Exact separation phenomenon for the eigenvalues of Information-Plus-Noise type models and application to spiked models

M. Capitaine

I M T Univ Toulouse 3, Equipe de Statistique et Probabilités, CNRS

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"Information plus noise" type model

$\sigma > 0$.
$n \in \mathbb{N}$, $N \in \mathbb{N}$, $n \leq N$.

$\{X_{ij}, i \in \mathbb{N}^*, j \in \mathbb{N}^*\}$ independent random complex variables with mean 0 and variance 1.
X_N: a $n \times N$ matrix such that $(X_N)_{ij} = X_{ij}$.

A_N: a $n \times N$ deterministic matrix.

$$M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*$$

Dozier-Silverstein (cas i.i.d) (2007), Xie (2012) proved that under the assumptions:

- $n/N \to c \in]0,1]$ when $N \to +\infty$,
- $\lim_{N\to+\infty} \frac{1}{N^2} \sum_{1\le i\le n; 1\le j\le N} \mathbb{E}\left(|X_{ij}|^2 \mathbf{1}_{|X_{ij}|\ge \eta\sqrt{N}}\right) = 0$,
- $\mu_{A_N A_N^*} = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(A_N A_N^*)} \to_{N\to+\infty} \nu$ weakly,

almost surely,

$$\mu_{M_N} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(M_N)} \to_{N\to+\infty} \mu_{\sigma,\nu,c} \text{ weakly.}$$

When $\nu = \delta_0$, $\mu_{\sigma,\delta_0,c}$ is the Marchenko-Pastur distribution:

$$\frac{d\mu_c}{dx}(x) = \frac{1}{2\pi c x}\sqrt{(b-x)(x-a)}\,1_{[a,b]}(x)$$

$a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$.

$\mu_{\sigma,\nu,c}$ is deterministic and caracterized by its Stieltjes transform: $g_{\mu_{\sigma,\nu,c}}$ defined for any $z \in \mathbb{C} \setminus \mathbb{R}$ by

$$g_{\mu_{\sigma,\nu,c}}(z) = \int_{\mathbb{R}} \frac{d\mu_{\sigma,\nu,c}(x)}{z - x},$$

which is the unique solution m among the Stieljes transforms of probability measures on $[0; +\infty[$, of the equation: $\forall z \in \mathbb{C}^+$,

$$m(z) = \int \frac{1}{(1 - \sigma^2 cm(z))z - \frac{t}{1-\sigma^2 cm(z)} - \sigma^2(1 - c)} d\nu(t).$$

Caracterization of $\mathrm{supp}(\mu_{\sigma,\nu,c})$ [Dozier-Silverstein 2007][C.2013]

$$\Phi_{\sigma,\nu,c} : \begin{cases} \mathbb{R} \setminus \mathrm{supp}(\nu) \to \mathbb{R} \\ x \mapsto x(1 + c\sigma^2 g_\nu(x))^2 + \sigma^2(1-c)(1 + c\sigma^2 g_\nu(x)) \end{cases} .$$

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$$\mathcal{E}_{\sigma,\nu,c} := \left\{ u \in \mathbb{R} \setminus \mathrm{supp}(\nu), \Phi'_{\sigma,\nu,c}(u) > 0, g_\nu(u) > -\frac{1}{\sigma^2 c} \right\} .$$

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$$\omega_{\sigma,\nu,c} : \begin{cases} \mathbb{R} \setminus \mathrm{supp}(\mu_{\sigma,\nu,c}) \to \mathbb{R} \\ x \mapsto x(1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(x))^2 - \sigma^2(1-c)(1 - \sigma^2 c g_{\mu_{\sigma,\nu,c}}(x)) \end{cases}$$

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$$\mathbb{R} \setminus \mathrm{supp}(\mu_{\sigma,\nu,c}) \underset{\Phi_{\sigma,\nu,c}}{\overset{\omega_{\sigma,\nu,c}}{\rightleftarrows}} \mathcal{E}_{\sigma,\nu,c}.$$

For any $y > x$ in $\mathcal{E}_{\sigma,\nu,c}$, $\Phi_{\sigma,\nu,c}(y) > \Phi_{\sigma,\nu,c}(x)$.

Proposition (C. 2013)

Assume that the support of ν is compact and has a finite number of connected components then there exists $p \in \mathbb{N}^*$ and $u_1 < v_1 < u_2 < \ldots < u_p < v_p$ (depending on σ, ν, c) such that

$$^{c}\mathcal{E}_{\sigma,\nu,c} = \cup_{l=1}^{p}[u_l; v_l].$$

We have moreover:

$supp(\nu) \subset \cup_{l=1}^{p}[u_l; v_l]$,

$\forall l \in \{1, \ldots, p\}$, $[u_l; v_l] \cap supp(\nu) \neq \emptyset$.

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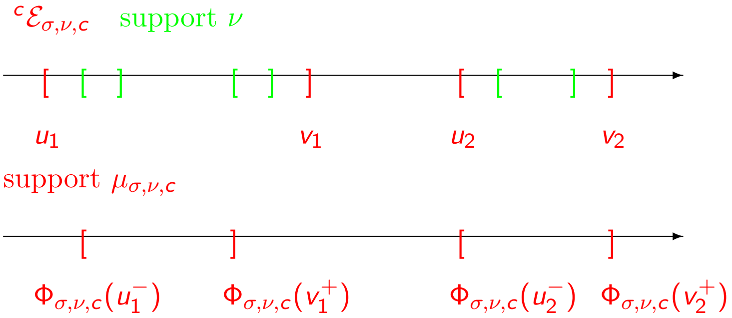
$$\mathbb{R} \setminus supp(\mu_{\sigma,\nu,c}) = \Phi_{\sigma,\nu,c}\left(\mathcal{E}_{\sigma,\nu,c}\right)$$

$$\text{For any } y > x \text{ in } \mathcal{E}_{\sigma,\nu,c}, \ \ \Phi_{\sigma,\nu,c}(y) > \Phi_{\sigma,\nu,c}(x).$$

$$\Longrightarrow supp(\mu_{\sigma,\nu,c}) = \cup_{l=1}^{p} [\Phi_{\sigma,\nu,c}(u_l^-); \Phi_{\sigma,\nu,c}(v_l^+)].$$



$$\Phi_{\sigma,\nu,c} : \begin{cases} \mathbb{R} \setminus \mathrm{supp}(\nu) \to \mathbb{R} \\ x \mapsto x(1 + c\sigma^2 g_\nu(x))^2 + \sigma^2(1-c)(1 + c\sigma^2 g_\nu(x)) \end{cases}.$$

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Aim of this talk: to describe almost surely for large N the spectrum of

$$M_N = (\sigma \frac{X_N}{\sqrt{N}} + A_N)(\sigma \frac{X_N}{\sqrt{N}} + A_N)^*.$$

.

Presentation of the results under the following assumptions:

$n/N \to c \in]0,1]$,

X_{ij} i.i.d , with mean 0, variance 1 and finite fourth moment.

$$A_N = \begin{pmatrix} a_1(N) & & & (0) \\ & & (0) & \\ & \ddots & & (0) \\ (0) & & & \\ & & a_n(N) & (0) \end{pmatrix} \quad (1)$$

$\sup_N \|A_N\| < +\infty$, $\mu_{A_N A_N^*} = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(A_N A_N^*)} \to_{N\to+\infty} \nu$ weakly, the support of ν is compact and has a finite number of connected components.

(actually the results are obtained under more general technical assumptions)

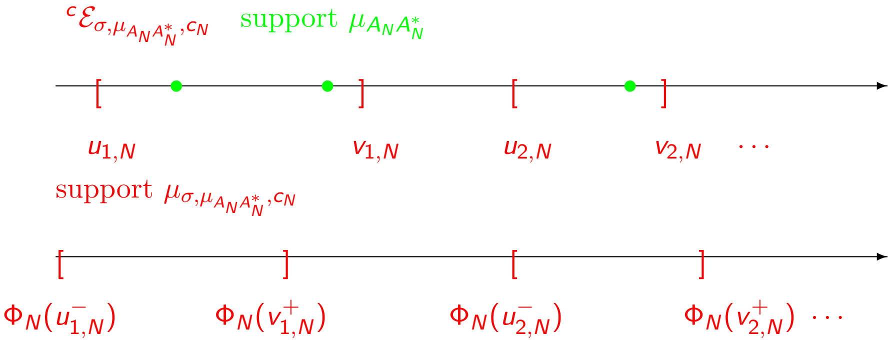
A deterministic probability measure playing a fundamental role in the study of the spectrum of M_N: $\mu_{\sigma,\mu_{A_N A_N^*},c_N}$, $(c_N := \frac{n}{N})$ caracterized by its Stieltjes transform:

$$z \in \mathbb{C} \setminus \mathbb{R}, \quad g_{\mu_{\sigma,\mu_{A_N A_N^*},c_N}}(z) = \int_{\mathbb{R}} \frac{d\mu_{\sigma,\mu_{A_N A_N^*},c_N}(x)}{z - x},$$

unique solution m among Stieljes transforms of probability measures on $[0; +\infty[$, of the equation: $\forall z \in \mathbb{C}^+$,

$$m(z) = \int \frac{1}{(1-\sigma^2 c_N m(z))z - \frac{t}{1-\sigma^2 c_N m(z)} - \sigma^2(1-c_N)} d\mu_{A_N A_N^*}(t).$$

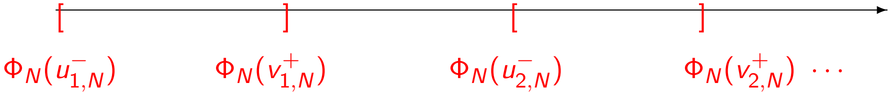
We have also $\mu_{\sigma,\mu_{A_N A_N^*},c_N} \rightarrow_{N \to +\infty} \mu_{\sigma,\nu,c}$ weakly.



$$\begin{aligned}\Phi_N(x) &:= \Phi_{\sigma,\mu_{A_NA_N^*},c_N}(x)\\ &= x(1+c_N\sigma^2 g_{\mu_{A_NA_N^*}}(x))^2+\sigma^2(1-c_N)(1+c_N\sigma^2 g_{\mu_{A_NA_N^*}}(x))\end{aligned}$$

$$\mathcal{E}_{\sigma,\mu_{A_NA_N^*},c_N} = \left\{u\in\mathbb{R}\setminus \operatorname{supp}(\mu_{A_NA_N^*}),\ \Phi'_{\sigma,\mu_{A_NA_N^*},c_N}(u)>0,\ g_{\mu_{A_NA_N^*}}(u)>-\frac{1}{\sigma^2 c_N}\right\}$$

support $\mu_{\sigma,\mu_{A_N A_N^*},c_N}$



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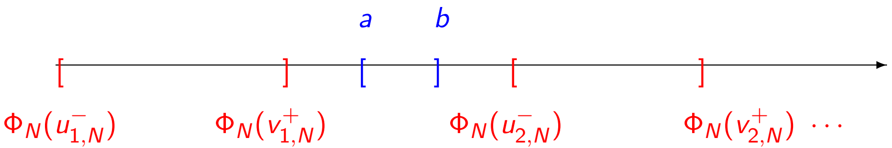

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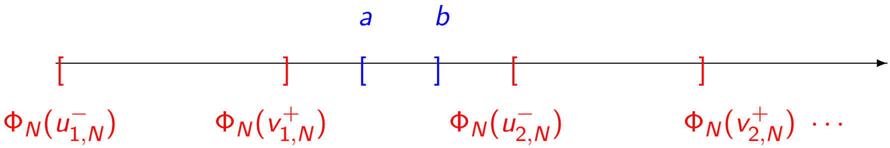


Theorem (Gaussian case: Vallet-Loubaton-Mestre (2010); non-gaussian case: Bai-Silverstein (2012))

$0 < a < b$. *If there exists* $0 < \delta < a$ *such that for all large* N, $]a-\delta; b+\delta[\subset \mathbb{R} \setminus \mathrm{supp}(\mu_{\sigma,\mu_{A_N A_N^*},c_N})$, *then*

$$\mathbb{P}[\textit{for } N \textit{ large enough}, \mathrm{spect}(M_N) \subset \mathbb{R} \setminus [a,b]] = 1.$$

Actually, in the non-gaussian case when A_N is not "diagonal", Bai and Silverstein has an extra technical condition (ii):

Theorem (Bai-Silverstein (2012))

Let $[a, b]$ be such that the couple of the following properties is satisfied:

- *(i) there exists $0 < \delta < a$ such that for all large N,* $]a - \delta; b + \delta[\subset \mathbb{R} \setminus \mathrm{supp}(\mu_{\sigma, \mu_{A_N A_N^*}, c_N})$
- *(ii) A_{Nj} denoting the matrix resulting from removing the j-th column from A_N, there exists $0 < \tau < \delta$ and a positive $d < 1$ such that for all N large, the number of j's with no eigenvalues of $N/(N-1) A_{Nj} A_{Nj}^*$ appearing in* $\omega_{\sigma, \mu_{A_N A_N^*}, c_N}(]a - \tau, b + \tau[)$ *is greater that $N - N^d$.*

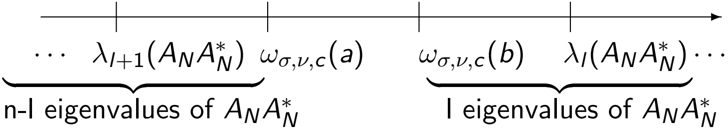
Then

$$\mathbb{P}[\textit{for all large } N, \mathrm{spect}(M_N) \subset \setminus [a, b]] = 1.$$

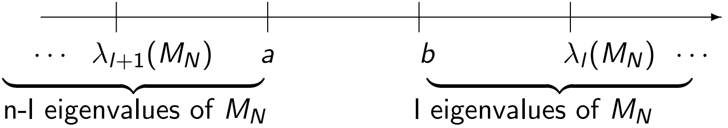
Exact separation phenomenon

$$[a, b] \longleftrightarrow [\omega_{\sigma,\nu,c}(a), \omega_{\sigma,\nu,c}(b)]$$

$$\text{gap in } \operatorname{Spect}(M_N) \longleftrightarrow \text{gap in } \operatorname{Spect}(A_N A_N^*)$$



Then, almost surely, for large enough N,



Theorem (gaussian case: Loubaton-Vallet 2010, non-gaussian case: C. 2013)

For such $[a;b]$ *satisfying moreover if* $c<1$, $\omega_{\sigma,\nu,c}(b)>0$, *for all large* N,

$$\omega_{\sigma,\nu,c}([a,b]) = [\omega_{\sigma,\nu,c}(a);\omega_{\sigma,\nu,c}(b)] \subset^c supp(\mu_{A_N A_N^*}).$$

With the convention $\lambda_0(M_N)=\lambda_0(A_N A_N^*)=+\infty$ *and* $\lambda_{N+1}(M_N)=\lambda_{N+1}(A_N^*)=-\infty$, *let* i_N *be in* $\{0,\ldots,n\}$ *such that*

$$\lambda_{i_N+1}(A_N A_N^*) < \omega_{\sigma,\nu,c}(a) \quad and \quad \lambda_{i_N}(A_N A_N^*) > \omega_{\sigma,\nu,c}(b).$$

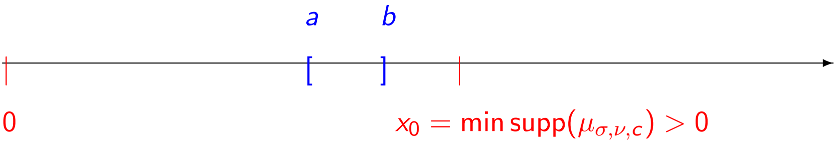
Then

$$P[\text{for all large } N, \lambda_{i_N+1}(M_N) < a \text{ and } \lambda_{i_N}(M_N) > b] = 1.$$

Remark

$c < 1$. *Let b be in $\mathbb{R} \setminus supp(\mu_{\sigma,\nu,c})$ such that $\omega_{\sigma,\nu,c}(b) \leq 0$. Then b is on the left hand side of $supp(\mu_{\sigma,\nu,c})$.*

$\Longrightarrow$ the restriction $\omega_{\sigma,\nu,c}(b) > 0$ means that we may not be allowed to consider an interval $[a, b]$ such that



Such an exact separation phenomenon was established:

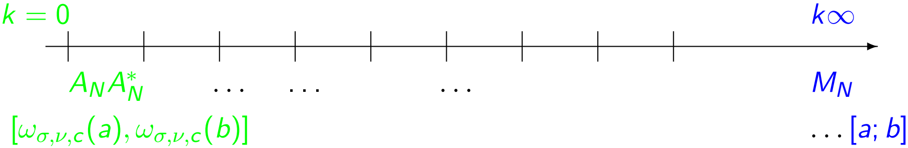
- sample covariance matrices $X_N \Sigma_N X_N^*$ by Bai-Siverstein 1999,

- deformed Wigner matrices $W_N + A_N$ by Capitaine-Donati-Martin-Féral-Février 2011.

key idea of the proof: To introduce a continuum of matrices between $A_N A_N^*$ and M_N:

$M_N^{(k)} := (\sigma_k \frac{X_N}{\sqrt{N}} + A_N)((\sigma_k \frac{X_N}{\sqrt{N}} + A_N)^*$, $\sigma_0 = 0$, $\sigma_k \uparrow \sigma$ when k goes to infinity and such that $\sigma_{k+1} - \sigma_k$ is small enough.

$$\Phi_{\sigma_k,\nu,c}(x) = x(1 + c\sigma_k^2 g_\nu(x))^2 + \sigma_k^2(1-c)(1 + c\sigma_k^2 g_\nu(x))$$



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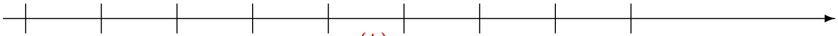

$A_N A_N^*$ ... ... $M_N^{(k)}$ ... M_N

$[\omega_{\sigma,\nu,c}(a), \omega_{\sigma,\nu,c}(b)]$ $[\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a)), \Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b))]$ $\ldots [a; b]$

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$A_N A_N^*$ $\quad \dots \quad \dots \quad$ $M_N^{(k)}$ $\dots$ $\qquad$ M_N

$[\omega_{\sigma,\nu,c}(a), \omega_{\sigma,\nu,c}(b)]$ $\quad$ $[\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a)), \Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b))]$ $\quad$ $\dots [a; b]$

- The intervals $[\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a)), \Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b))]$ split the spectra of $M_N^{(k)}$ exactly in the same way.

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$$\Phi_{\sigma_k,\nu,c}(x) = x(1 + c\sigma_k^2 g_\nu(x))^2 + \sigma_k^2(1-c)(1 + c\sigma_k^2 g_\nu(x))$$

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$A_N A_N^*$ $\ldots$ $\ldots$ $M_N^{(k)}$ $\ldots$ M_N

$[\omega_{\sigma,\nu,c}(a), \omega_{\sigma,\nu,c}(b)]$ $[\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a)), \Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b))]$ $\ldots [a; b]$

- The intervals $[\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a)), \Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b))]$ split the spectra of $M_N^{(k)}$ exactly in the same way.
- For k large enough, the interval $[\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a)), \Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b))]$ splits the spectrum of $M_N^{(k)}$ as $[a; b]$ splits the spectrum of M_N.

Lemma

There exists $m_{a,b} > 0$ such that for all k,

$$\Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(b)) - \Phi_{\sigma_k,\nu,c}(\omega_{\sigma,\nu,c}(a)) \geq m_{a,b}.$$

Tool:

Lemma

Let B and C be two $n \times N$ complex matrices, $n \leq N$. For any pair of integers j, k such that $1 \leq j, k \leq n$ and $j + k \leq n + 1$, we have

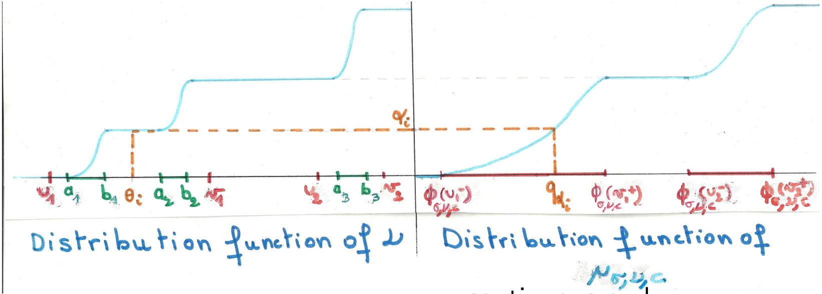
$$\sqrt{\lambda_{j+k-1}[(B+C)(B+C)^*]} \leq \sqrt{\lambda_j(BB^*)} + \sqrt{\lambda_k(CC^*)}.$$

Consequence

Corollary

$\forall l \in \{1, \dots, p\}$,

$$\mu_{\sigma,\nu,c}([\Phi_{\sigma,\nu,c}(u_l^-); \Phi_{\sigma,\nu,c}(v_l^+)]) = \nu([u_l; v_l])$$



continuous and strictly increasing on $]\Phi_{\sigma,\nu,c}(u_l^-); \Phi_{\sigma,\nu,c}(v_l)^+[$

When $A_N := 0$,

Theorem

(Geman 1980) (Bai-Yin-Krishnaiah 1988) (Bai-Silverstein-Yin 1988)

$$\lambda_1\left(\frac{X_N X_N^*}{\sqrt{N}}\right) \to (1+\sqrt{c})^2 \text{ a.s when } N \to +\infty.$$

$$\lambda_n\left(\frac{X_N X_N^*}{N}\right) \to (1-\sqrt{c})^2 \text{ a.s when } N \to +\infty.$$

$\Longrightarrow$ Almost surely, for all large N, no eigenvalues of $\frac{X_N X_N^*}{\sqrt{N}}$ outside a neighborhood of the support of the limiting spectral measure μ_c.

EXISTENCE OF OUTLIERS WHEN $A_N \neq 0$?

Spiked models

$$A_N = \begin{pmatrix} a_1 & & & & & (0) \\ & (0) & \ddots & (0) & & \\ & & & a_r & & \\ & & & & b_{r+1}(N) & \\ & & (0) & & \ddots & (0) \\ & & & & b_N(N) & (0) \end{pmatrix} \text{ deterministic.}$$

$\mu_{A_N A_N^*} \to \nu$ weakly . The support of ν is compact, with a finite number of connected components.

The eigenvalues of $A_N A_N^*$:

- $N - r$ eigenvalues $\beta_i(N) = |b_i(N)|^2$ such that

$$\max_{i=r+1}^{N} \operatorname{dist}(\beta_i(N), \operatorname{supp}(\nu)) \to_{N\to\infty} 0$$

- a **finite** number J of **fixed** eigenvalues (independent of N) **(spikes)** $\theta_1 > \ldots > \theta_J$, $\forall i = 1, \ldots, J$, $\theta_i \notin \operatorname{supp}(\nu)$, θ_i with a fixed multiplicity k_i, $\sum_i k_i = r$. ($\theta_i = |a_j|^2$ for some j).

The equation satisfied by the Stieltjes transform of $\mu_{\sigma,\nu,c}$ leads to

$$
\begin{aligned}
& czg_{\mu_{\sigma,\nu,c}}(z)^2 + (1-c)g_{\mu_{\sigma,\nu,c}}(z) \\
& \qquad = c\omega_{\sigma,\nu,c}(z)g_\nu(\omega_{\sigma,\nu,c}(z))^2 + (1-c)g_\nu(\omega_{\sigma,\nu,c}(z)).
\end{aligned}
$$

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$$= c\omega_{\sigma,\nu,c}(z)g_\nu(\omega_{\sigma,\nu,c}(z))^2 + (1-c)g_\nu(\omega_{\sigma,\nu,c}(z)).$$

The intuition is that for large N,
$$cxg_{\mu_{M_N}}(x)^2 + (1-c)g_{\mu_{M_N}}(x)$$

$$\approx \quad cxg_{\mu_{\sigma,\mu_{A_N A_N^*},c}}(x)^2 + (1-c)g_{\mu_{\sigma,\mu_{A_N A_N^*},c}}(x)$$

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$$\approx \quad c\omega_{\sigma,\nu,c}(x)g_{\mu_{A_NA_N^*}}(\omega_{\sigma,\nu,c}(x))^2 + (1-c)g_{\mu_{A_NA_N^*}}(\omega_{\sigma,\nu,c}(x))$$

$\Longrightarrow$ they will be eigenvalues of M_N that separate from the bulk whenever some of the equations $\omega_{\sigma,\nu,c}(x) = \theta_j$, $j = 1, \ldots, J$, admits a solution outside the support of $\mu_{\sigma,\nu,c}$.

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$$\mathbb{R} \setminus \mathrm{supp}(\mu_{\sigma,\nu,c}) \underset{\Phi_{\sigma,\nu,c}}{\overset{\omega_{\sigma,\nu,c}}{\rightleftarrows}} \mathcal{E}_{\sigma,\nu,c}.$$

$$\Downarrow$$

$\omega_{\sigma,\nu,c}(x) = \theta_j$ admits a solution outside the support of $\mu_{\sigma,\nu,c}$ if and only if $\theta_j \in \mathcal{E}_{\sigma,\nu,c}$ i.e $\Phi'_{\nu,\sigma,c}(\theta_j) > 0, g_\nu(\theta_j) > -\frac{1}{\sigma^2 c}$.
There is only one such a solution which is $\Phi_{\sigma,\nu,c}(\theta_j)$.

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There is only one such a solution which is $\Phi_{\sigma,\nu,c}(\theta_j)$.

This intuition leads us to introduce the following set

$$\Theta_{\sigma,\nu,c} = \left\{ \theta \in \{\theta_1, \ldots, \theta_J\}, \Phi^{'}_{\sigma,\nu,c}(\theta) > 0, g_\nu(\theta) > -\frac{1}{\sigma^2 c} \right\},$$

and for any θ in $\Theta_{\sigma,\nu,c}$, $\rho_\theta = \Phi_{\sigma,\nu,c}(\theta) \notin \mathrm{supp}(\mu_{\sigma,\nu,c})$.

$\forall \theta \in \Theta_{\sigma,\nu,c}$,
$\rho_\theta := \Phi_{\sigma,\nu,c}(\theta) \notin \text{supp}(\mu_{\sigma,\nu,c})$, $(\mathbb{R} \setminus \text{supp}(\mu_{\sigma,\nu,c}) = \Phi_{\sigma,\nu,c}(\mathcal{E}_{\sigma,\nu,c}))$.

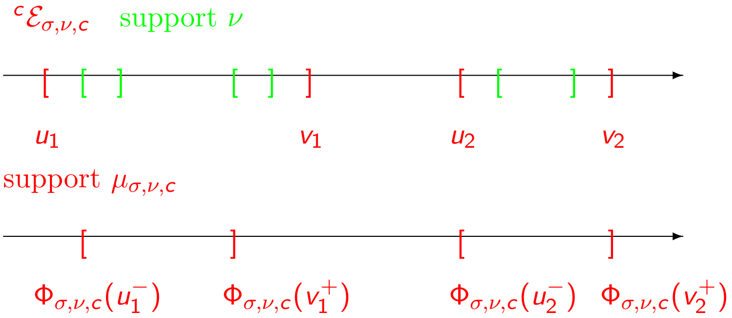
$$\mathcal{S} := \text{ supp } (\mu_{\sigma,\nu,c}) \cup \{\rho_\theta, \theta \in \Theta_{\sigma,\nu,c}\}.$$

Theorem (C.2013)

- *If* $c = 1$,
  $\mathbb{P}[\textit{for all large } N, \mathrm{spect}(\mathrm{M_N}) \subset \{\mathrm{x}, \textit{dist}(\mathrm{x}, \mathcal{S}) \leq \epsilon\}] = 1.$
- *If* $c < 1$, *then,* $\forall \epsilon > 0$,
  $\mathbb{P}[\textit{for all large } N, \mathrm{spect}(\mathrm{M_N}) \subset \{\mathrm{x}, \textit{dist}(\mathrm{x}, \mathcal{S} \cup \{0\}) \leq \epsilon\}] = 1.$
- *If* $c < 1$ *and if moreover* $\theta_J > 0$ *and* $0 \in \mathcal{E}_{\sigma,\nu,c}$, *then,* $\forall \epsilon > 0$,
  $\mathbb{P}[\textit{for all large } N, \mathrm{spect}(\mathrm{M_N}) \subset \{\mathrm{x}, \textit{dist}(\mathrm{x}, \mathcal{S}) \leq \epsilon\}] = 1.$

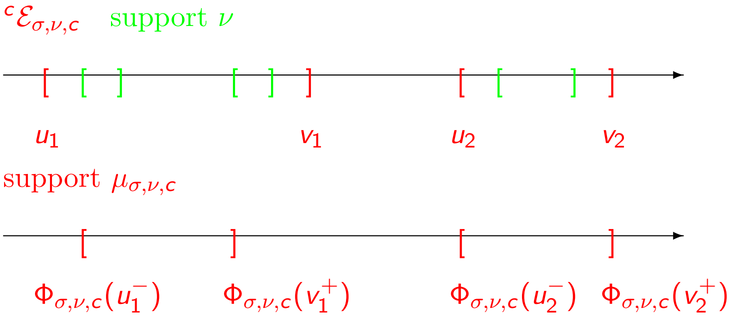
$n_{i-1}+1, \ldots, n_{i-1}+k_i$: descending ranks of θ_i among the eigenvalues $A_N A_N^*$.

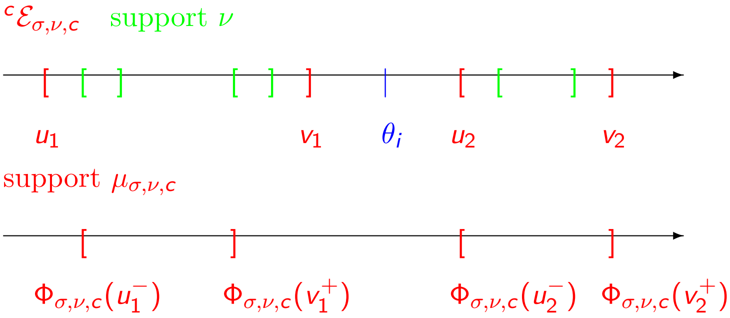
Complete description of the convergence of $\lambda_{n_{i-1}+1}(M_N), \ldots, \lambda_{n_{i-1}+k_i}(M_N)$ depending on the location of θ_i with respect to $\mathcal{E}_{\sigma,\nu,c}$ and the connected components of $\mathrm{support}\ \nu$.

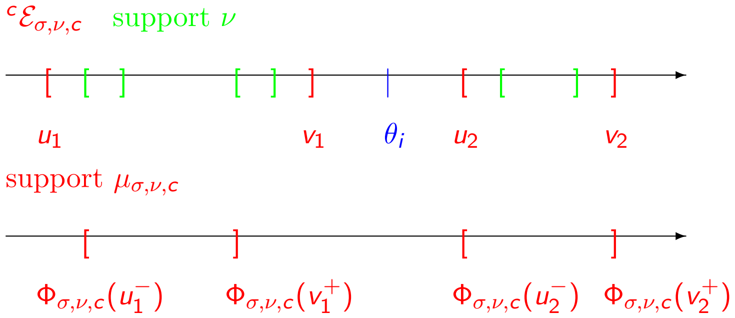


$$\Phi_{\sigma,\nu,c} : \begin{cases} \mathbb{R} \setminus \mathrm{supp}(\nu) \to \mathbb{R} \\ x \mapsto x(1 + c\sigma^2 g_\nu(x))^2 + \sigma^2(1-c)(1 + c\sigma^2 g_\nu(x)) \end{cases} .$$

$$\mathcal{E}_{\sigma,\nu,c} := \left\{ u \in \mathbb{R} \setminus \mathrm{supp}(\nu), \Phi'_{\sigma,\nu,c}(u) > 0, g_\nu(u) > -\frac{1}{\sigma^2 c} \right\}.$$

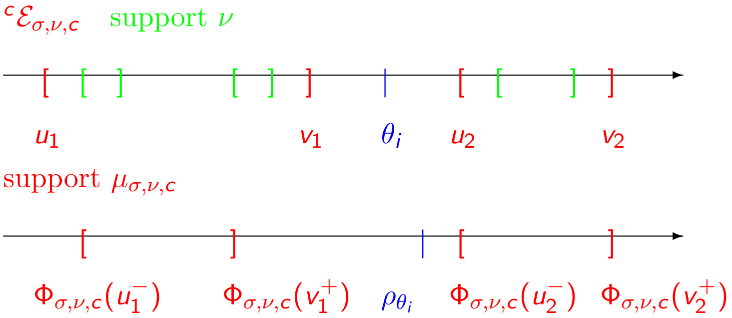

${}^c\mathcal{E}_{\sigma,\nu,c}$
support ν
u_1
v_1
u_2
v_2
support $\mu_{\sigma,\nu,c}$
$\Phi_{\sigma,\nu,c}(u_1^-)$
$\Phi_{\sigma,\nu,c}(v_1^+)$
$\Phi_{\sigma,\nu,c}(u_2^-)$
$\Phi_{\sigma,\nu,c}(v_2^+)$


${}^c\mathcal{E}_{\sigma,\nu,c}$
support ν
u_1
v_1
θ_i
u_2
v_2
support $\mu_{\sigma,\nu,c}$
$\Phi_{\sigma,\nu,c}(u_1^-)$
$\Phi_{\sigma,\nu,c}(v_1^+)$
$\Phi_{\sigma,\nu,c}(u_2^-)$
$\Phi_{\sigma,\nu,c}(v_2^+)$



Theorem (C.2013)

$n_{i-1}+1,\dots,n_{i-1}+k_i$: *descending ranks of* θ_i *among the eigenvalues of* $A_N A_N^*$.
If $\theta_i \in \mathcal{E}_{\sigma,\nu,c}$ *and if* $c<1$ $\theta_i \neq 0$, *the* k_i *eigenvalues* $(\lambda_{n_{i-1}+j}(M_N),\ 1 \le j \le k_i)$ *converge a.s outside the support of* $\mu_{\sigma,\nu,c}$ *towards*
$\rho_{\theta_i} = \Phi_{\sigma,\nu,c}(\theta_i) = \theta_i(1+c\sigma^2 g_\nu(\theta_i))^2 + \sigma^2(1-c)(1+c\sigma^2 g_\nu(\theta_i))$.



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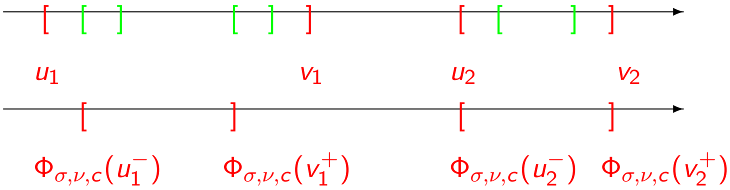
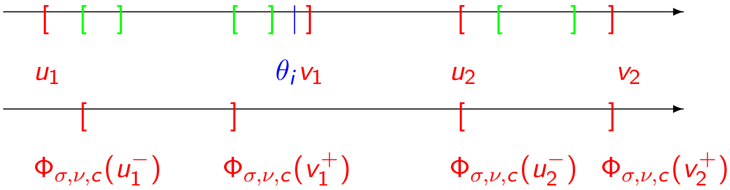
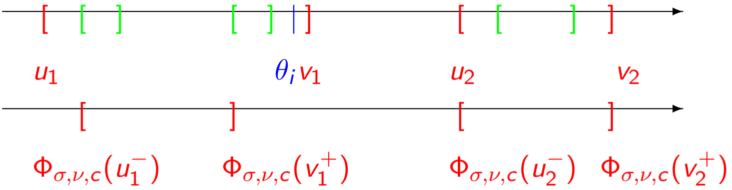
Convergence of outliers

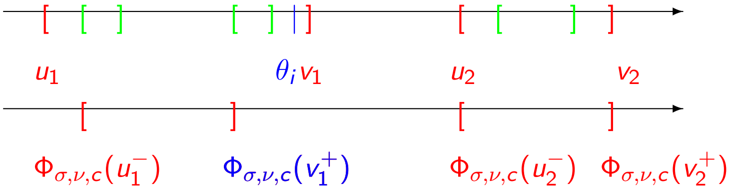



u_1
θ_i v_1
u_2
v_2
$\Phi_{\sigma,\nu,c}(u_1^-)$
$\Phi_{\sigma,\nu,c}(v_1^+)$
$\Phi_{\sigma,\nu,c}(u_2^-)$
$\Phi_{\sigma,\nu,c}(v_2^+)$



Theorem (C. 2013)

${}^c\mathcal{E}_{\sigma,\nu,c} = \bigcup_{l=1}^{p} [u_l, v_l]$. *Assume* $\theta_i \in {}^c\mathcal{E}_{\sigma,\nu,c}$, $\theta_i \in [u_{l_i}, v_{l_i}]$ *and* θ_i *is on the right hand side of any connected component of* $\mathrm{supp}(\nu)$ *which is included in* $[u_{l_i}, v_{l_i}]$. *Let* $n_{i-1}+1, \ldots, n_{i-1}+k_i$ *be the descending ranks of* θ_i *among the eigenvalues of* $A_N A_N^*$.
Then $(\lambda_{n_{i-1}+j}(M_N), 1 \leq j \leq k_i)$ *converge a.s towards* $\Phi_{\sigma,\nu,c}(v_{l_i})$ *which is a boundary point of the support of* $\mu_{\sigma,\nu,c}$.



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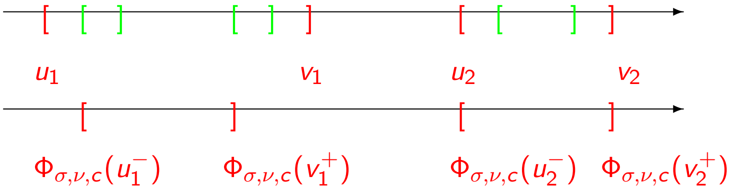
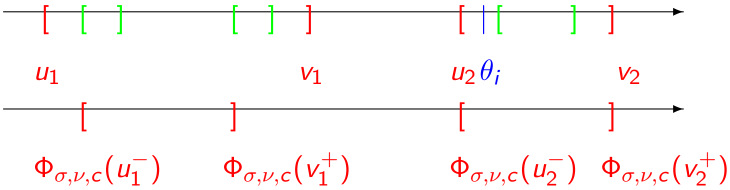
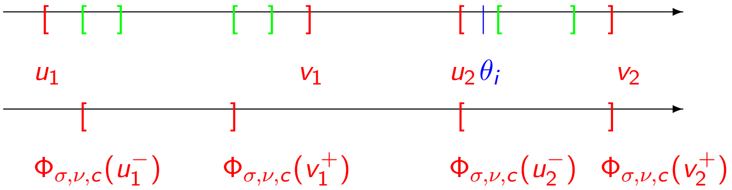

u_1
v_1
u_2
v_2
$\Phi_{\sigma,\nu,c}(u_1^-)$
$\Phi_{\sigma,\nu,c}(v_1^+)$
$\Phi_{\sigma,\nu,c}(u_2^-)$
$\Phi_{\sigma,\nu,c}(v_2^+)$

Convergence of outliers





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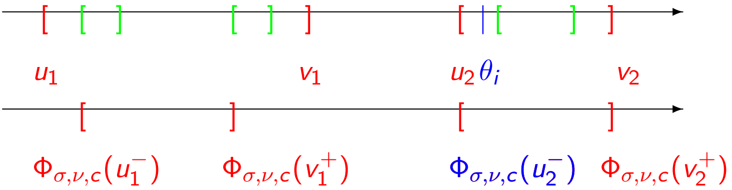
- *If* $u_{l_i} > 0$ *then* $(\lambda_{n_{i-1}+j}(M_N),\ 1 \le j \le k_i)$ *converge a.s towards* $\Phi_{\sigma,\nu,c}(u_{l_i}^-)$
- *If* $l_i = 1$ *and* $\Phi_{\sigma,\nu,c}(u_1^-) = 0$ *then* $(\lambda_{n_{i-1}+j}(M_N),\ 1 \le j \le k_i)$ *converge a.s towards 0.*



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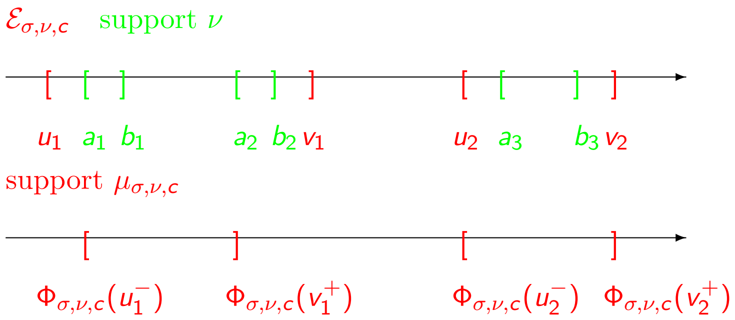
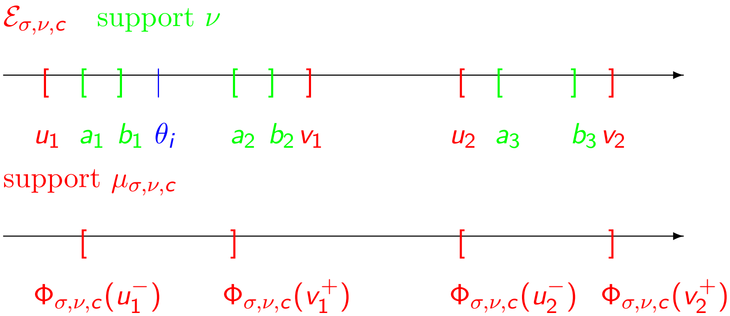
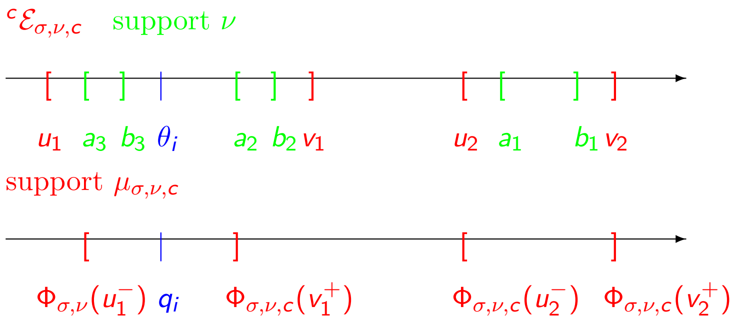

$\mathcal{E}_{\sigma,\nu,c}$ support ν
u_1 a_1 b_1 a_2 b_2 v_1 u_2 a_3 b_3 v_2
support $\mu_{\sigma,\nu,c}$
$\Phi_{\sigma,\nu,c}(u_1^-)$ $\Phi_{\sigma,\nu,c}(v_1^+)$ $\Phi_{\sigma,\nu,c}(u_2^-)$ $\Phi_{\sigma,\nu,c}(v_2^+)$

Convergence of outliers

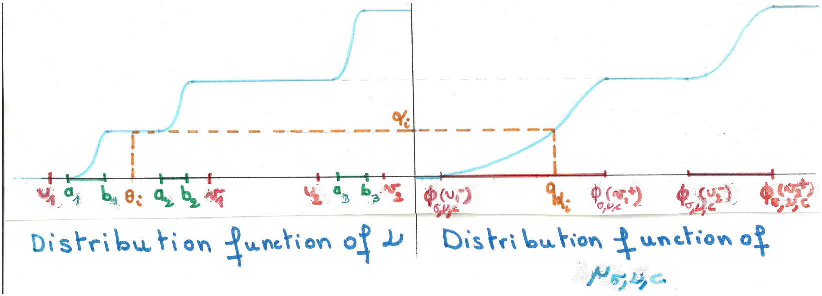




Theorem (C. 2013)

If θ_i is between two connected components of $\mathrm{supp}(\nu)$ which are included in $[u_{l_i}; v_{l_i}]$ then the k_i eigenvalues $(\lambda_{n_{i-1}+j}(M_N)$, $1 \leq j \leq k_i)$ converge a.s towards q_i defined by $\mu_{\sigma,\nu,c}(]-\infty, q_i]) = \nu(]-\infty, \theta_i])$.

Convergence of outliers



continuous and strictly increasing on $]\Phi_{\sigma,\nu,c}(u_l^-); \Phi_{\sigma,\nu,c}(v_l)^+[$

Rectangular free convolution

- (a) A and B are independent $n \times p$ matrices,
- (b) A or B is invariant, in law, under multiplication, on the right and on the left, by any unitary matrix,
- (c) there exists μ_1 and μ_2, two laws on $\mathbb{R}^+$ such that, for the weak convergence in probability, we have $\frac{1}{n} \sum_{s \text{ sing. val. of } A} \delta_s \to \mu_1$, $\frac{1}{n} \sum_{s \text{ sing. val. of } B} \delta_s \to \mu_2$, as n, p go to infinity with $n/p \to c$.

Theorem (Benaych-Georges)

Under Hypotheses (a), (b) and (c) above, there is a non random law μ on $\mathbb{R}^+$, depending only on μ_1, μ_2 and c, such that for the weak convergence in probability, we have $\frac{1}{n} \sum_{s \text{ sing. val. of } A+B} \delta_s \to \mu$. The law μ denoted by $\mu_1 \boxplus_c \mu_2$ is called the rectangular free convolution with ratio c of the laws μ_1 and μ_2.

Choosing Gaussian entries for X_N, we can deduce that

$$\sqrt{\mu_{\sigma,\nu,c}} = \sqrt{\nu} \boxplus_c \sqrt{\sigma^2 \mu_c}$$

where μ_c is the Marchenko-Pastur distribution:

$$\frac{d\mu_c}{dx}(x) = \frac{1}{2\pi c x}\sqrt{(b-x)(x-a)}\,1_{[a,b]}(x)$$

$a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$, and for any probability measure on $\mathbb{R}^+$ and any $\alpha > 0$, $\sqrt{\alpha\tau}$ denotes the pushfoward of τ by the map $x \mapsto \sqrt{\alpha x}$.

(F. Benaych-Georges):rectangular R-transform with ratio c

τ probability measure on $\mathbb{R}^+$; $c \in]0;1]$.

$$M_\tau(z) = \int_{\mathbb{R}^+} \frac{t^2 z}{1 - t^2 z} d\tau(t).$$

$$H_\tau^{(c)}(z) := z\left(cM_\tau(z) + 1\right)\left(M_\tau(z) + 1\right).$$

$$C_\tau^{(c)}(z) = T^{(c)-1}\left(\frac{z}{{H_\tau^{(c)}}^{-1}(z)}\right), \quad \text{for} \quad z \neq 0;$$

$$C_\tau^{(c)}(0) = 0.$$

$$T^{(c)}(z) = (cz + 1)(z + 1).$$

Rectangular subordination

Theorem (Belinschi-Benaych-Georges-Guionnet)

Assume that the rectangular R-transform $C_\tau^{(c)}$ of τ extends analytically to $\mathbb{C} \setminus \mathbb{R}^+$; this happens for example if τ is $\boxplus_c$ infinitely divisible. Then there exist two unique meromorphic functions ω_1, ω_2 on $\mathbb{C} \setminus \mathbb{R}^+$ so that

$$H_\tau^{(c)}(\omega_1(z)) = H_\nu^{(c)}(\omega_2(z)) = H_{\tau \boxplus_c \nu}^{(c)}(z),$$

$\omega_j(\overline{z}) = \overline{\omega_j(z)}$ *and* $\lim_{x \uparrow 0} \omega_j(x) = 0$, $j \in \{1; 2\}$.

The equation satisfied by the Stieltjes transform of $\mu_{\sigma,\nu,c}$ leads to

$$
\begin{aligned}
& czg_{\mu_{\sigma,\nu,c}}(z)^2 + (1-c)g_{\mu_{\sigma,\nu,c}}(z) \\
& \qquad = c\omega_{\sigma,\nu,c}(z)g_\nu(\omega_{\sigma,\nu,c}(z))^2 + (1-c)g_\nu(\omega_{\sigma,\nu,c}(z)).
\end{aligned}
$$

The equation satisfied by the Stieltjes transform of $\mu_{\sigma,\nu,c}$ leads to

$$czg_{\mu_{\sigma,\nu,c}}(z)^2 + (1-c)g_{\mu_{\sigma,\nu,c}}(z)$$

$$= c\omega_{\sigma,\nu,c}(z)g_\nu(\omega_{\sigma,\nu,c}(z))^2 + (1-c)g_\nu(\omega_{\sigma,\nu,c}(z)).$$

Since $H^{(c)}_{\sqrt{\tau}}(z) = \frac{c}{z}g_\tau(\frac{1}{z})^2 + (1-c)g_\tau(\frac{1}{z})$, this may be rewritten

$$H^{(c)}_{\sqrt{\mu_{\sigma,\nu,c}}}\left(\frac{1}{z}\right) = H^{(c)}_{\sqrt{\nu}}\left(\frac{1}{\omega_{\sigma,\nu,c}(z)}\right) \quad \text{and then} \quad \omega_2(z) = \frac{1}{\omega_{\sigma,\nu,c}(1/z)}.$$

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Results in the lineage of various works on the convergence of outliers of "spiked Hermitian models":

Sample covariance matrices:

Baik-BenArous-Péché (2005) (gaussian case, finite rank perturbation)

Bai-Yao (2008), Rao-Silverstein (2010).

Deformed Wigner matrices:

Péché (2006) (gaussian case, finite rank perturbation),

Féral-Péché (2007) (perturbation of rank 1) ,

Capitaine-Donati-Martin-Féral (2009) (perturbation of finite rank),

Capitaine-Donati-Martin-Féral-Février (2011)

Pizzo, Renfrew, Soshnikov (2011-2012) (perturbation of finite rank)

Deformed unitarily invariant models: Benaych-Georges-Rao (2011) (perturbation of finite rank),

Belinschi-Bercovici-Capitaine-Février (2012).

Information-plus-noise:

Loubaton-Vallet (2010) (gaussian case , perturbation of finite rank)

As to deformations of non Hermitian models:

Bounded rank perturbation of i.i.d matrices Tao (2010)

Low rank perturbations of large elliptic random matrices O'Rourke-Renfrew (2013)

Bounded rank perturbation of a bi-unitarily invariant model Benaych-Georges-Rochet (2013)

Perturbation of i.i.d matrices forthcoming paper with Charles Bordenave