
<th>Plank + WP + highL + BAO</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Best fit</td>
<td>68% limits</td>
<td>Best fit</td>
<td>68% limits</td>
</tr>
<tr>
<td>( \Omega_0 h^2 )</td>
<td>0.022242</td>
<td>0.02217± 0.00033</td>
<td>0.022161</td>
<td>0.02214± 0.00024</td>
</tr>
<tr>
<td>( \Omega_c h^2 )</td>
<td>0.11805</td>
<td>0.1186± 0.0031</td>
<td>0.11889</td>
<td>0.1187± 0.0017</td>
</tr>
<tr>
<td>( \Omega_L )</td>
<td>0.6964</td>
<td>0.693± 0.019</td>
<td>0.6914</td>
<td>0.692± 0.0010</td>
</tr>
<tr>
<td>( \Sigma_8 )</td>
<td>0.8285</td>
<td>0.823± 0.018</td>
<td>0.8288</td>
<td>0.826± 0.0012</td>
</tr>
<tr>
<td>( H_0 )</td>
<td>68.14</td>
<td>67.9± 1.5</td>
<td>67.77</td>
<td>67.80± 0.77</td>
</tr>
<tr>
<td>Age/Gyr</td>
<td>13.784</td>
<td>13.796± 0.058</td>
<td>13.7965</td>
<td>13.798± 0.037</td>
</tr>
</tbody>
</table>

Table 1: Cosmological parameters values for the Planck-only best-fit 6 parameter ΛCDM model (Planck temperature data plus lensing) and for the Planck best-fit cosmology including external data sets (Planck temperature data, lensing, WMAP polarization [WP] at low multipoles, high-\( l \) experiments, and BAO, labelled [Planck + WP + highL + BAO] in Planck Collaboration XVI (2013)). The table 1 is taken from [29].

In the succeeding sections, all calculations are performed with the following values of cosmological parameters: \( \Omega_M = 0.308 \), \( \Omega_L = 0.692 \), \( H_0 = 67.8 \) [1, 3, 15, 16, 29].
Chapter 4

Cosmic parallax in the LTB metric

One of the possibilities to apply the precise measurement of angles which we will consider is the investigation of the cosmic parallax. After comparing the theoretical predictions of the cosmic parallax in the LTB model with the practical measurements, one can look for alternative to the ΛCDM cosmological models that work well with all existing observations but do not require the homogeneity of space. Therefore, the cosmic parallax in the LTB space is a good method to investigate the structure of space. Using the CP one can investigate the large-scale deviations from homogeneity. However, in this chapter we will only look at the general mathematical basis for the derivation of the cosmic parallax in LTB and we will not perform the estimation of the magnitudes of the possible parallax.

4.1 The derivation of the cosmic parallax in the LTB metric

The cosmic parallax $\Delta t \gamma$ is defined as the temporal change of the angular separation between two sources. That means

$$\Delta t \gamma \equiv \gamma_2 - \gamma_1. \quad (4.1)$$

In the following we consider an off-center observer in a coordinate system with the origin in the center of the LTB metric, which describes an isotropic
but inhomogeneous universe with respect to the center $C$. In figure 2.2 we can see the overall scheme describing the possible variation of the angular separation of two sources $a$ and $b$. Space expands isotropically with respect to the center $C$ and anisotropically with respect to the off-center observer $O$. Here $r, \theta, \phi$ are the comoving spherical coordinates. Because of the symmetry of the chosen spherically symmetric model, objects move radially outwards with respect to the center, keeping $r, \theta, \phi$ constant. First of all, we consider the expansion with respect to the center in the FLRW universe which we observe by the off-center observer at the distance $X_{\text{obs}}$ from the center. The comoving coordinates $r$ and $r_{\text{obs}}$ correspond to the physical distances $X$ and $X_{\text{obs}}$, respectively. Now we can set up the relation between the observer’s line-of-sight angle $\xi$, the coordinates of the sources $X$, the coordinate of the off-center observer $X_{\text{obs}}$ and the angle $\theta$. It is

$$
\cos \xi = \frac{X \cos \theta - X_{\text{obs}}}{\sqrt{X^2 + X_{\text{obs}}^2 - 2X_{\text{obs}}X \cos \theta}}. 
$$

(4.2)

For simplicity, we choose such sources that have the same $\phi$ coordinate. First, we consider a pair of sources at locations $a_1, b_1$ which are in the same plane as the axis $CO$ and are separated by the angle $\gamma_1$. After some time $\Delta t$ due to the expansion of space, both sources move to the positions $a_2, b_2$ which lie in the same plain with the angular separation $\gamma_2$. The physical distances $X$ and $X_{\text{obs}}$ change like $\Delta_t X$ and $\Delta_t X_{\text{obs}}$, which can be described by using the Hubble law as

$$
\Delta_t X = X H(t_0, X) \Delta t \equiv X H_x \Delta t,
$$

(4.3)

where

$$
X(r) \equiv \int_0^r \frac{1}{r} \frac{\dot{a}}{a} dr' = \int_0^r a(t_0, r') dr',
$$

(4.4)

thus for FLRW models $X_{F RW} = a(t_0)r$. According to the figure 2.2, we can rewrite the definition (4.1) as

$$
\Delta_t \gamma \equiv \gamma_2 - \gamma_1 = (\xi_{b2} - \xi_{a2}) - (\xi_{b1} - \xi_{a1}).
$$

(4.5)

For two sources $a$ and $b$ at distances $X$ much larger than $X_{\text{obs}}$ we define a parameter $s$ as following

$$
s \equiv \frac{X_{\text{obs}}}{X} \ll 1.
$$

(4.6)
4.1. THE DERIVATION OF THE COSMIC PARALLAX IN LTB

Using (4.2) and the definition (4.6) of the parameter \( s \) we obtain

\[
\xi = \arccos \left( \frac{X \cos \theta - X_{\text{obs}}}{\sqrt{X^2 + X_{\text{obs}}^2 - 2X_{\text{obs}}X \cos \theta}} \right) = \arccos \left( \frac{\cos \theta - s}{\sqrt{1 + s^2 - 2s \cos \theta}} \right),
\]

where \( s^2 \to 0 \) because of the formula (4.6). In the next step, one can use the Taylor expansion of the first order with respect to \( s \). The result is shown in the next formula.

\[
P_f(s) = f(s)|_{s=0} + f'(s)|_{s=0}s + \mathcal{O}(s^2) = \theta + s \sin \theta = \xi,
\]

where

\[
f(s)|_{s=0} = \arccos \left( \frac{\cos \theta - s}{\sqrt{1 + s^2 - 2s \cos \theta}} \right) |_{s=0} = \theta,
\]

and

\[
f'(s)|_{s=0} = \frac{-1}{\sqrt{1 - \left( \frac{\cos \theta - s}{\sqrt{1 - 2s \cos \theta}} \right)^2}} \left( -\sqrt{1 - 2s \cos \theta} + \frac{(\cos \theta - s) \cos \theta}{2\sqrt{1 - 2s \cos \theta}} \right) |_{s=0}
\]

\[
= \frac{1 - \cos^2 \theta}{\sqrt{1 - \cos^2 \theta}} = \sin \theta.
\]

Therefore, each term in (4.5) is given by

\[
\xi_{b1} = \theta_{b1} + \sin \theta_{b1} \frac{X_{\text{obs}1}}{X_{b1}}, \quad \xi_{a1} = \theta_{a1} + \sin \theta_{a1} \frac{X_{\text{obs}1}}{X_{a1}};
\]

\[
\xi_{b2} = \theta_{b2} + \sin \theta_{b2} \frac{X_{\text{obs}2}}{X_{b2}}, \quad \xi_{a2} = \theta_{a2} + \sin \theta_{a2} \frac{X_{\text{obs}2}}{X_{a2}}.
\]

Since the expansion is radially with respect to the center, the angle \( \theta_{a1} \) is the same as \( \theta_{a2} \) for every time and the angle \( \theta_{b1} \) is the same as \( \theta_{b2} \) for every time. Therefore, using (4.5) and (4.11) we obtain the following relation of the CP

\[
\Delta_{\gamma \gamma} = \sin \theta_b \left( \frac{X_{\text{obs}2}}{X_{b2}} - \frac{X_{\text{obs}1}}{X_{b1}} \right) - \sin \theta_a \left( \frac{X_{\text{obs}2}}{X_{a2}} - \frac{X_{\text{obs}1}}{X_{a1}} \right).
\]

(4.12)
According to the Hubble law, we obtain

\[
\Delta t = \gamma = \sin \theta_b \left( \frac{\Delta t H_{obs}}{H_{Xb}} \right) - \sin \theta_a \left( \frac{\Delta t H_{obs}}{H_{Xa}} \right)
\]

\[
\approx \Delta t X_{obs} \left[ \sin \theta_b \left( \frac{H_{obs} - H_{Xb}}{X_b} \right) - \sin \theta_a \left( \frac{H_{obs} - H_{Xa}}{X_a} \right) \right].
\]

(4.13)

In the FLRW metric, \( H_{obs} = H_{Xa} = H_{Xb} \). Therefore, the cosmic parallax vanishes, unless the sources are not comoving with the Hubble flow, e.g. because they are gravitationally connected. We will consider these two cases later in the exact formalism [11, 12, 13].

The derivation which was performed above includes many assumptions and thus it is not an exact calculation of the CP in LTB model. To obtain the exact result we should consider the full relativistic propagation of the light. The way of photons is determined by the geodesic equation

\[
\frac{d^2 x^\rho}{d\lambda^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0
\]

(4.14)
or alternatively, we can calculate it using the Euler-Lagrange equation

\[
\frac{d}{d\lambda} \frac{dL}{d\dot{x}} - \frac{dL}{dx} = 0,
\]

(4.15)

where \( \lambda \) is an affine parameter and \( L \) is the Lagrange function which is given by

\[
L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \frac{1}{2} \left[ -c^2 \left( \frac{dt}{d\lambda} \right)^2 + \frac{(R')^2}{1 + \beta(r)} \left( \frac{dr}{d\lambda} \right)^2 + R^2 \left( \frac{d\theta}{d\lambda} \right)^2 \right].
\]

(4.16)

Using the Euler-Lagrange equations and the 4-velocity identity

\[
g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0,
\]

(4.17)

we obtain four differential equations of the second order.

1: \( x = t \)

\[
c^2 \frac{d^2 t}{d\lambda^2} + \frac{R' \dot{R}'}{1 + \beta(r)} \left( \frac{dr}{d\lambda} \right)^2 + R \dot{R} \left( \frac{d\theta}{d\lambda} \right)^2 = 0.
\]

(4.18)
2: \( x = r \)

\[
\frac{d^2r}{d\lambda^2} + \left( \frac{R''}{R'} - \frac{\beta'}{(2 + 2\beta)} \right) \left( \frac{dr}{d\lambda} \right)^2 + \frac{2\dot{R}' \, dr \, d\theta}{R' \, d\lambda \, d\lambda} - (1 + \beta) \frac{R}{R'} \left( \frac{d\theta}{d\lambda} \right)^2 = 0, \quad (4.19)
\]

3: \( x = \theta \)

\[
\frac{d}{d\lambda} (R^2 \frac{d\theta}{d\lambda}) = 0 \Rightarrow R^2 \frac{d\theta}{d\lambda} = \text{const}, \quad (4.20)
\]

The formula (4.20) can be interpreted as the conservation of angular momentum \( J = R^2 \frac{d\theta}{d\lambda} \).

4: \( g_{\mu \nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \)

\[
-c^2 \left( \frac{dt}{d\lambda} \right)^2 + \frac{(R')^2}{1 + \beta(r)} \left( \frac{dr}{d\lambda} \right)^2 + R^2 \left( \frac{d\theta}{d\lambda} \right)^2 = 0. \quad (4.21)
\]

Now we consider a light ray that hits the observer at an angle \( \xi \), see figure 4.1. We choose the initial condition in the way that the photons hit the observer that lies at \( r = r_0 \) and \( \theta = 0 \), at the time \( t_0 \). An auxiliary photon that moves radially with respect to the center on the \( x \)-axis has a following normalized 3-velocity vector:

\[
\vec{v} = \frac{\sqrt{1 + \beta}}{R'} (1, 0, 0). \quad (4.22)
\]

The spatial components of the vector \( u^\mu \) can be found if we build a tangent to the photon path at the time \( t_0 \). After the normalization, \( g_{ij} u^i u^j = 1 \), we obtain

\[
\vec{u} = \frac{1}{c \, dt} \left( \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, 0 \right). \quad (4.23)
\]

Using the inner product of \( \vec{v} \) and \( \vec{u} \) we obtain a relation for the angle \( \xi \)

\[
\cos \xi = g_{ij} u^i v^j = -\frac{R'}{c\sqrt{1 + \beta}} \frac{d\lambda \, dr}{dt \, d\lambda}. \quad (4.24)
\]
The four differential equations of the second order can be redefined into five differential equations of the first order and can be solved with the chosen initial condition. If it is possible to make assumptions for the function $\beta(r)$ and for the scale function $R(r, t)$ by other theories, one can calculate the CP in the exact formalism using the solution of the differential equations. A simpler way to estimate the CP is using the formula (4.13) which was derived through the Euclidean approximation. Such an estimation was performed, for example, in [11], where the estimation of the distance between the center and the off-center observer comes from [20, 21].

If provided that the estimated values of the CP have a measurable magnitude, the homogeneity of the space can be indirectly verified by the application of the proposed theory. Any significant deviation from homogeneity will encourage the search for alternative models to the standard one [11, 12, 13, 16, 18].

In the next chapter, we will work with the cosmic parallax in the FLRW universe and for this, we will use the formulas derived here.
Chapter 5

Cosmic parallax in the FLRW metric

In this chapter the possibility to apply the precise measurements of an angle for the determination of the cosmic parallax in FLRW models, and especially in the ΛCDM model, will be discussed. First, the cosmic parallax in the flat FLRW universe will be examined by the application of an exact mathematical formalism, then the cosmic parallax in the ΛCDM model for gravitationally bound sources will be calculated.

5.1 Cosmic parallax in flat FLRW metric for comoving sources

First of all, to determine the CP in flat FLRW models we go back to the formula (4.13) which was obtained by an approximation. As it has already mentioned, in the FLRW universe \( H_{\text{obs}} = H_X = H_{Xb} \), and therefore, the cosmic parallax is equal to zero in this formula. Admittedly, in the derivation of the formula (4.13) a lot of assumptions were made, thus, we want to investigate the cosmic parallax in the flat FLRW models, if the exact calculation will be considered.

Because of the singularity of the spherical coordinates at the point \( r = 0 \), we first consider an off-center observer who is placed at the distance \( r_0 \) from the center of the chosen coordinate system. Figure 5.1 shows rays of light hitting the observer \( O \) who is at a distance \( r_0 \) from the origin. For simplicity, we choose the coordinate system in the way that one of the axes goes through
CHAPTER 5. COSMIC PARALLAX IN THE FLRW METRIC

the center $C$, the observer $O$ and one of the sources.

Analogous to the LTB universe, the Euler-Lagrange equation (4.15) can be used to calculate the light path in flat FLRW models. The Lagrange function for such models is given by

$$L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \frac{1}{2} \left[ -c^2 \left( \frac{dt}{d\lambda} \right)^2 + a(t)^2 \left( \frac{dr}{d\lambda} \right)^2 + r^2 a(t)^2 \left( \frac{d\theta}{d\lambda} \right)^2 \right].$$

$$\tag{5.1}$$

![Figure 5.1: Rays of light hitting the observer $O$ at an angle $\gamma$.](image)

Using the Euler-Lagrange equation and the 4-velocity identity, we obtain the following relations

1: $x = t$

$$c^2 \frac{d^2 t}{d\lambda^2} + a(t)\dot{a}(t) \left[ \left( \frac{dr}{d\lambda} \right)^2 + r^2 \left( \frac{d\theta}{d\lambda} \right)^2 \right] = 0,$$

$$\tag{5.2}$$

2: $x = r$

$$\frac{d^2 r}{d\lambda^2} + 2H \frac{dr}{d\lambda} \frac{d\theta}{d\lambda} - r \left( \frac{d\theta}{d\lambda} \right)^2 = 0,$$

$$\tag{5.3}$$
3. \( x = \theta \)

\[
\frac{d}{d\lambda} \left( r^2 a(t)^2 \left( \frac{d\theta}{d\lambda} \right) \right) = 0 \Rightarrow r^2 a(t)^2 \left( \frac{d\theta}{d\lambda} \right) = \text{const} =: J, \tag{5.4}
\]

4. \( g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \)

\[
-c^2 \left( \frac{dt}{d\lambda} \right)^2 + a(t)^2 \left( \frac{dr}{d\lambda} \right)^2 + r^2 a(t)^2 \left( \frac{d\theta}{d\lambda} \right)^2 = 0. \tag{5.5}
\]

The last equation will be used to normalize the vector \( \vec{u} \). We assume that the observer is at rest and the sources move away from the observer due to the space expansion. The coordinate system is chosen in the way that the source \( a_1 \) moves away from the observer along the axis \( x \) and the coordinate \( \phi \) is constant. The spatial direction of \( \vec{v} \) and \( \vec{u} \) can be obtained, like in the LTB universe, if we construct the tangent to the photon path. Thus, the spatial components of the unit vector for the source \( a_1 \) in flat FLRW models are given by

\[
\vec{v} = \frac{1}{a(t)} \frac{d\lambda}{dt} \left( \frac{dr_a}{d\lambda}, 0, 0 \right). \tag{5.6}
\]

and the spatial components of the unit vector for the source \( b_1 \) are given by

\[
\vec{u} = \frac{1}{c} \frac{d\lambda}{dt} \left( \frac{dr_b}{d\lambda}, \frac{d\theta}{d\lambda}, 0 \right) \tag{5.7}
\]

Using the inner product of \( v^j \) and \( u^i \), we obtain a relation for the angle \( \gamma \) in flat FLRW models:

\[
\cos \gamma = g_{ij} u^i v^j = \frac{a(t)}{c} \left( \frac{dr_b}{dt} \right). \tag{5.8}
\]

Using the definition of the FLRW metric (3.13), the fact that for every flat model \( k = 0 \) and that for the light \( ds^2 = 0 \), the relation (5.8) can be rewritten as

\[
\frac{a(t) dr_b}{c \frac{dt}{d\lambda}} = \sqrt{1 - \frac{a(t)^2 r^2}{c^2} \left( \frac{d\theta}{dt} \right)^2}. \tag{5.9}
\]

The term \( \left( \frac{d\theta}{dt} \right)^2 \) can be expanded as \( \left( \frac{d\phi}{d\lambda} \right)^2 \left( \frac{d\lambda}{dt} \right)^2 \) and according to the 4-velocity identity (5.5), we obtain
\[
\left(\frac{d\theta}{d\lambda}\right)^2 \left(\frac{d\lambda}{dt}\right)^2 = \left(\frac{d\theta}{d\lambda}\right)^2 \left[ \frac{c^2}{a(t)^2 \left(\frac{dr}{d\lambda}\right)^2 + r^2 a(t)^2 \left(\frac{d\theta}{d\lambda}\right)^2} \right]. \tag{5.10}
\]

Now, the equations (5.9) and (5.10) can be inserted into the equation (5.8) and the result is given by

\[
\cos \gamma = \sqrt{1 - a(t)^2 r^2 \left(\frac{d\theta}{d\lambda}\right)^2 \left[ \frac{1}{a(t)^2 \left(\frac{dr}{d\lambda}\right)^2 + r^2 a(t)^2 \left(\frac{d\theta}{d\lambda}\right)^2} \right]} \tag{5.11}
\]

After simplifying the formula (5.11) we obtain

\[
\cos \gamma = \sqrt{1 - \frac{1}{\left(\frac{dr}{d\lambda}\right)^2 + 1}}. \tag{5.12}
\]

Because of the property of light to propagate on a straight line, the following approach is used

\[
r(\lambda) = \frac{-r_0 m}{\sin \theta(\lambda) - m \cos \theta(\lambda)}, \tag{5.13}
\]

where \(r_o\) is the coordinate of the observer with respect to the origin, \(\theta\) is the angle between the source, the origin and the observer, \(\lambda\) is a parameter, which is choosen as \(\lambda \in [0; 1]\) and \(m\) is the gradient which is given by

\[
m = \frac{-r_q \sin \theta_q}{r_0 - r_q \cos \theta_q}, \tag{5.14}
\]

with the initial conditions \(r(0) = r_0\), \(r(1) = r_q\) and \(\theta(0) = 0\), \(\theta(1) = \theta_q\). Thus, the relation (5.13) describes a straight line which goes from the observer to the source. The derivative of the equation (5.13) is

\[
\frac{dr(\lambda)}{d\lambda} = \frac{r_0 m (\cos \theta + m \sin \theta)}{(\sin \theta - m \cos \theta)^2} \left(\frac{d\theta}{d\lambda}\right). \tag{5.15}
\]
5.1. COSMIC PARALLAX FOR GRAVITATIONALLY BOUND SOURCES

Using the equations (5.4), (5.13), (5.14) and (5.12), the relation of the angle $\gamma$ is given by

$$\cos \gamma = \sqrt{1 - \left[ \frac{1}{\left( \frac{(\cos \theta + m \sin \theta)^2}{(\sin \theta - m \cos \theta)^2} + 1 \right)} \right]}.$$  \hfill (5.16)

Using the initial condition $\theta(0) = 0$ and the definition (5.14), we obtain the following relation for the angle $\gamma$

$$\cos \gamma = \sqrt{1 - \left[ \frac{1}{m^2 + 1} \right]} = \frac{1}{\sqrt{1 + m^2}} \text{ = const.}$$ \hfill (5.17)

In this relation it can be seen that the angle between two sources in each flat FLRW model has a dependence on the initial angle of the source $\theta_q$, its radial coordinate $r_q$ and the distance $r_0$ between the observer and the center of the coordinate system, which means that the angle $\gamma$ is constant in time. According to the definition of the CP (4.1) and to the fact that $\gamma$ is constant in time, it is obvious that the cosmic parallax in such models does not exist. These findings correspond to our expectations because otherwise, the isotropy of space would be violated. The exception constitute the sources which are connected by gravity.

In the next section, the cosmic parallax of two sources, which are gravitationally bound, will be calculated [15, 16, 34].

5.2 Cosmic parallax in the $\Lambda$CDM model for gravitationally bound sources

As it was shown above, there is no cosmic parallax effect in a homogeneous and isotropic universe for two comoving sources that are not bound gravitationally. In this section, a CP in the $\Lambda$CDM model will be estimated, using the assumption that the sources are gravitationally bound, e.g. are of type FRII (Fanaroff-Riley Class II). The FRII sources are radio galaxies with an active galactic nucleus (AGN) which are very luminous at radio wavelengths. In FRII sources one can observe two jets which are directed oppositely each other. These sources have weak jets but bright hotspots at the end of the lobes. The goal is to investigate the cosmic parallax between the hotspots
CHAPTER 5. COSMIC PARALLAX IN THE FLRW METRIC

with respect to us. The graphical representation of the CP of two bound sources in the \( \Lambda \)CDM model is shown in figure 5.2.

In a FLRW universe the formula of the CP is simpler than in the LTB universe. Since, per definition, the cosmic parallax is the temporal change of the angular separation between two sources, and each point in the \( \Lambda \)CDM model can be considered as a central one, the relation of the CP (4.1) can be rewritten as following

\[
\Delta_t \gamma = \gamma_2 - \gamma_1 = \theta_2 - \theta_1.
\] (5.18)

According to figure 2.2 and to figure 5.2 we can see that in the FLRW space there is not difference between \( \theta \) and \( \gamma \). (Remember that \( \theta_1 := \theta_{b1} - \theta_{a1} \) and \( \theta_2 := \theta_{b2} - \theta_{a2} \))

\[\text{center } C = \text{observer } O\]

Figure 5.2: Graphical representation of the CP in the \( \Lambda \)CDM model. Rays of light hit the observer O at an angle \( \gamma_1 \) at time \( t_1 \) and at an angle \( \gamma_2 \) at time \( t_2 \).

To obtain the relation of an angle, the equation (3.13) must be transformed with respect to the angle \( \theta \). Using the assumption that we only
observe small angles, see figure 2.1, we obtain
\[ \theta = \frac{l}{d_A} = \frac{l(1+z)}{a_0 r}, \quad (5.19) \]
where \( l \) is a distance between two sources, which are on the same shell (which means that they have the same redshift with respect to the observer) and \( d_A \) is an angular distance. The denominator \( a_0 r \) for the \( \Lambda \)CDM is given by the Friedmann equation as
\[ a_0 r = c \int_0^z \frac{dz'}{H(z')} = \frac{c}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}}, \quad (5.20) \]
where \( H_0 = h \times 100 \left[ \frac{\text{km}}{\text{s Mpc}} \right] \) is the Hubble parameter today, \( h \) is dimensionless Hubble parameter and \( \Omega_M \) is the matter density parameter (dark plus barionic) today.

By inserting the formula (5.20) into (5.19) we obtain the relation of the angle \( \theta \) in units of [as]
\[ \theta(z) = 68.755 \left( \frac{l}{\text{Mpc}} \right) h \int_0^z \frac{(1+z)}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}}. \quad (5.21) \]
Thus, according to the equation (5.21) the CP is
\[ \Delta_t \theta = \alpha \left( \frac{1+z_2}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}} - \frac{1+z_1}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}} \right), \quad (5.22) \]
where \( z_2 = z_1 + \Delta_t z \) and \( \alpha = 68.755 \left( \frac{l}{\text{Mpc}} \right) h \)

To estimate the cosmic parallax one must find out how the redshift changes with respect to time. Using the definition of the redshift, we obtain the following formula for the change of the redshift in time:
\[ \Delta_t z = z_2 - z_1 = \frac{a(t_0) + \Delta t_0}{a(t_s) + \Delta t_s} - \frac{a(t_0)}{a(t_s)}, \quad (5.23) \]
where \( t_s \) is the emission time of light from the source and \( t_0 \) is the observation time. Using the Taylor expansion, we get
\[ \Delta_t z = \Delta t_0 \left( \frac{\dot{a}(t_0) - \dot{a}(t_s)}{a(t_s)} \right) + \mathcal{O}\left( \frac{\Delta t_0}{t_0} \right)^2 = H_0 \Delta t_0 \left( 1 + z - \frac{H(z)}{H_0} \right). \quad (5.24) \]
CHAPTER 5. COSMIC PARALLAX IN THE FLRW METRIC

The results of the calculation of the redshift change $\Delta t z$ for different $z$ and for $\Delta t_0=10$ years are presented in table 3 in the appendix.

Alternatively, because of the small values of $\Delta t z$ one can use the Taylor expansion of equation (5.22):

$$P_f(\Delta t z) = f(\Delta t z) \bigg|_{\Delta t z=0} + f'(\Delta t z) \bigg|_{\Delta t z=0} \Delta t z + O(\Delta t z)^2$$ \hspace{1cm} (5.25)

Thus, one obtains

$$f(\Delta t z) \bigg|_{\Delta t z=0} = \alpha \left( \frac{1 + z + \Delta t z}{\sqrt{\Omega_M(1+z')^3+\Omega_\Lambda}} - \frac{1 + z}{\sqrt{\Omega_M(1+z')^3+\Omega_\Lambda}} \right) \bigg|_{\Delta t z=0} = 0$$ \hspace{1cm} (5.26)

$$f'(\Delta t z) \bigg|_{\Delta t z=0} = \alpha \left( \frac{\int_{0}^{z+\Delta t z} dz' \sqrt{\Omega_M(1+z')^3+\Omega_\Lambda} - (1+z+z')}{\sqrt{\Omega_M(1+z')^3+\Omega_\Lambda}} \right) \bigg|_{\Delta t z=0} \Delta t z$$

$$= \alpha \left( \frac{\int_{0}^{z} dz' \sqrt{\Omega_M(1+z')^3+\Omega_\Lambda} - (1+z)}{\sqrt{\Omega_M(1+z')^3+\Omega_\Lambda}} \right) \bigg|_{\Delta t z=0} \Delta t z$$ \hspace{1cm} (5.27)

Using the equation (5.25), we obtain

$$\Delta t \theta = \alpha \Delta t z \left( \frac{\int_{0}^{z} dz' \sqrt{\Omega_M(1+z')^3+\Omega_\Lambda} - (1+z)}{\sqrt{\Omega_M(1+z')^3+\Omega_\Lambda}} \right) \bigg|_{\Delta t z=0} \Delta t z + O(\Delta t z)^2$$ \hspace{1cm} (5.28)

The integrals in the denominator of the equation (5.22) are not solvable analytically. Therefore, for gravitationally bound sources which are positioned on the same shell and at the distance of 1Mpc from each other, the cosmic parallax in the $\Lambda$CDM model has to be calculated numerically, using the values of the redshift change $\Delta t z$ from table 3 in appendix.
5.2. **COSMIC PARALLAX FOR GRAVITATIONALLY BOUND SOURCES** 45

The results of the cosmic parallaxes with respect to $z$ are shown also in table 3. The same results can be obtained by using the formula (5.28). A graphical representation of the CP with respect to $z$ is shown in figure 5.3.

![Graphical representation of the cosmic parallax with respect to $z$ in the $\Lambda$CDM model for two gravitationally bound sources at a distance of 1Mpc.](image)

Figure 5.3: Graphical representation of the cosmic parallax with respect to $z$ in the $\Lambda$CDM model for two gravitationally bound sources at a distance of 1Mpc.

The logarithmical representation of the absolute values of the cosmic parallax with respect to the redshift is shown in figure 5.4. From the values in table 3 we can see that the magnitude of the cosmic parallax after 10 years is very small and not measurable for most redshifts. The best possible accuracy of the modern technique, for example, the accuracy of RadioAstron is approximately 7$\mu$as, that means that only the sources lying at $z < 0.003$ can be used for the investigation of the CP. The period of 10 years was chosen, based on the idea of real-time cosmology which was briefly introduced in chapter 2.4. In the formula (5.28), using the relation (5.24), we can see that the time dependence of the CP is linear. That means that by doubling the observation time, the values of the parallax will be doubled and we will obtain measurable values of the CP for the sources, positioned at the more distant redshifts. In the figure 5.5 the angle under which an object of fixed physical size is seen at different redshifts in the $\Lambda$CDM model is shown. According to the figure 5.5, we would expect the change of the sign of the resulting CP,
approximately from $z = 1.5$. The sign change does not happen, since from approximately $z = 1.5$, $\Delta_t z$ also changes the sign [15, 16].

Figure 5.4: Logarithmical representation of the CP with respect to $z$ in the $\Lambda$CDM model for two gravitationally bound sources, placed at distance 1 Mpc.

Figure 5.5: Angle with respect to $z$ in the $\Lambda$CDM model for two gravitationally bound sources, placed at distance 1 Mpc.
5.2. COSMIC PARALLAX FOR GRAVITATIONALLY BOUND SOURCES

In addition, if considering sources of the type FRII, one has to keep in mind that the jets dilate nearly with the speed of light which can have a great influence on the magnitude of the CP. This influence will be estimated in the next step. The distance between the hotspots changes mostly according to

\[ l_2 = l_1 + 2 \, c \, t = l_1 + 6.132 \times 10^{-6} \text{Mpc}. \]  \hspace{1cm} (5.29)

where \( l_2 \) is the distance between two hotspots after 10 years and \( 10 \text{yr} \times c = 3.066 \times 10^{-6} \text{Mpc} \) which means that the formula (5.22) can be rewritten as

\[
\Delta \theta = \frac{68.755h}{\text{Mpc}} \left( \int_0^{z_2} \frac{l_2(1 + z_2)}{\Omega_m(1+z')^3+(1-\Omega_m)} \, dz' - \int_0^{z_1} \frac{l_1(1 + z_1)}{\Omega_m(1+z')^3+(1-\Omega_m)} \, dz' \right). \hspace{1cm} (5.30)
\]

![Figure 5.6: The black line shows the logarithm of the cosmic parallax, taking into account the expansion of the jets with respect to \( z \) in the \( \Lambda \text{CDM} \) model for two gravitationally bound sources which are at initial distance \( l_1 = 1 \text{ Mpc} \) from each other. The blue line is the logarithm of the cosmic parallax with respect to \( z \) in the same model for two gravitationally bound sources which are at distance 1 Mpc from each other, which remains constant over the time.](image)
The results of the calculation of the cosmic parallax, taking into account
the expansion of the jets, are shown in appendix in table 4 and a graphical
representation of this is given in figure 5.6.

If comparing the values from table 3 and table 4 in appendix, we can
see that the difference between the values of CP amounts several orders of
magnitude, see also the figure 5.6. This means that the effect of the extension
of the jets dominates the CP and hence, such a source of the type FRII
cannot serve the purpose. For the investigation of the cosmic parallax one
needs sources which are bound by gravity and which do not change their size.

5.3 Estimation of the secular parallax

In addition to the effect of the expansion of jets, the cosmic parallax is
influenced by an annual and secular parallax. In this section, the secular
parallax will be estimated and the results will be compared with the CP
from section 5.2. The secular parallax is a parallax which is caused by the
motion of our solar system in the Milky Way. In a lot of recent astrometrical
articles the Galactic parameters have been estimated, see e.g. [25]. These
estimations give us the distance $r$ to the galactic center as $8.4 \pm 0.6 \text{kpc}$ and
a circular rotation speed $V$ of $254 \pm 16 \text{km/s}$.

In figure 5.8 we can see a schematic representation of the secular parallax.
In this scheme the points D and C are the gravitational bound sources which
are at rest, the point A gives the position of the solar system in the Milky
Way at time $t_0$ and B shows the position of the solar system in the Milky
Way after 10 years. From figure 5.8 we can see that the secular parallax reads

$$\Delta t \theta_s = \theta_1 - (\theta_2 + \theta_3).$$  \hspace{1cm} (5.31)

Due to straightforward geometry, the angular change between the moving
solar system and the sources, which were assumed at rest, can be calculated.
As an example, we carry out the calculation of the secular parallax for two
sources which lie at the redshift $z = 0.1$ and which have the distance of 1Mpc
from each other. For this purpose we need to know the physical distance of
the chosen redshift. The physical distance for the $\Lambda$CDM model is calculated
by using the FLRW metric and the following definition of the physical distance

$$X_{ph} = a_0 r = \frac{a_0 c}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_m (1 + z')^3 + \Omega_A}}.$$  \hspace{1cm} (5.32)
5.3. ESTIMATION OF THE SECULAR PARALLAX

Since the integral is not analytically solvable, we solve it numerically by using the program "Mathematica". The results are shown in table 5 in appendix.

In the next step one can calculate all the parameters which remain constant for different $X_{ph}$. The first one is $l_a$ which tells us how far the solar system travels on the arc with respect to the initial position after 10 years

$$l_a = t V = 8.010144 \times 10^{10} \text{km} = 0.002596 \text{pc}. \quad (5.33)$$

Knowing the length of the arc which the solar system travels on, one can compute the angle $\alpha$, see figure 5.8. It is given by

$$l_a = \pi r \frac{\alpha}{180^\circ} \Rightarrow \alpha = 1.7707 \times 10^{-5} \circ = 63.745 \text{mas}. \quad (5.34)$$

Thus, the length $l$ is

$$\sin \left( \frac{\alpha}{2} \right) \approx \frac{\alpha}{2} = \frac{l}{2r} \Rightarrow l = 0.002565 \text{pc}. \quad (5.35)$$

From figure 5.8 one can see that the angle $\beta = \frac{\alpha}{2}$ is given by

$$\beta = 8.854 \times 10^{-6} \circ = 31.874 \text{mas}. \quad (5.36)$$

The lengths $AK$ and $BK$ are given by

$$AK = l \cos \beta = 2.565 \times 10^{-3} \text{pc}, \quad (5.37)$$
$$BK = l \sin \beta = 3.964 \times 10^{-10} \text{pc}. \quad (5.38)$$

Using equation (5.31) and figure 5.8 we obtain the following relation for the secular parallax after 10 years

$$\Delta t \theta_s = 2 \arctan \left( \frac{CF}{FA} \right) - \left[ \arctan \left( \frac{CE}{EB} \right) + \arctan \left( \frac{DE}{EB} \right) \right], \quad (5.39)$$

where $FA = \sqrt{X_{ph}^2 - CF^2}$ and $X_{ph}$ is the physical distance to the sources, $X_{ph} = CA = DA$, $CF$ is the half of the distance between two sources and it is chosen to be equal to 0.5Mpc, $CE = 0.5 \text{Mpc} - AK$, $EB = FA + BK$ and $DE = 0.5 \text{Mpc} + AK$. So the formula (5.39) is
\[ \Delta t \theta_s = 2 \arctan \left( \frac{5 \times 10^5}{\sqrt{X_{ph}^2 - 0.5^2}} \right) - \arctan \left( \frac{5 \times 10^5 - 2.565 \times 10^{-3}}{\sqrt{X_{ph}^2 - 0.5^2 + 3.964 \times 10^{-10}}} \right) - \arctan \left( \frac{5 \times 10^5 + 2.565 \times 10^{-3}}{\sqrt{X_{ph}^2 - 0.5^2 + 3.964 \times 10^{-10}}} \right), \]

(5.40)

where the \( X_{ph} \) is expressed in units of pc.

The secular parallax can be calculated, using formula (5.40). The results of the secular parallaxes and the values of the physical distances which depend on the redshift for the \( \Lambda \)CDM model can be found in table 5. The graphical representation of the secular parallax, dependent on the redshift of the chosen sources, is shown in figure 5.7.

Figure 5.7: The red line is the graphical representation of the secular parallax depending on \( z \) for the \( \Lambda \)CDM model and the blue line is the graphical representation of the absolute values of the cosmic parallax of two bound sources at a distance of 1Mpc from each other depending on \( z \) for the \( \Lambda \)CDM model.
Figure 5.8: Schematic representation of the secular parallax. The lengths in the image do not have the correct ratio.
In figure 5.7 one can see that the further away the source is, the smaller the secular parallax is. On the other hand, the cosmic parallax at large distances is too small to be measured. This means that the cosmic parallax can be studied at the sources that are located at small distances from us ($z < 0.003$), taking into account the secular parallax.
Chapter 6

Estimation of cosmological parameters

In this chapter, the precise measurement of angles will be applied to estimate cosmological parameters. For this purpose we first consider the mathematical tools which allow the estimation of parameters.

6.1 Likelihood estimation method

The maximum likelihood estimation method of parameters is a popular statistical method. Due to its advantages over some other estimation techniques like the least squares method and the torque, the likelihood estimator is the most important principle for estimating the parameters of a distribution function. This method is used to create a model, based on statistical data, and to provide estimates of the model parameters. The likelihood method is based on the assumption that all the information about the statistical sample is contained in the likelihood function.

We assume that we have a set of N data \( \mathbf{x} = (x_1, x_2, ..., x_N) \) that is described by a joint probability density function \( f(\mathbf{x}; \theta) \), where \( \theta = (\theta_1, \theta_2, ..., \theta_M) \) is a set of unknown parameters for which we need to find estimates. The likelihood function is defined by the probability density function that is evaluated with the data \( \mathbf{x} \), but viewed as a function of the parameters. Thus, \( L(\theta) = f(\mathbf{x}; \theta) \). Since the random variables \( x_i \) are independent, the likelihood function is given by
\[ L(\theta) = \prod_{i=1}^{N} f(x_i; \theta). \] (6.1)

The idea of the maximum likelihood method is that the estimation of the parameter \( \theta \) is given by some vector \( \tilde{\theta} \) which maximizes the likelihood function. That means
\[ L(x_1, x_2, ..., x_N; \tilde{\theta}) = \max_{\theta} L(x_1, x_2, ..., x_N; \theta). \] (6.2)

The necessary condition to maximize a function is
\[
\begin{align*}
\frac{\partial}{\partial \theta_1} L(x_1, x_2, ..., x_N; \theta) &= 0, \\
\frac{\partial}{\partial \theta_2} L(x_1, x_2, ..., x_N; \theta) &= 0, \\
& \cdots \\
\frac{\partial}{\partial \theta_N} L(x_1, x_2, ..., x_N; \theta) &= 0.
\end{align*}
\] (6.3)

These relations are called the likelihood equations. Once they are solved, it must be checked by using the sufficient condition that the found values are really the maxima. For the search of the maxima it is easier to use the natural logarithm of the likelihood function instead of the likelihood function. That means that we use the following equations instead of the equations (6.3)
\[
\begin{align*}
\frac{\partial}{\partial \theta_1} \ln L(x_1, x_2, ..., x_N; \theta) &= 0, \\
\frac{\partial}{\partial \theta_2} \ln L(x_1, x_2, ..., x_N; \theta) &= 0, \\
& \cdots \\
\frac{\partial}{\partial \theta_N} \ln L(x_1, x_2, ..., x_N; \theta) &= 0.
\end{align*}
\] (6.4)

The disadvantage of the likelihood method is that we only obtain the values which give us the best possible fit for a chosen cosmological model and we do not know whether this fit is good in the absolute sense.

These theoretical ideas can be carried out in a given example. Here, a two-variables case is considered. The probability density function is chosen to be a Gaussian with the standard deviation \( \sigma \) and the expectation value \( \mu \). The Gaussian distribution is given by
6.1. LIKELIHOOD ESTIMATION METHOD

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}. \]  

Using (6.5), the likelihood function is built as follows:

\[ L(x_1, x_2, ..., x_N; \mu, \sigma^2) = \prod_{i=1}^{N} f(x_i) = \left( \frac{1}{\sigma \sqrt{2\pi}} \right)^N \prod_{i=1}^{N} \exp\left\{ -\frac{(x_i - \mu)^2}{2\sigma^2} \right\}. \]  

(6.6)

It is more convenient to use the natural logarithm of the likelihood function. So \( l_L \) is defined in the following way

\[ l_L := \ln L(x_1, x_2, ..., x_N; \mu, \sigma^2) = - \frac{N}{2} \ln (\sigma^2) - \frac{N}{2} \ln (2\pi) - \sum_{i=1}^{N} \frac{(x_i - \mu)^2}{2\sigma^2}. \]  

(6.7)

According to the necessary condition, equation (6.4), we obtain

\[ \begin{align*}
\frac{\partial}{\partial \mu} l_L(x; \mu; \sigma^2) &= \sum_{i=1}^{N} \frac{x_i - \mu}{\sigma^2} = 0, \\
\frac{\partial}{\partial \sigma^2} l_L(x; \mu; \sigma^2) &= \sum_{i=1}^{N} \frac{(x_i - \mu)^2}{2(\sigma^2)} - \frac{N}{2\sigma^2} = 0.
\end{align*} \]  

(6.8)

After the calculation of (6.8) we obtain

\[ \begin{align*}
\tilde{\mu} &= \frac{1}{N} \sum_{i=1}^{N} x_i, \\
\tilde{\sigma}^2 &= \frac{1}{N} \sum_{i=1}^{N} (x_i - \tilde{\mu})^2.
\end{align*} \]  

(6.9)

**Remark:** In the following we will drop the tilde over \( \mu \) and \( \sigma \).

To verify that we actually have found a maximum of the likelihood function for the values of \( \mu \) and \( \sigma^2 \), the second derivative of \( l_L \) can be calculated. So we obtain

\[ \begin{align*}
\frac{\partial^2}{\partial \mu^2} l_L(x; \mu; \sigma^2) &= -\frac{N}{\sigma^4}, \\
\frac{\partial^2}{\partial \sigma^2} l_L(x; \mu; \sigma^2) &= -\frac{1}{(\sigma^2)^3} \sum_{i=1}^{N} (x_i - \mu)^2 + \frac{N}{2(\sigma^2)^2}, \\
\frac{\partial^2}{\partial \mu \partial \sigma^2} l_L(x; \mu; \sigma^2) &= -\frac{1}{(\sigma^2)^2} \sum_{i=1}^{N} (x_i - \mu),
\end{align*} \]  

(6.10)
By using the relations (6.9) the second derivatives are given by

\[
\begin{align*}
\frac{\partial^2}{\partial \mu^2} l_L(x; \mu; \sigma^2) &= -\frac{N}{\sigma^2}, \\
\frac{\partial^2}{\partial (\sigma^2)^2} l_L(x; \mu; \sigma^2) &= -\frac{N}{2(\sigma^2)^2}, \\
\frac{\partial^2}{\partial \mu \partial \sigma^2} l_L(x; \mu; \sigma^2) &= 0.
\end{align*}
\] (6.11)

We can see from (6.11) that the second derivatives are negative, hence we really found maxima of the likelihood function [2, 14, 19, 27, 28].

### 6.2 Fisher Information Matrix

The second mathematical tool is the Fisher matrix analysis. The Fisher matrix analysis leads to an estimation of the best possible accuracy of a certain model’s parameters. If assuming some cosmological model \( \mu(p) \) and considering a set of cosmological parameters \( p_i \), where \( i = 1, 2, ..., N \), then the Fisher information matrix is given as

\[
F_{ij} := \frac{\partial^2 l_L}{\partial p_i \partial p_j}.
\] (6.12)

The Fisher information matrix for the Gaussian distribution can be calculated by using the formula (6.7) and the definition (6.12). Thus, we obtain

\[
F_{ij} = \sum_{k=1}^{N} \frac{(x_k - \mu(p))}{\sigma^2} \frac{\partial^2 \mu(p)}{\partial p_i \partial p_j} - \sum_{i=1}^{N} \frac{1}{\sigma^2} \frac{\partial \mu(p)}{\partial p_i} \frac{\partial \mu(p)}{\partial p_j}.
\] (6.13)

Due to equation (6.9) the first term in formula (6.13) is equal to zero, therefore, the Fisher matrix for the Gaussian distribution is given by

\[
F_{ij} = -\sum_{k=1}^{N} \frac{1}{\sigma^2} \frac{\partial \mu(p)}{\partial p_i} \frac{\partial \mu(p)}{\partial p_j}.
\] (6.14)

The best possible \( 1\sigma \) error on the parameter \( p_i \) is given by the square root of the eigenvalues of the covariance matrix \( C \) which is defined by

\[
C = -F^{-1}.
\] (6.15)

The square root of the eigenvalues of the covariance matrix also gives us the axes lengths of the error ellipses. The eigenvectors of the covariance matrix...
show us the direction of the axes of the error ellipses with respect to the coordinate frame. It is known from probability theory that for normally distributed random variables 68.3\% of the realizations lie in the interval ±1σ and 95.4\% lie in the interval ±2σ. The axes lengths of the 95\% probability error ellipse can be obtained by multiplying the axes lengths of the 68\% probability error ellipse by the factor \(\sqrt{6.17}\) [2, 14, 18, 19, 27, 28].

6.3 Fisher matrix analysis of angular sizes in the \(ΛCDM\) model

The Fisher matrix analysis can be used to estimate the determination accuracy of the parameters for a selected model. In this section the Fisher matrix analysis is performed for the estimation of cosmological parameters with the assumption of using the data of the angular size measurement. The Fisher matrix will be calculated for different assumptions: the first one is for an one-dimensional case, which means that only one cosmological parameter is variable, and the second one is for the case that two parameters are variable simultaneously. First of all, the likelihood function for the \(ΛCDM\) model should be established. For this purpose, the formula for an angle in the \(ΛCDM\) model (5.21) from the chapter 5.2 is inserted into the definition of the \(\ln L\) (6.7), so we obtain

\[
l_L = -\frac{N}{2} \ln (\sigma^2) - \frac{N}{2} \ln (2\pi) - \sum_{i=1}^{N} \left( \theta_i - \frac{68.755 p_1 (1 + z) \int \frac{dz'}{\sqrt{p_2 (1 + z')^3 (1 - p_2)}}}{2 \sigma^2} \right)^2 \]

where \(\theta_i\) is the i-th angle measurement, \(N\) is the number of measurements, \(\sigma\) is the standard deviation of performed measurements, \(p_1\) and \(p_2\) are cosmological parameters whose accuracy should be determined and which are chosen to have fictive values of

\[
p_1 \equiv \left( \frac{l}{\text{Mpc}} \right) h = 0.678, \quad p_2 \equiv \Omega_m = 0.308.
\]

First, we calculate the Fisher matrix with the assumption that one of the parameter can be varied and the other is fixed.
CHAPTER 6. ESTIMATION OF COSMOLOGICAL PARAMETERS

Case 1: \( p_1 \) is variable, \( p_2 \) is constant.

Using equation (6.16) the first derivative of \( l_L \) is given by

\[
\frac{\partial l_L}{\partial p_1} = \frac{1}{\sigma^2} \sum_{i=1}^{N} \left( \frac{\theta_i - 68.755p_1(1 + z)}{\int_0^z \frac{dz'}{\sqrt{p_2(1+z')^3}+(1-p_2)}} \right) \int_0^z \frac{68.755(1 + z)}{\sqrt{p_2(1+z')^3}+(1-p_2)} .
\] (6.19)

Because of the maximum likelihood formalism the first derivative must be equal to 0. Thus, the following relation is obtained

\[
\sum_{i=1}^{N} \theta_i = \frac{68.755p_1N(1 + z)}{\int_0^z \frac{dz'}{\sqrt{p_2(1+z')^3}+(1-p_2)}}.
\] (6.20)

Thus, according to the equations (6.14), (6.19) and (6.20), we obtain the Fisher matrix for the fixed \( p_2 \) and variable \( p_1 \). It is

\[
F_{11} = \frac{\partial^2 l_L}{\partial p_1^2} = -\frac{N}{\sigma^2} \frac{68.755^2 (1 + z)^2}{\left( \int_0^z \frac{dz'}{\sqrt{p_2(1+z')^3}+(1-p_2)} \right)^2}.
\] (6.21)

Using the definition (6.15) the covariance matrix is given by

\[
C_{11} = \frac{\sigma^2}{N} \frac{\left( \int_0^z \frac{dz'}{\sqrt{p_2(1+z')^3}+(1-p_2)} \right)^2}{68.755^2(1 + z)^2}.
\] (6.22)

Thus, the best possible 1σ error on \( p_1 \), which is given according to \( \Delta p_1 = \sqrt{C_{11}} \), is equal to

\[
\Delta p_1 = \frac{\sigma}{\sqrt{N}} \frac{\int_0^z \frac{dz'}{\sqrt{p_2(1+z')^3}+(1-p_2)}}{68.755(1 + z)}.
\] (6.23)

The results of the calculation for different \( z \) are shown in table 6.

Case 2: \( p_2 \) is variable, \( p_1 \) is constant.

After an analogous calculation we obtain the estimation of accuracy for \( p_2 \). It is given by
6.3. FISHER MATRIX ANALYSIS IN THE $\Lambda$CDM MODEL

$$\Delta p_2 = \frac{\sigma}{\sqrt{N}} 68.755 p_1 (1 + z) \int_0^z \frac{d'z'}{(p_2 (1 + z') + (1 - p_2))} \cdot$$

(6.24)

The results for different $z$ are also shown in table 6 in appendix. The graphical representation of the 1σ errors of the parameters $p_1$ and $p_2$ are shown in the figure 6.1.

![Figure 6.1](image)

Figure 6.1: Estimated error of the parameters $p_1$, $p_2$ for two sources at a distance of 1Mpc from each other and at a distance of $z = 0.1$ from observer (top left), $z = 1$ (top right), $z = 4$ (bottom left) and $z = 8$ (bottom right)

The parameters in the figure 6.1 are plotted, assuming the observational accuracy of the VLA FIRST survey (VLA Faint Images of Radio Sky at Twenty cm). In this survey about $N = 10^6$ radio sources were observed with a precision of $\sigma = 5\text{as}$. We assume that the radio sources, which we consider here, are distributed as it is shown in table 2. This distribution of the radio
sources in the universe can be found in [38].

<table>
<thead>
<tr>
<th>( z )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_z )</td>
<td>( 10^3 )</td>
<td>( 10^3 )</td>
<td>( 10^3 )</td>
<td>( 2 \times 10^4 )</td>
<td>( 1.25 \times 10^5 )</td>
<td>( 7.5 \times 10^4 )</td>
<td>( 6 \times 10^4 )</td>
</tr>
</tbody>
</table>

| \( z \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| \( N_z \) | \( 3 \times 10^4 \) | \( 10^4 \) | \( 5 \times 10^5 \) | \( 10^4 \) | \( 10^3 \) | \( 10^4 \) | \( 10^2 \) |

Table 2: Distribution of the radio sources [38], where \( N \) is the number of radio sources at the given redshift \( z \).

It should be noted that all further calculations of the \( 1\sigma \) and \( 2\sigma \) errors of the parameters \( p_1 \) and \( p_2 \) were made, assuming that the measurement accuracy of \( \sigma = 5\)as applies to each redshifts.

**Case 3:** \( p_1 \) and \( p_2 \) are variable.

In the next step we assume that both parameters \( p_1 \) and \( p_2 \) are variable. That means that we consider a two-dimensional case and the following \( 2 \times 2 \) Fisher matrix is obtained

\[
F = -\sum_z \frac{N_z}{\sigma^2} \left( \left( \frac{\partial \theta(z)}{\partial p_1} \right)^2 \left( \frac{\partial \theta(z)}{\partial p_2} \right)^2 \right) .
\]  

(6.25)

For the calculation of the Fisher matrix for two unknown parameters we need at least two angle measurements at different redshifts. These redshifts can be chosen freely. So we consider three different classes of bins in the redshift:

**The first class** contains the following bins of redshift: \([0.1,0.2], [0.5,1], [2,3], [4,5], [6,7], [8,9]\).

**The second class** contains the following bins of redshift: \([0.5,1], [0.5,1.5], [0.5,2], [0.5,2.5], [0.5,3],[0.5,4], [0.5,5], [0.5,6]\).

**The third case** contains only one bin of five different redshifts which are considered simultaneously: \([1, 2, 3, 4, 5]\).
The Fisher matrix for the first two classes is given as following:

\[
F = -\frac{1}{\sigma^2} \left( N_{l_z} \left( \frac{\partial \theta}{\partial p_1} \right)_{l_z}^2 + N_{h_z} \left( \frac{\partial \theta}{\partial p_1} \right)_{h_z}^2 + N_{l_z} \frac{\partial \theta}{\partial p_1} \frac{\partial \theta}{\partial p_2} \bigg|_{l_z} + N_{h_z} \frac{\partial \theta}{\partial p_1} \frac{\partial \theta}{\partial p_2} \bigg|_{h_z} \right),
\]

where \( l_z \) is the low redshift boundary of the given bin and \( h_z \) is the high redshift boundary of the given bin. \( N_{l_z} \) is the number of sources measured at the low redshift of the given bin and \( N_{h_z} \) is the number of sources measured at the high redshift of the given bin.

In table 7 in appendix we can see the obtained Fisher matrices and the covariance matrices for the first class of bins. To obtain the error ellipses, the eigenvalues and the eigenvectors of the covariance matrix, as well as the 1\( \sigma \) and the 2\( \sigma \) errors of the parameters \( p_1 \) und \( p_2 \) must be calculated. The results of these calculations are shown in table 8. In table 9 we can see the Fisher and the covariance matrices for the second class of redshift bins and in table 10, the results of the calculation of the eigenvalues and the eigenvectors of the covariance matrix of the second class are shown. The Fisher matrix for the third class contains five summands in each entry. The results of this are shown in table 11 und table 12.

The graphical representation of the error ellipses for the three different classes are shown in figures 6.2 - 6.4. The figure 6.2 was plotted, assuming the observational accuracy of the VLA FIRST survey and the distribution of the radio sources in the universe which was described in table 2.

If one uses the accuracy of the MOJAVE survey (Monitoring Of Jets in Active galactic nuclei with the VLBA Experiments), one obtains \( N \approx 0.5 \times 10^3 \) and \( \sigma \approx 10^{-5} \)as. That means that the error ellipses of this accuracy are at least four order of magnitude higher than with the values of VLA. Using the accuracy of the RadioAstron (see section 1.3.3), we obtain that the accuracy of the estimated parameters \( h \) and \( \Omega_M \) is at least two orders of magnitude higher than with the values of MOJAVE VLBA. That means, if we would plot the error ellipses with the accuracy of MOJAVE VLBA, they would be smaller by a factor of about \( 10^4 \) than those which are shown here in figures 6.2 - 6.4. The conclusion is that, using the measurement of an angle with the accuracy of the missions VLBA or RadioAstron, one can determine the cosmological parameters with very high precision.
CHAPTER 6. ESTIMATION OF COSMOLOGICAL PARAMETERS

Figure 6.2: 1σ and 2σ error ellipses for the first class of the red shift bins with the accuracy of VLA FIRST survey data.

Figure 6.3: 1σ and 2σ error ellipses for the second class of the red shift bins with the accuracy of VLA FIRST survey data.
If we analyze the results in tables 6, 8, 10 and 12, we can notice that there is no correlation between the values of the 1σ and 2σ errors which were calculated by the variation of one parameter and the values that were found by the simultaneous variation of the two parameters. The reason is that in the case of the variation of one parameter, the calculation was made, using the exact knowledge of the fixed parameter.

By comparing the results of calculations of three classes of redshifts, we can see that the most accurate estimation of the parameters is obtained if the measurements of angles are performed at the five different redshifts which are considered simultaneously. From figures 6.2 and 6.3 one can see that by using the redshift bin of the first class, the parameter $p_2$ can be determined more accurately than $p_1$. The parameter $p_1$ can be determined more accurately than $p_2$ if we use the redshift bin of the second class.

However, one should notice that the astrophysics of jets have not been yet understood to a level that would allow us to know their precisely sizes. This stops us from obtaining super-precise measurements of the cosmological parameters by means of radio observations.
Chapter 7

Conclusion

Due to the rapid development of technology in recent decades, new ways have been found in every area of physics, especially, in astronomy and cosmology. Using modern facilities like the VLBA or RadioAstron, it has become possible to carry out angle measurements with very high precision.

The aim of this thesis was to investigate some applications in cosmology of the precise angle measurement theoretically. The first possible application which was treated here was to study the homogeneity of space on the large scales. The homogeneity of space is impossible to check in a direct way. Therefore, the precise measurement of angles provides a good chance to investigate the homogeneity indirectly by using the cosmic parallax effect. For this purpose, two types of calculations were performed. First of all, two comoving sources which were placed at a distance of 1Mpc from each other were investigated. Depending on the result of the measurement of the angle between these sources and the observer, there are two ways to proceed: a) If we "live" in an inhomogeneous universe, we will measure a cosmic parallax which is caused by the inhomogeneous expansion of space. The magnitude of the cosmic parallax was estimated, for example, in [18]. b) If we "live" in a homogeneous universe, we expect that there is no cosmic parallax for two comoving sources. In this case, two gravitationally bound sources can be investigated to find a change of angular separation. The theoretical prediction of the change of the angular separation in the $\Lambda$CDM model was calculated, thus, this method is an independent proof of the $\Lambda$CDM model. Significant deviations from the expected values of the CP in the $\Lambda$CDM would require the consideration of alternative models.

The second application of precise angle measurements, which has been
discussed here, is the estimation of the cosmological parameters in the ΛCDM model. The accuracy of the estimation of the parameters Ω_M and h was determined with the precision of the VLA FIRST survey, MOJAVE and RadioAstron. Due to the calculated accuracy it was found that the precise angle measurement is also a good method for the determination of the cosmological parameters. However, we still need to identify good standard rules of the Mpc scale to make this option come true.
Chapter 8

Appendix

<table>
<thead>
<tr>
<th>z</th>
<th>$\Delta_iz$</th>
<th>$\Delta_i\theta$ [arcsec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>$3.72547 \times 10^{-13}$</td>
<td>$-1.73666 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.003</td>
<td>$1.11616 \times 10^{-12}$</td>
<td>$-5.78120 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.01</td>
<td>$3.70332 \times 10^{-12}$</td>
<td>$-1.72628 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.03</td>
<td>$1.09625 \times 10^{-11}$</td>
<td>$-5.67633 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.07</td>
<td>$2.48926 \times 10^{-11}$</td>
<td>$-2.36418 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.09</td>
<td>$3.15650 \times 10^{-11}$</td>
<td>$-1.81152 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.1</td>
<td>$3.48284 \times 10^{-11}$</td>
<td>$-1.61795 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$9.01465 \times 10^{-11}$</td>
<td>$-4.51525 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$1.27592 \times 10^{-10}$</td>
<td>$-2.15330 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$1.48984 \times 10^{-10}$</td>
<td>$-1.14825 \times 10^{-8}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$1.56126 \times 10^{-10}$</td>
<td>$-6.13505 \times 10^{-9}$</td>
</tr>
<tr>
<td>1</td>
<td>$1.54859 \times 10^{-10}$</td>
<td>$-4.38390 \times 10^{-9}$</td>
</tr>
<tr>
<td>1.3</td>
<td>$1.33737 \times 10^{-10}$</td>
<td>$-1.23466 \times 10^{-9}$</td>
</tr>
<tr>
<td>1.5</td>
<td>$1.06595 \times 10^{-10}$</td>
<td>$-2.55238 \times 10^{-10}$</td>
</tr>
<tr>
<td>2</td>
<td>$-9.23694 \times 10^{-13}$</td>
<td>$-6.01899 \times 10^{-12}$</td>
</tr>
<tr>
<td>3</td>
<td>$-3.58296 \times 10^{-10}$</td>
<td>$-4.52152 \times 10^{-9}$</td>
</tr>
<tr>
<td>4</td>
<td>$-8.73328 \times 10^{-10}$</td>
<td>$-1.27249 \times 10^{-8}$</td>
</tr>
<tr>
<td>5</td>
<td>$-1.52359 \times 10^{-9}$</td>
<td>$-2.34163 \times 10^{-8}$</td>
</tr>
<tr>
<td>6</td>
<td>$-2.29492 \times 10^{-9}$</td>
<td>$-3.61168 \times 10^{-8}$</td>
</tr>
<tr>
<td>7</td>
<td>$-3.17722 \times 10^{-9}$</td>
<td>$-5.05686 \times 10^{-8}$</td>
</tr>
<tr>
<td>8</td>
<td>$-4.16272 \times 10^{-9}$</td>
<td>$-6.6605 \times 10^{-8}$</td>
</tr>
<tr>
<td>9</td>
<td>$-5.24518 \times 10^{-9}$</td>
<td>$-8.4106 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 3: Change of redshift in 10 years for different initial values of $z$ in the $\Lambda$CDM model and cosmic parallax with respect to $z$ in the $\Lambda$CDM model for two gravitationally bound sources, placed at a distance of 1Mpc from each other.
<table>
<thead>
<tr>
<th>$z$</th>
<th>$\Delta t \theta$ [arcsec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.286183</td>
</tr>
<tr>
<td>0.003</td>
<td>0.095629</td>
</tr>
<tr>
<td>0.01</td>
<td>0.028940</td>
</tr>
<tr>
<td>0.03</td>
<td>0.009882</td>
</tr>
<tr>
<td>0.07</td>
<td>0.004442</td>
</tr>
<tr>
<td>0.09</td>
<td>0.003537</td>
</tr>
<tr>
<td>0.1</td>
<td>0.003220</td>
</tr>
<tr>
<td>0.3</td>
<td>0.001335</td>
</tr>
<tr>
<td>0.5</td>
<td>0.000975</td>
</tr>
<tr>
<td>0.7</td>
<td>0.000834</td>
</tr>
<tr>
<td>0.9</td>
<td>0.000765</td>
</tr>
<tr>
<td>1</td>
<td>0.000745</td>
</tr>
<tr>
<td>1.3</td>
<td>0.000712</td>
</tr>
<tr>
<td>1.5</td>
<td>0.000706</td>
</tr>
<tr>
<td>2</td>
<td>0.000714</td>
</tr>
<tr>
<td>3</td>
<td>0.000777</td>
</tr>
<tr>
<td>4</td>
<td>0.000861</td>
</tr>
<tr>
<td>5</td>
<td>0.000953</td>
</tr>
<tr>
<td>6</td>
<td>0.001048</td>
</tr>
<tr>
<td>7</td>
<td>0.001146</td>
</tr>
<tr>
<td>8</td>
<td>0.001243</td>
</tr>
<tr>
<td>9</td>
<td>0.001342</td>
</tr>
</tbody>
</table>

Table 4: Cosmic parallax, taking into account jets expansion of the speed of light with respect to $z$ in the $\Lambda$CDM model for two gravitationally bound sources which are at initial distance $l_1 = 1\text{Mpc}$ from each other.
<table>
<thead>
<tr>
<th>$z$</th>
<th>$X_{ph}[Mpc]$</th>
<th>$\Delta \theta_s[arcsec]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>4.42400</td>
<td>573.6470</td>
</tr>
<tr>
<td>0.003</td>
<td>13.2650</td>
<td>21.99820</td>
</tr>
<tr>
<td>0.01</td>
<td>44.1450</td>
<td>0.599158</td>
</tr>
<tr>
<td>0.03</td>
<td>131.818</td>
<td>$2.25124 \times 10^{-2}$</td>
</tr>
<tr>
<td>0.07</td>
<td>304.661</td>
<td>$1.82352 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.1</td>
<td>432.074</td>
<td>$6.39277 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.3</td>
<td>1231.86</td>
<td>$2.75855 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.5</td>
<td>1945.88</td>
<td>$6.99869 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.7</td>
<td>2579.39</td>
<td>$3.00479 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.9</td>
<td>3140.79</td>
<td>$1.66436 \times 10^{-6}$</td>
</tr>
<tr>
<td>1</td>
<td>3397.34</td>
<td>$1.31507 \times 10^{-6}$</td>
</tr>
<tr>
<td>1.3</td>
<td>4083.69</td>
<td>$7.57193 \times 10^{-7}$</td>
</tr>
<tr>
<td>1.5</td>
<td>4481.73</td>
<td>$5.72833 \times 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>5315.37</td>
<td>$3.43372 \times 10^{-7}$</td>
</tr>
<tr>
<td>3</td>
<td>6515.54</td>
<td>$1.86429 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>7346.63</td>
<td>$1.30047 \times 10^{-7}$</td>
</tr>
<tr>
<td>5</td>
<td>7963.69</td>
<td>$1.02999 \times 10^{-7}$</td>
</tr>
<tr>
<td>6</td>
<td>8444.59</td>
<td>$8.56307 \times 10^{-8}$</td>
</tr>
<tr>
<td>7</td>
<td>8832.79</td>
<td>$7.48293 \times 10^{-8}$</td>
</tr>
<tr>
<td>8</td>
<td>9154.62</td>
<td>$6.72116 \times 10^{-8}$</td>
</tr>
<tr>
<td>9</td>
<td>9427.02</td>
<td>$6.15520 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 5: Physical distance and secular parallax with respect to $z$ in the ΛCDM model for two gravitationally bound sources, placed at a distance of 1Mpc from each other.
<table>
<thead>
<tr>
<th>$z$</th>
<th>$\frac{\sqrt{N}}{\sigma} \Delta p_1$</th>
<th>$\frac{\sqrt{N}}{\sigma} \Delta p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.001291</td>
<td>0.261656</td>
</tr>
<tr>
<td>0.2</td>
<td>0.002300</td>
<td>0.121477</td>
</tr>
<tr>
<td>0.5</td>
<td>0.004264</td>
<td>0.041603</td>
</tr>
<tr>
<td>1</td>
<td>0.005584</td>
<td>0.018330</td>
</tr>
<tr>
<td>2</td>
<td>0.005824</td>
<td>0.008493</td>
</tr>
<tr>
<td>3</td>
<td>0.005354</td>
<td>0.005569</td>
</tr>
<tr>
<td>4</td>
<td>0.004830</td>
<td>0.004161</td>
</tr>
<tr>
<td>5</td>
<td>0.004363</td>
<td>0.003328</td>
</tr>
<tr>
<td>6</td>
<td>0.003965</td>
<td>0.002774</td>
</tr>
<tr>
<td>7</td>
<td>0.003629</td>
<td>0.002380</td>
</tr>
<tr>
<td>8</td>
<td>0.003344</td>
<td>0.002084</td>
</tr>
</tbody>
</table>

Table 6: Best possible 1σ error on $p_1$ and $p_2$ with respect to $z$ in the ΛCDM model (for $p_2$ or $p_1$ fixed, respectively).
<table>
<thead>
<tr>
<th>$z$ bin</th>
<th>Fisher matrix</th>
<th>Covariance matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1-0.2</td>
<td>$-\frac{1}{\sigma^2}$ \begin{pmatrix} 7.87501 \times 10^{10} &amp; 4.90337 \times 10^9 \ 4.90337 \times 10^9 &amp; 3.39964 \times 10^8 \end{pmatrix}</td>
<td>$\sigma^2$ \begin{pmatrix} 1.24567 \times 10^{-10} &amp; -1.79666 \times 10^{-9} \ -1.79666 \times 10^{-9} &amp; 2.88555 \times 10^{-8} \end{pmatrix}</td>
</tr>
<tr>
<td>0.5-1</td>
<td>$-\frac{1}{\sigma^2}$ \begin{pmatrix} 1.19149 \times 10^{10} &amp; 2.55435 \times 10^9 \ 2.55435 \times 10^9 &amp; 1.30848 \times 10^9 \end{pmatrix}</td>
<td>$\sigma^2$ \begin{pmatrix} 1.18280 \times 10^{-10} &amp; -3.72116 \times 10^{-11} \ -3.72116 \times 10^{-11} &amp; 4.03196 \times 10^{-11} \end{pmatrix}</td>
</tr>
<tr>
<td>2-3</td>
<td>$-\frac{1}{\sigma^2}$ \begin{pmatrix} 3.25768 \times 10^9 &amp; 1.94555 \times 10^9 \ 1.94555 \times 10^9 &amp; 1.16669 \times 10^9 \end{pmatrix}</td>
<td>$\sigma^2$ \begin{pmatrix} 7.49123 \times 10^{-8} &amp; -1.24922 \times 10^{-7} \ -1.24922 \times 10^{-7} &amp; 2.00173 \times 10^{-7} \end{pmatrix}</td>
</tr>
<tr>
<td>4-5</td>
<td>$-\frac{1}{\sigma^2}$ \begin{pmatrix} 6.91387 \times 10^8 &amp; 4.91040 \times 10^8 \ 4.91040 \times 10^8 &amp; 3.48890 \times 10^8 \end{pmatrix}</td>
<td>$\sigma^2$ \begin{pmatrix} 2.25481 \times 10^{-9} &amp; -1.60207 \times 10^{-9} \ -1.60207 \times 10^{-9} &amp; 3.17479 \times 10^{-9} \end{pmatrix}</td>
</tr>
<tr>
<td>6-7</td>
<td>$-\frac{1}{\sigma^2}$ \begin{pmatrix} 1.3952 \times 10^8 &amp; 1.05634 \times 10^8 \ 1.05634 \times 10^8 &amp; 7.99858 \times 10^7 \end{pmatrix}</td>
<td>$\sigma^2$ \begin{pmatrix} 7.25192 \times 10^{-5} &amp; -9.61691 \times 10^{-5} \ -9.61691 \times 10^{-5} &amp; 1.27019 \times 10^{-5} \end{pmatrix}</td>
</tr>
<tr>
<td>8-9</td>
<td>$-\frac{1}{\sigma^2}$ \begin{pmatrix} 9.98678 \times 10^7 &amp; 7.75569 \times 10^7 \ 7.75569 \times 10^7 &amp; 6.02311 \times 10^7 \end{pmatrix}</td>
<td>$\sigma^2$ \begin{pmatrix} 7.52281 \times 10^{-4} &amp; -9.68678 \times 10^{-4} \ -9.68678 \times 10^{-4} &amp; 1.24734 \times 10^{-3} \end{pmatrix}</td>
</tr>
</tbody>
</table>

Table 7: Fisher matrices and covariance matrices for the redshift bins of the first class.
<table>
<thead>
<tr>
<th>$z$ bins</th>
<th>eigenvalues</th>
<th>eigenv</th>
<th>$1\sigma$ errors</th>
<th>$2\sigma$ errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1-0.2</td>
<td>$\sigma^22.89670 \times 10^{-8}$</td>
<td>$\begin{pmatrix} -0.06 \ 1 \end{pmatrix}$</td>
<td>$\Delta p_1 = \sigma 1.70197 \times 10^{-4}$</td>
<td>$\Delta p_1 = \sigma 4.22761 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^21.26493 \times 10^{-11}$</td>
<td></td>
<td>$\Delta p_2 = \sigma 3.55700 \times 10^{-6}$</td>
<td>$\Delta p_2 = \sigma 8.83400 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.5-1</td>
<td>$\sigma^24.07001 \times 10^{-10}$</td>
<td>$\begin{pmatrix} -0.13 \ 1 \end{pmatrix}$</td>
<td>$\Delta p_1 = \sigma 2.01980 \times 10^{-5}$</td>
<td>$\Delta p_1 = \sigma 5.01710 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^21.13500 \times 10^{-10}$</td>
<td></td>
<td>$\Delta p_2 = \sigma 1.06540 \times 10^{-5}$</td>
<td>$\Delta p_2 = \sigma 2.64630 \times 10^{-5}$</td>
</tr>
<tr>
<td>2-3</td>
<td>$\sigma^22.83859 \times 10^{-7}$</td>
<td>$\begin{pmatrix} -0.59 \ 1 \end{pmatrix}$</td>
<td>$\Delta p_1 = \sigma 5.32784 \times 10^{-4}$</td>
<td>$\Delta p_1 = \sigma 1.32341 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^22.26201 \times 10^{-10}$</td>
<td></td>
<td>$\Delta p_2 = \sigma 1.50400 \times 10^{-5}$</td>
<td>$\Delta p_2 = \sigma 3.73590 \times 10^{-5}$</td>
</tr>
<tr>
<td>4-5</td>
<td>$\sigma^24.38160 \times 10^{-9}$</td>
<td>$\begin{pmatrix} -0.75 \ 1 \end{pmatrix}$</td>
<td>$\Delta p_1 = \sigma 6.61940 \times 10^{-5}$</td>
<td>$\Delta p_1 = \sigma 1.64422 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^21.04800 \times 10^{-9}$</td>
<td></td>
<td>$\Delta p_2 = \sigma 3.23730 \times 10^{-5}$</td>
<td>$\Delta p_2 = \sigma 8.04120 \times 10^{-5}$</td>
</tr>
<tr>
<td>6-7</td>
<td>$\sigma^21.99833 \times 10^{-4}$</td>
<td>$\begin{pmatrix} -0.75 \ 1 \end{pmatrix}$</td>
<td>$\Delta p_1 = \sigma 1.4136 \times 10^{-2}$</td>
<td>$\Delta p_1 = \sigma 3.51136 \times 10^{-2}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^24.55580 \times 10^{-9}$</td>
<td></td>
<td>$\Delta p_2 = \sigma 6.74970 \times 10^{-5}$</td>
<td>$\Delta p_2 = \sigma 1.67658 \times 10^{-4}$</td>
</tr>
<tr>
<td>8-9</td>
<td>$\sigma^21.99961 \times 10^{-3}$</td>
<td>$\begin{pmatrix} -0.78 \ 1 \end{pmatrix}$</td>
<td>$\Delta p_1 = \sigma 4.47170 \times 10^{-2}$</td>
<td>$\Delta p_1 = \sigma 1.11107 \times 10^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^26.24616 \times 10^{-9}$</td>
<td></td>
<td>$\Delta p_2 = \sigma 7.90330 \times 10^{-5}$</td>
<td>$\Delta p_2 = \sigma 1.96313 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 8: Results for eigenvalues, eigenvectors and $1\sigma$, $2\sigma$ errors for the redshift bins of the first class.
<table>
<thead>
<tr>
<th>$z$ bin</th>
<th>Fisher matrix</th>
<th>covariance matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5-1</td>
<td>$\frac{-1}{\sigma^2}$ ((1.19149 \times 10^{10}, 2.55435 \times 10^9)) ((2.55435 \times 10^9, 1.30848 \times 10^9))</td>
<td>$\sigma^2$ ((1.18280 \times 10^{-10}, -3.72116 \times 10^{-11})) ((-3.72116 \times 10^{-11}, 4.03196 \times 10^{-11}))</td>
</tr>
<tr>
<td>0.5-1.5</td>
<td>$\frac{-1}{\sigma^2}$ ((9.09966 \times 10^9, 3.08834 \times 10^9)) ((3.08834 \times 10^9, 1.20532 \times 10^9))</td>
<td>$\sigma^2$ ((8.42796 \times 10^{-10}, -2.15946 \times 10^{-9})) ((-2.15946 \times 10^{-9}, 6.36277 \times 10^{-9}))</td>
</tr>
<tr>
<td>0.5-2</td>
<td>$\frac{-1}{\sigma^2}$ ((7.71095 \times 10^9, 2.54409 \times 10^9)) ((2.54409 \times 10^9, 1.01934 \times 10^9))</td>
<td>$\sigma^2$ ((7.34568 \times 10^{-10}, -1.83335 \times 10^{-9})) ((-1.83335 \times 10^{-9}, 5.55676 \times 10^{-9}))</td>
</tr>
<tr>
<td>0.5-2.5</td>
<td>$\frac{-1}{\sigma^2}$ ((7.40422 \times 10^9, 2.45966 \times 10^9)) ((2.45966 \times 10^9, 1.02721 \times 10^9))</td>
<td>$\sigma^2$ ((6.60248 \times 10^{-10}, -1.58096 \times 10^{-9})) ((-1.58096 \times 10^{-9}, 4.75912 \times 10^{-9}))</td>
</tr>
<tr>
<td>0.5-3</td>
<td>$\frac{-1}{\sigma^2}$ ((6.54624 \times 10^9, 1.96505 \times 10^9)) ((1.96505 \times 10^9, 7.44840 \times 10^8))</td>
<td>$\sigma^2$ ((7.34216 \times 10^{-10}, -1.93702 \times 10^{-9})) ((-1.93702 \times 10^{-9}, 6.45287 \times 10^{-9}))</td>
</tr>
<tr>
<td>0.5-4</td>
<td>$\frac{-1}{\sigma^2}$ ((5.32846 \times 10^9, 1.58148 \times 10^9)) ((1.58148 \times 10^9, 5.08233 \times 10^8))</td>
<td>$\sigma^2$ ((9.92731 \times 10^{-10}, -3.08911 \times 10^{-9})) ((-3.08911 \times 10^{-9}, 1.15801 \times 10^{-8}))</td>
</tr>
<tr>
<td>0.5-5</td>
<td>$\frac{-1}{\sigma^2}$ ((5.76244 \times 10^9, 1.47316 \times 10^9)) ((1.47316 \times 10^9, 4.38142 \times 10^8))</td>
<td>$\sigma^2$ ((1.23569 \times 10^{-9}, -4.15475 \times 10^{-9})) ((-4.15475 \times 10^{-9}, 1.62518 \times 10^{-8}))</td>
</tr>
<tr>
<td>0.5-6</td>
<td>$\frac{-1}{\sigma^2}$ ((5.56335 \times 10^9, 1.32943 \times 10^9)) ((1.32943 \times 10^9, 3.34412 \times 10^8))</td>
<td>$\sigma^2$ ((3.59300 \times 10^{-10}, -1.42837 \times 10^{-8})) ((-1.42837 \times 10^{-8}, 5.97739 \times 10^{-8}))</td>
</tr>
</tbody>
</table>

Table 9: Fisher matrix and covariance matrix for the redshift bins of the second class.
<table>
<thead>
<tr>
<th>$z$ bins</th>
<th>eigenvalues</th>
<th>eigenvec</th>
<th>1σ errors</th>
<th>2σ errors</th>
</tr>
</thead>
</table>
| 0.5-1    | $\sigma^2 4.07001 \times 10^{-10}$  
$\sigma^2 1.13500 \times 10^{-10}$ | $(-0.13)$  
$1$ | $\Delta p_1 = \sigma 2.01980 \times 10^{-5}$  
$\Delta p_1 = \sigma 1.06540 \times 10^{-5}$ | $\Delta p_2 = \sigma 5.01710 \times 10^{-5}$  
$\Delta p_2 = \sigma 2.64630 \times 10^{-5}$ |
| 0.5-1.5  | $\sigma^2 7.10718 \times 10^{-9}$  
$\sigma^2 9.83839 \times 10^{-11}$ | $(-0.35)$  
$1$ | $\Delta p_1 = \sigma 8.43040 \times 10^{-5}$  
$\Delta p_1 = \sigma 9.91900 \times 10^{-6}$ | $\Delta p_2 = \sigma 2.09407 \times 10^{-4}$  
$\Delta p_2 = \sigma 2.46380 \times 10^{-5}$ |
| 0.5-2    | $\sigma^2 6.17461 \times 10^{-9}$  
$\sigma^2 1.16709 \times 10^{-10}$ | $(-0.34)$  
$1$ | $\Delta p_1 = \sigma 7.85790 \times 10^{-5}$  
$\Delta p_1 = \sigma 1.08030 \times 10^{-5}$ | $\Delta p_2 = \sigma 1.95186 \times 10^{-4}$  
$\Delta p_2 = \sigma 2.68350 \times 10^{-5}$ |
| 0.5-2.5  | $\sigma^2 5.29805 \times 10^{-9}$  
$\sigma^2 1.21320 \times 10^{-10}$ | $(-0.34)$  
$1$ | $\Delta p_1 = \sigma 7.27880 \times 10^{-5}$  
$\Delta p_1 = \sigma 1.10150 \times 10^{-5}$ | $\Delta p_2 = \sigma 1.80801 \times 10^{-4}$  
$\Delta p_2 = \sigma 2.73600 \times 10^{-5}$ |
| 0.5-3    | $\sigma^2 7.04721 \times 10^{-9}$  
$\sigma^2 1.39876 \times 10^{-10}$ | $(-0.31)$  
$1$ | $\Delta p_1 = \sigma 8.39480 \times 10^{-5}$  
$\Delta p_1 = \sigma 1.18270 \times 10^{-5}$ | $\Delta p_2 = \sigma 2.08523 \times 10^{-4}$  
$\Delta p_2 = \sigma 2.93770 \times 10^{-5}$ |
| 0.5-4    | $\sigma^2 1.24155 \times 10^{-8}$  
$\sigma^2 1.57328 \times 10^{-10}$ | $(-0.27)$  
$1$ | $\Delta p_1 = \sigma 1.11425 \times 10^{-4}$  
$\Delta p_2 = \sigma 1.25430 \times 10^{-5}$ | $\Delta p_2 = \sigma 2.76774 \times 10^{-4}$  
$\Delta p_2 = \sigma 3.11560 \times 10^{-5}$ |
### Table 10: Results for eigenvalues, eigenvectors and $1\sigma$, $2\sigma$ errors for the redshift bins of the second class.

<table>
<thead>
<tr>
<th>$z$ bins</th>
<th>eigenvalues</th>
<th>eigenvector</th>
<th>$1\sigma$ errors</th>
<th>$2\sigma$ errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5-5</td>
<td>$\sigma^21.73240 \times 10^{-8}$</td>
<td>$(0.26)$</td>
<td>$\Delta p_1 = \sigma1.31621 \times 10^{-4}$</td>
<td>$\Delta p_1 = \sigma3.26939 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^21.62791 \times 10^{-10}$</td>
<td>1</td>
<td>$\Delta p_2 = \sigma1.27590 \times 10^{-5}$</td>
<td>$\Delta p_2 = \sigma3.16930 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.5-6</td>
<td>$\sigma^26.31969 \times 10^{-8}$</td>
<td>$(0.24)$</td>
<td>$\Delta p_1 = \sigma2.51390 \times 10^{-4}$</td>
<td>$\Delta p_1 = \sigma6.24440 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^21.70012 \times 10^{-10}$</td>
<td>1</td>
<td>$\Delta p_2 = \sigma1.30390 \times 10^{-5}$</td>
<td>$\Delta p_2 = \sigma3.23880 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

### Table 11: Fisher matrix and covariance matrix for the redshift bins of the third class.

<table>
<thead>
<tr>
<th>$z$ bin</th>
<th>Fisher matrix</th>
<th>covariance matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 5</td>
<td>$-\frac{1}{\sigma^2}$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td></td>
<td>$\begin{pmatrix} 1.03642 \times 10^{10} &amp; 4.98169 \times 10^9 \ 4.98169 \times 10^9 &amp; 2.52532 \times 10^9 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1.86286 \times 10^{-9} &amp; -3.67486 \times 10^{-9} \ -3.67486 \times 10^{-9} &amp; 7.64537 \times 10^{-9} \end{pmatrix}$</td>
</tr>
</tbody>
</table>

### Table 12: Results for eigenvalues, eigenvectors and $1\sigma$, $2\sigma$ errors for the redshift bins of the third class.

<table>
<thead>
<tr>
<th>$z$ bin</th>
<th>eigenvalue</th>
<th>eigenvector</th>
<th>$1\sigma$ errors</th>
<th>$2\sigma$ errors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 5</td>
<td>$\sigma^29.43001 \times 10^{-9}$</td>
<td>$(0.49)$</td>
<td>$\Delta p_1 = \sigma9.71080 \times 10^{-5}$</td>
<td>$\Delta p_1 = \sigma2.41210 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$\sigma^27.82262 \times 10^{-11}$</td>
<td>1</td>
<td>$\Delta p_2 = \sigma8.84500 \times 10^{-6}$</td>
<td>$\Delta p_2 = \sigma2.19690 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Bibliography


[34] Y.B.Zel’dovich, I.D. Novikov, The Structure and Evolution of the Universe (The University of Chicago Press, 1983)


Declaration

I hereby certify that this thesis is my own work, that it has not been previously submitted to an examination office in the identical or a similar form and that all information sources used have been referenced.

Bielefeld, April 10, 2014
Acknowledgements

I would like to give my thanks to Prof. Dr. D.J. Schwarz for this interesting topic of my master thesis and for his excellent supervision. I am most grateful to Stefan de Boer and Andrey Shikhov for the proofreading. A special thanks goes to Nick Diederich for his support and interesting discussions during the whole study and to all my friends who accompanied me all the time, especially, to Anna Aguf, Rene Wegner and Nataliya Schrodke. I would like to express my deep gratitude to my family for their support: to my parents and especially to my husband.