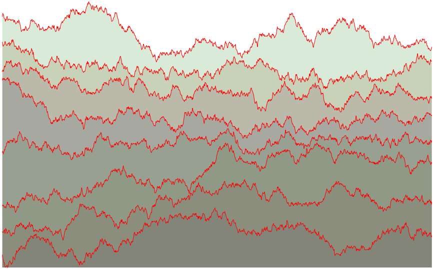
On Mesoscopic Equilibrium in Dyson's Brownian Motion



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Based on joint work with Kurt Johansson arXiv:1312.4295

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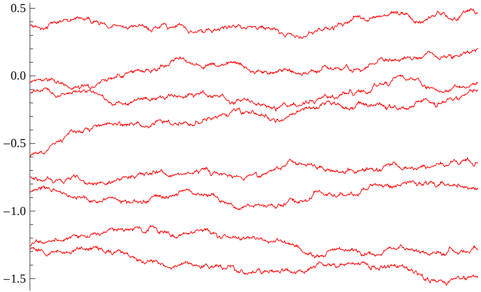
Dyson's Brownian motion

In 1962 Dyson introduced the stochastic evolution on n particles with locations $x_1, \ldots, x_n \in \mathbb{R}$ driven by

$$\mathrm{d}x_i = \sqrt{\frac{2}{n\beta}}\mathrm{d}B_i - x_i\mathrm{d}t + \frac{1}{n}\sum_{j\neq i}\frac{\mathrm{d}t}{x_i - x_j}$$

for $i = 1, \ldots, n$



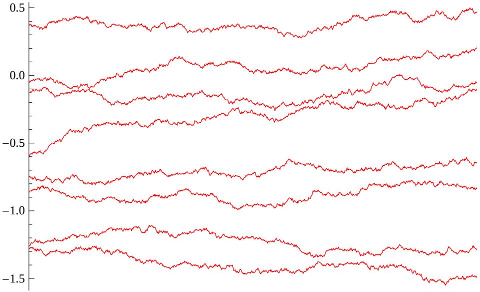
Simulation for $\beta = 2$.

Dyson's Brownian motion with $\beta = 2$

For $\beta = 2$ the locations $x_1(t), \dots, x_n(t)$ have the same evolution as the eigenvalues of a Hermitian matrix $M_n(t)$ for which the independent entries perform an Ornstein-Uhlenbeck process, i.e.

$$P_t(H, H') = C_{n,t} \exp\left(-n\frac{\mathrm{Tr}(H - \mathrm{e}^{-t}H')^2}{1 - \mathrm{e}^{-2t}}\right).$$

for the transition from H' to H.



Dyson's Brownian motion with $\beta = 2$

Hence, at time t the $x_j(t)$ have the same distribution as the eigenvalues of

$$M_n(t) = \mathrm{e}^{-t}\Xi_n + \sqrt{1-\mathrm{e}^{-2t}}X_n$$

where X_n is a GUE matrix: a hermitian matrix with independent (up to symmetry) normally distributed entries with

$$\mathbb{E}(X_n)_{ij} = 0, \qquad \mathbb{E}|(X_n)_{ij}|^2 = 1/2n$$

and Ξ_n is a diagonal matrix

$$\Xi_n = \mathrm{diag}(\xi_1^{(n)}, \ldots, \xi_n^{(n)}),$$

where $\xi_j^{(n)}$ are the initial positions of the particles.

Dyson's conjecture

Clearly, $M_n(t) \longrightarrow X_n$ as $t \longrightarrow \infty$ for any choice of Ξ_n. But how long does it take for the system to forget about the initial points and to see GUE limiting statistics in the leading order as $n \longrightarrow \infty$?

Dyson conjectured that for generic initial points it depends on the scale we look at:

- Microscopic scale: We have sine-universality for $n \longrightarrow \infty$ and $nt_n \longrightarrow \infty$.
  (Guhr, Müller-Groeling, Forrester, Nagao, ...)
  Important theme in the recent proofs of Wigner universality
- Macroscopic scale: long range GUE statistics for $n \longrightarrow \infty$ and $t_n \longrightarrow \infty$ simultaneously.
  (Cabanal-Duvillard, Bender, Israelsson, ...)

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But what about the intermediate scales?

Mesoscopic linear statistics

Scaling:

Consider intervals of length $n^{-\alpha}$ around a given point x_*

- $\alpha = 0 \Longrightarrow$ macroscopic scale
- $\alpha = 1 \Longrightarrow$ microscopic scale.
- $0 < \alpha < 1 \Longrightarrow$ mesoscopic scales

Mesoscopic linear statistics:

For f be sufficiently smooth with compact support and $0 < \alpha < 1$

$$Y_n(f) = f(n^{\alpha}(x_j(t) - x_*)).$$

Only depends on $x_j(t)$ with distance $\mathcal{O}(n^{-\alpha})$ to x_*.

What is the behavior of $Y_n(t)$ as $n \to \infty$ when we scale $t \sim n^{-\gamma}$.

Equilibrium situation – GUE statistics

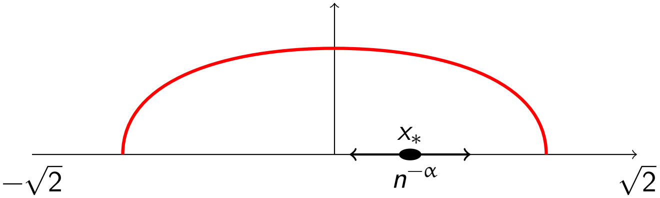
GUE and mesoscopic linear statistics

Let

- f be compactly supported sufficiently smooth
- $0 < \alpha < 1$
- $x_* \in (-\sqrt{2}, \sqrt{2})$.

Linear statistic for eigenvalues $\lambda, \dots, \lambda_n$ of GUE matrix

$$S_n(f) = \sum_{j=1}^{n} f(n^{\alpha}(\lambda_j - x_*)).$$



Mesoscopic Central Limit Theorem for GUE

Then as $n \to \infty$ it is expected that

$$S_n(f) - \mathbb{E}S_n(f) \to N(0, \sigma_\infty(f)^2),$$

where

$$\sigma_\infty(f)^2 = \frac{1}{4\pi^2} \iint \left(\frac{f(u) - f(v)}{u - v} \right)^2 \mathrm{d}u\mathrm{d}v.$$

Note that $\sigma_\infty(f)^2$ does not depend on α and is scale invariant.

Similar results:

- GOE Boutet-de-Monvel / Khorunzhy '99
- Wigner matrices Boutet-de-Monvel / Khorunzhy '99
- Classical compact groups Soshnikov '00

Questions

Let f be sufficiently smooth, $0 < \alpha < 1$ and $x_* \in (-\sqrt{2}, \sqrt{2})$.

$$Y_n(f) = f(n^\alpha(x_j(t) - x_*)).$$

and scale

$$t = \frac{\tau}{\sqrt{2 - x_*^2 n^\gamma}}, \qquad 0 < \gamma < 1.$$

(For $\gamma = 0$, there is a transition on the microscopic scale)
(For $\gamma = 1$, there is a transition on the macroscopic scale)

Then we answer the following questions

- For what regimes do we see a CLT of GUE type
- How does the transition come about?
- What is the relation between γ and α?

Initial points

We consider two situations

1. Deterministic sequences of initial points $\xi_j^{(n)}$
2. Sequences such that $\xi_j^{(n)}$ are independent random variables. In this case, we will see a transition from a classical CLT for independent random variables to a CLT for GUE.

Deterministic initial points

Regularity I

- We look at eigenvalues $x_j(t)$ of

$$\mathrm{e}^{-t}\Xi_n + \sqrt{\mathrm{e}^{-2t} - 1}X_n$$

for $\Xi_n = \mathrm{diag}(\xi_1^{(n)}, \dots, \xi_n^{(n)})$ with $\xi_n^{(n)}$ a fixed sequence.

- We will assume that $x_j(0) = \xi_j^{(n)}$ are such that we have the weak limit

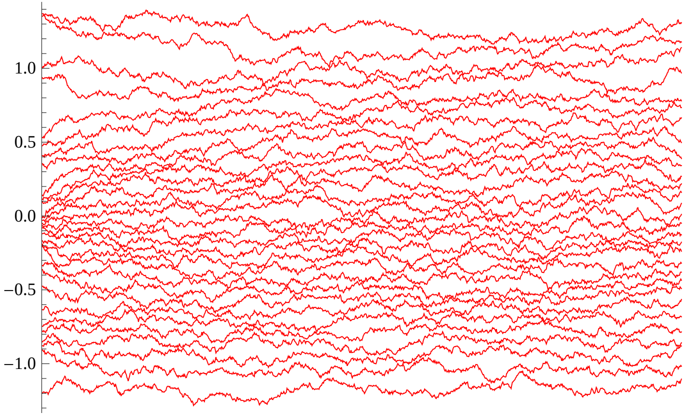
$$\frac{1}{n}\sum_{j=1}^{n} \delta_{\xi_j^{(n)}} \to \frac{1}{\pi}\sqrt{2 - x^2}\mathrm{d}x$$

as $n \to \infty$.

- This assumption is not essential, but convenient: for all $0 \leq t \leq \infty$ we have

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{x_j(t)} \to \frac{1}{\pi}\sqrt{2 - x^2}\mathrm{d}x,$$

i.e. semi-circle at all times.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0.5
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Regularity II

- We define

$$\mathcal{C}(U,A,\delta) = \Big\{ \xi = \left(\xi^{(n)}\right)_{n\in\mathbb{N}} \mid$$

$$\sup_{w:\mathrm{Im}\, w \geq 1/n,\ \mathrm{Re}\, w \in U} \sqrt{\frac{\mathrm{Im}\, w}{n}} \left| \sum_{j=1}^{n} \left(\frac{1}{w - \xi_j^{(n)}} - \frac{1}{\pi} \int \frac{\sqrt{2-\xi^2}}{w-\xi}\, \mathrm{d}\xi \right) \right| \leq A n^{\delta} \Big\}.$$

- Regularity condition: For fixed sufficiently small $\delta > 0$, $A > 0$ and U an open interval around x_*, we will consider only $\xi \in \mathcal{C}(U,A,\delta)$. This means that distribution of initial points is well-behaved near x_*.

- If we choose the initial point independently from the semi-circle law, then the regularity condition is satisfied with probability 1 for any $\delta > 0, A > 0$ and $U = \mathbb{R}$.

Limiting variance

Theorem (D-Johansson '13)

Let $f \in C_c^1(\mathbb{R})$. Then, as $n \to \infty$, the limiting behavior of variance of the linear statistic $Y_n(f)$ is given by

$$\operatorname{Var} Y_n(f) = \begin{cases} \sigma_\infty(f)^2 + o(1) & \alpha > \gamma, \\ \sigma_\tau(f)^2 + o(1), & \alpha = \gamma, \\ o(1), & \alpha < \gamma. \end{cases}$$

uniformly for $\xi \in \mathcal{C}(U, A, \delta)$, where

$$\sigma_\infty(f)^2 := \frac{1}{4\pi^2} \iint \left(\frac{f(u) - f(v)}{u - v} \right)^2 \mathrm{d}u \mathrm{d}v,$$

$$\sigma_\tau(f)^2 := \frac{\tau}{2\pi^2} \iint \left(\frac{f(u) - f(v)}{u - v} \right)^2 \frac{2\tau}{(u-v)^2 + 4\tau^2} \mathrm{d}u \mathrm{d}v.$$

Central Limit Theorem

Theorem (D-Johansson '13)

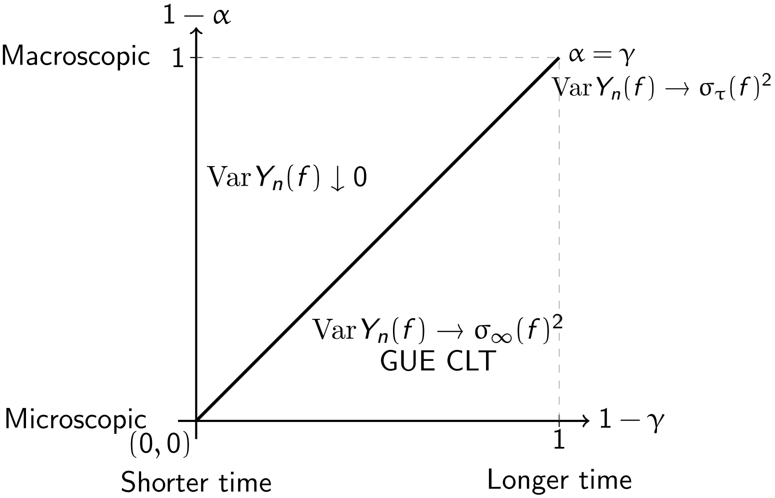
Let $f \in C_c^1(\mathbb{R})$. *Then, as* $n \to \infty$,

$$\mathbb{E}\left[\exp \lambda(Y_n(f) - \mathbb{E} Y_n(f))\right] = \begin{cases} e^{\frac{1}{2}\lambda^2 \sigma_\infty(f)^2}(1 + o(1)), & \alpha > \gamma \\ e^{\frac{1}{2}\lambda^2 \sigma_\tau(f)^2}(1 + o(1)), & \alpha = \gamma \end{cases}$$

uniformly for λ *in a small neighborhood of the origin and* $\xi \in \mathcal{C}(U, A, \delta)$. *Hence,*

$$Y_n(f) - \mathbb{E} Y_n(f) \xrightarrow{\mathcal{D}} \begin{cases} N(0, \sigma_\infty(f)^2), & \alpha > \gamma, \\ N(0, \sigma_\tau(f)^2), & \alpha = \gamma. \end{cases},$$

$\alpha\gamma$-*diagram*



Overview of the proof

- Computation of asymptotic behavior of the variance, based on the determinantal structure of the correlation functions.
- We prove the Central Limit Theorem by using the loop equations. In the spirit of Johansson '99 and Ameur-Hedenmalm-Makarov '11, but now for <u>mesoscopic</u> scales.
- An essential input in the loop equations is a concentration inequality for linear statistics, that proves a local semi-circle law for all mesoscopic scales. Two ingredients:
  - A generalization of the concentration inequality for determinantal point processes with self-adjoint kernel Breuer-D' 13. Works on all scales, but only for functions with compact support.
  - A concentration inequality due to Herbst ' for probability measures satisfying the log-Sobolev inequality. Allows unbounded support, but only macrosopic scale.

Determinantal point process

- For fixed $t > 0$ and $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$ the eigenvalues $x_1(t), \ldots, x_n(t)$ form a determinantal point process with kernel

$$K_n(x, y; t) = \frac{n}{\sinh t (2\pi \mathrm{i})^2} \oint_\Sigma \mathrm{d}z \int_\Gamma \mathrm{d}w \frac{\mathrm{e}^{\frac{n}{1-\mathrm{e}^{-2t}}(\mathrm{e}^{-t}w - x)^2}}{\mathrm{e}^{\frac{n}{1-\mathrm{e}^{-2t}}(\mathrm{e}^{-t}z - y)^2}} \prod_{j=1}^n \frac{w - \xi_j^{(n)}}{z - \xi_j^{(n)}} \frac{1}{w - z},$$

  where the contour Σ is a counter clockwise oriented simple contour surrounding the poles $\xi_1^{(n)}, \ldots, \xi_n^{(n)}$, and Σ is a contour that connects $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ and lies at the right of Σ.
- Variance for a linear statistic

$$\mathrm{Var} \sum_{j=1}^n g(x_j(t)) = \int g(x)^2 K_n(x, x; t) \mathrm{d}x - \iint g(x) g(y) K_n(x, y; t) K_n(y, x; t) \mathrm{d}x \mathrm{d}y.$$

Concentration inequality

Proposition (D-Johansson '13)

There exists constants $d_1, d_2 > 0$ *such that for* n *sufficiently large we have*

$$|\log \mathbb{E}\left[\exp \lambda\left(Y_n(f) - \mathbb{E}[Y_n(f)]\right)\right]| \leq d_2 |f|_{\mathcal{L}_w}^2 |\lambda|^2,$$

for $f \in \mathcal{L}_w$*, complex* λ *such that* $|\lambda| \leq 1/(d_1 \|f\|_\infty)$ *and* $(\xi^{(n)})_{n\in\mathbb{N}} \in \mathcal{C}(U, A, \delta)$*. Here*

$$\mathcal{L}_w := \Big\{ f : \mathbb{R} \to \mathbb{R} \mid \lim_{x\to\infty} f(x) = 0 \ \textit{and} \\ |f|_{\mathcal{L}_w} := \sup_{x,y\in\mathbb{R}} \sqrt{1+x^2}\sqrt{1+y^2} \left| \frac{f(x)-f(y)}{x-y} \right| < \infty \Big\}$$

Local semi-circle law

By applying the exponential version of the Chebyshev inequality follows that any $\varepsilon > 0$ the probability that

$$|Y_n(f)(t) - \mathbb{E}[Y_n(f)(t)]| > n^{\varepsilon}$$

is exponentially small.

In other words: starting from initial points $\xi \in \mathcal{C}(U, A, \delta)$, for times $t >> n^{-1}$ the eigenvalue distribution is very close to the semi-circle law at all mesoscopic scales. In this sense, we have local semi-circle law analogous to the recent results for Wigner matrices by Erdős, Schlein, Yau, Yin,...

Random initial points

Random initial points

- We look at eigenvalues $x_j(t)$ of

$$\mathrm{e}^{-t}\Xi_n + \sqrt{\mathrm{e}^{-2t} - 1}X_n$$

with $\Xi_n = \mathrm{diag}(\xi_1^{(n)}, \ldots, \xi_n^{(n)})$ with $\xi_j^{(n)}$ independent points from the semi-circle law and X_n a GUE matrix.

- Equivalently, we consider a probability measure on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\frac{1}{n!\pi^n} \det\left(K_n(x_i, x_j; t)\right)_{i,j=1}^n \prod_{j=1}^n \sqrt{2 - \xi_j^2}\mathrm{d}x_1 \cdots \mathrm{d}x_n \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_n.$$

Let $\mathbb{E}_{K_n}$ denote expectation with respect to the determinantal point process with kernel K_n for given $\xi^{(n)}$ and $\mathbb{E}_\xi$ denote expectation with respect to the random initial points.

- We study the limiting behavior of

$$Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_f$$

  as $n \to \infty$.
- The key in the analysis is that we can split

$$Y_n(f) - \mathbb{E}_{K_n} Y_n(f) + \mathbb{E}_{K_n} Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_n(f)$$

  and analyze the two separately:
  - ξ is regular with probability one and the limiting laws for

$$Y_n(f) - \mathbb{E}_{K_n} Y_n(f)$$

    do not depend on ξ
  - $\mathbb{E}_{K_n} Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_n(f)$ only depends on ξ.

CLT for shorter times scales

Theorem (D-Johansson '13)

Let $f \in C_c^\infty(\mathbb{R})$.

- *If $\alpha < \gamma$, then*

$$\frac{1}{n^{(1-\alpha)/2}}\left(Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_n(f)\right) \to N(0, \pi^{-1}\sqrt{2 - x_*^2}\|f\|_2^2),$$

in distribution as $n \to \infty$.

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  in distribution as $n \to \infty$.

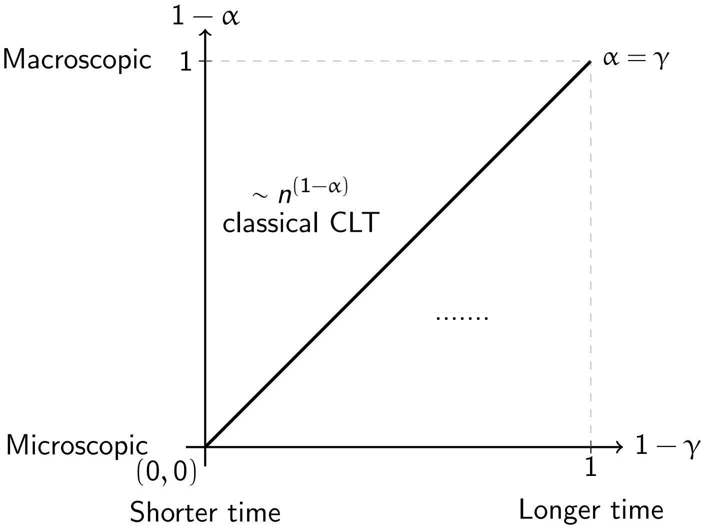
- *If $\alpha = \gamma$ then*

$$\frac{1}{n^{(1-\alpha)/2}}\left(Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_n(f)\right) \to N(0, \pi^{-1}\sqrt{2 - x_*^2}\|\mathcal{P}_\tau f\|_2^2),$$

  in distribution as $n \to \infty$, where

$$\mathcal{P}_\tau g(x) = \frac{1}{\pi}\int g(y)\frac{\tau}{(x-y)^2 + \tau^2}\mathrm{d}y,$$

CLT for shorter times scales



CLT for relatively longer times scales

Theorem (D-Johansson '13)

Let $f \in C_c^\infty(\mathbb{R})$ *such that* $\mu_0(f) = \int_{\mathbb{R}} f(x)\mathrm{d}x \neq 0$.

CLT for relatively longer times scales

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Let $f \in C_c^\infty(\mathbb{R})$ *such that* $\mu_0(f) = \int_\mathbb{R} f(x)\mathrm{d}x \neq 0$.

- *If* $\gamma < \alpha < \frac{1}{2}(1+\gamma)$, *then*

$$\frac{1}{n^{(1-2\alpha+\gamma)/2}}\left(Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_n(f)\right) \to N(0, \frac{\mu_0(f)^2}{2\pi^2\tau}),$$

  in distribution as $n \to \infty$.

CLT for relatively longer times scales

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  in distribution as $n \to \infty$.

- *If* $\alpha = \frac{1}{2}(1+\gamma)$ *then*

$$Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_n(f) \to N(0, \frac{\mu_0(f)^2}{2\pi^2\tau} + \sigma_\infty(f)^2),$$

CLT for relatively longer times scales

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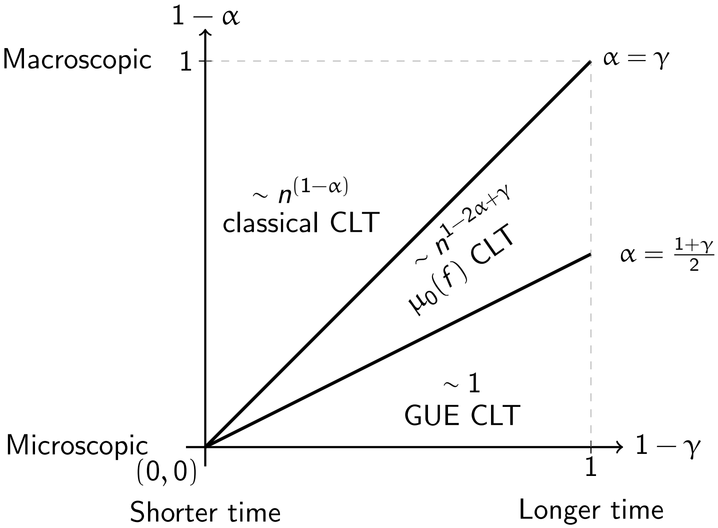
$\alpha\gamma$-*diagram when* $\mu_0(f) \neq 0$



Vanishing low moments

Theorem (D-Johansson '13)

Let $f \in C_c^\infty(\mathbb{R})$ *and* $p \in \mathbb{N}$ *be such that* $\mu_0(f) = \ldots = \mu_{p-1}(f) = 0$ *and* $\mu_p(f) \neq 0$. *Set*

$$S_p(f) = \frac{\sqrt{2 - x_*^2}(2p)!\mu_p(f)^2}{2\pi^2(p!)^2 4^{2p}\tau^{2p+1}}$$

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1. *If* $\gamma < \alpha < \frac{1}{2p+2}((2p+1)\gamma + 1)$, *then, as* $n \to \infty$,

$$\frac{1}{n^{(1-\alpha+(2p+1)(\gamma-\alpha))/2}}\left(Y_n(f) - \mathbb{E}_{\xi^{(n)}}\mathbb{E}_{K_n} Y_n(f)\right) \to N(0, S_p(f)),$$

Vanishing low moments

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2. *If* $\alpha = \frac{1}{2p+2}((2p+1)\gamma + 1)$, *then, as* $n \to \infty$,

$$Y_n(f) - \mathbb{E}_{\xi^{(n)}}\mathbb{E}_{K_n} Y_n(f) \to N(0, S_p(f) + \sigma_\infty(f)^2),$$

Vanishing low moments

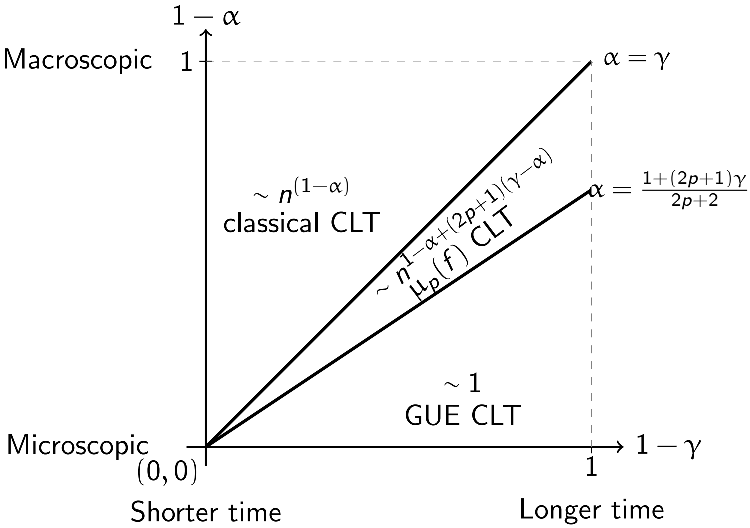
Theorem (Continued....)

1. *If* $1 > \alpha > \frac{1}{2p+2}\left((2p+1)\gamma+1\right)$, *then, as* $n \to \infty$,

$$Y_n(f) - \mathbb{E}_\xi \mathbb{E}_{K_n} Y_n(f) \to N(0, \sigma_\infty(f)^2),$$

in distribution.

$\alpha\gamma$-diagram



Further remarks

- It should be possible, with some effort, to remove the condition that the initial points converge to the semi-circle.
- A similar picture should hold for general β
- Even $\beta = \infty$ is interesting for random initial points. In that case, we have deterministic trajectories and only capture the regularizing effect.
- The class of test function in the linear statistics is surely not optimal.

Thank you for your attention!