Perturbed GUE minors and Warren Process with drifts.

IX Brunel-Bielefeld Workshop on RMT

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Based (mainly) on arXiv:1006.3946 & 1212.5534 with René Frings.

Outline

1. GUE: - minor process
   - DBM process
2. Warren Process and connections to (1)
3. Generalization to perturbed GUE and Warren process with drifts

① GUE: $N \times N$ Hermitian matrices with

$$\mathbb{P}(H \in dH) = \text{const}\, e^{-\frac{\mathrm{Tr}(H^2)}{2}} dH,$$

$\iff H_{ii} \sim \mathcal{N}(0,1),\ \mathrm{Re}(H_{ij}) \sim \mathcal{N}(0,\tfrac{1}{2}),\ \mathrm{Im}(H_{ij}) \sim \mathcal{N}(0,\tfrac{1}{2})$ ⊕ independent

Eigenvalues' distribution:

$$\mathbb{P}(\lambda \in d\lambda) = \text{const}\,\big(\Delta_N(\lambda)\big)^2 \prod_{k=1}^{N} e^{-\frac{\lambda_k^2}{2}} d\lambda_k$$

↑ Vandermonde determinant

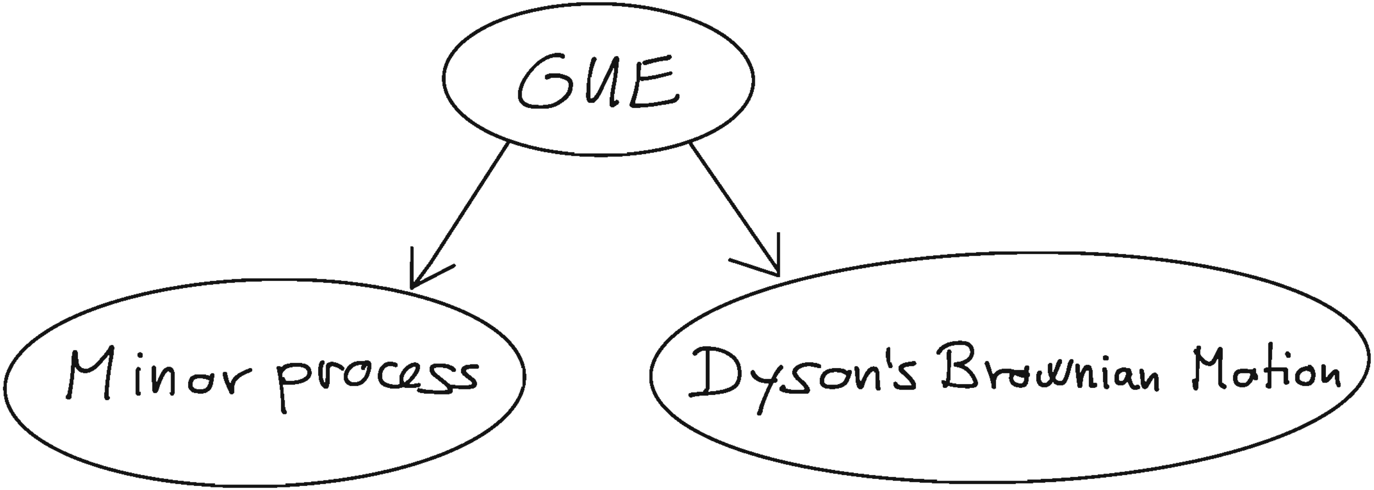
Eigenvalues' point process

$\xi = \sum_{k=1}^{N} \delta_{\lambda_k}$ is **determinantal**: $\rho^{(n)}(x_1,\ldots,x_n) = \det\big(K_N(x_i,x_j)\big)_{i,j=1}^{n}$

with

$$K(x_1,x_2) = \frac{1}{(2\pi i)^2} \oint_{|z|=\varepsilon/2} dz \int_{i\mathbb{R}+\varepsilon} dw\, \frac{e^{\frac{w^2}{2} - x_1 w}}{e^{\frac{z^2}{2} - x_2 z}} \frac{w^N}{z^N} \frac{1}{w-z}.$$

Two natural ways to add a dimension to the GUE eigenvalues' process



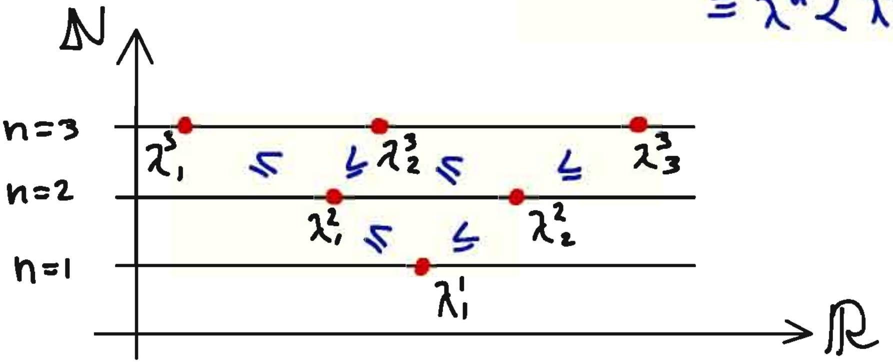
GUE minor process

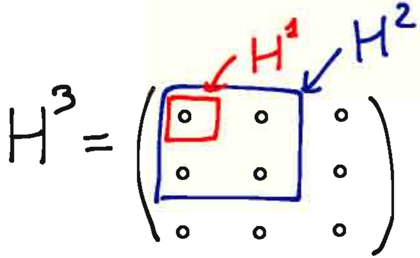
Let $H^n = (H_{ij})_{i,j=1}^{n}$ be the $n \times n$ submatrix of H.

Let $\lambda^n \equiv (\lambda_1^n \le \lambda_2^n \le \cdots \le \lambda_n^n)$ be the eigenvalues of H^n.

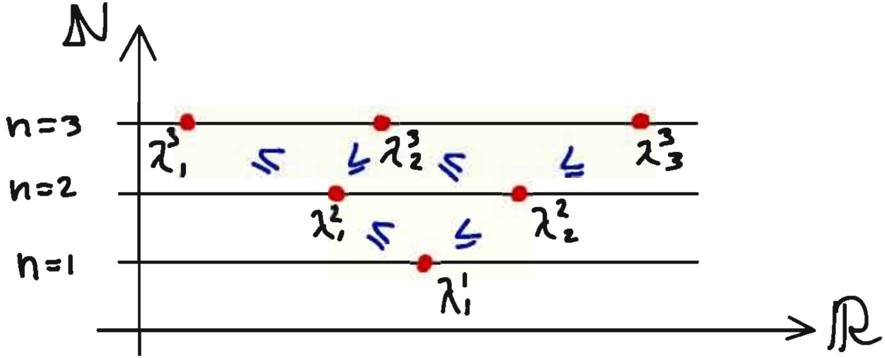
Thm (Baryshnikov '01): Conditioned on λ^N, the eigenvalues $\{\lambda_k^n, 1 \le k \le n \le N-1\}$ are uniformly distributed on

$$\{\lambda_k^n, 1 \le k \le n \le N-1 \mid \underbrace{\lambda_1^{n+1} \le \lambda_1^n \le \lambda_2^{n+1} \le \cdots \le \lambda_{n+1}^{n+1}}_{\equiv\, \lambda^n \prec \lambda^{n+1} \text{ (interlace)}}, n = 1, \ldots, N-1\}$$





GUE minor process



Thm (Johansson, Nordenstam '06)

The eigenvalues point process for the GUE minors is **determinantal** with correlation Kernel on $\mathbb{R}\times\{1,\dots,N\}$

$$K(x_1,n_1;x_2,n_2) = -\frac{1}{2\pi i}\int_{i\mathbb{R}+\varepsilon} dw\, \frac{e^{-(x_1-x_2)w}}{w^{n_2-n_1}}\,\mathbb{1}_{(n_1<n_2)}$$

$$+\frac{1}{(2\pi i)^2}\oint_{|z|=\varepsilon/2} dz \int_{i\mathbb{R}+\varepsilon} dw\, \frac{e^{\frac{w^2}{2}-x_1 w}}{e^{\frac{z^2}{2}-x_2 z}}\,\frac{w^{n_1}}{z^{n_2}}\,\frac{1}{w-z}.$$

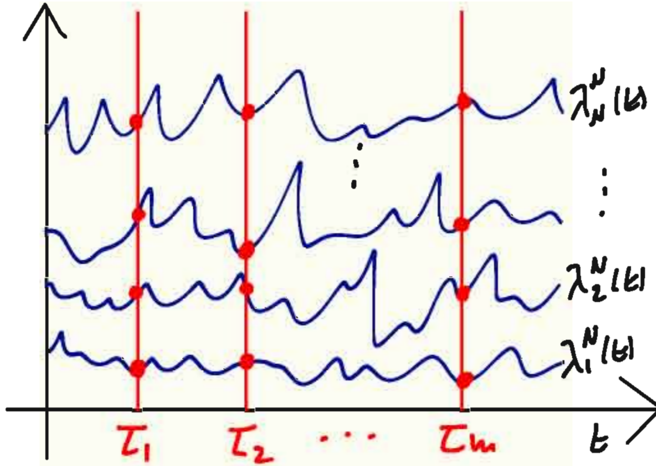
Dyson's Brownian Motion

Let $H(t)$ be a $N\times N$ matrix with Brownian entries:

$$H_{ij}(t) = \begin{cases} b_{ii}(t), & 1\le i\le N, \\ \frac{1}{\sqrt{2}}\left[b_{ij}(t) + i\,\tilde{b}_{ij}(t)\right], & 1\le i<j\le N, \\ \frac{1}{\sqrt{2}}\left[b_{ij}(t) - i\,\tilde{b}_{ij}(t)\right], & 1\le j<i\le N, \end{cases}$$

with $b_{ij}, \tilde{b}_{ij}$ independent standard BM.

Let $\lambda_1^N(t) \le \lambda_2^N(t) \le \dots \le \lambda_N^N(t)$ be the eigenvalues of $H(t)$



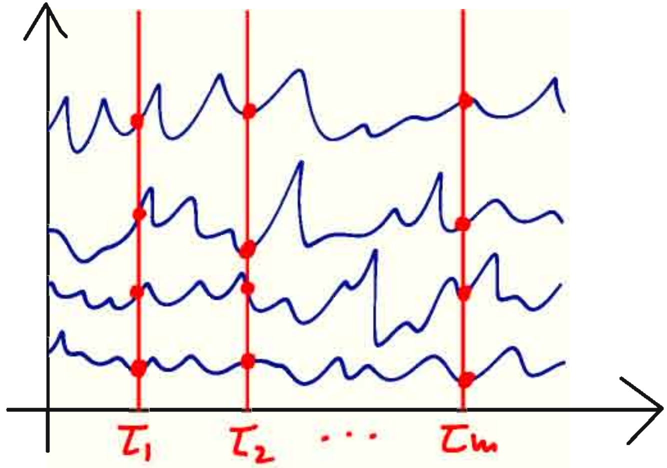
Eigenvalues' evolution (Dyson '62)

$$d\lambda_i^N(t) = \sum_{j\ne i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)}\,dt + db_i(t)$$

for $i=1,\dots,N$; b_i are indep. std. BM.

(I.e., they are BM with log potential)

Dyson's Brownian Motion

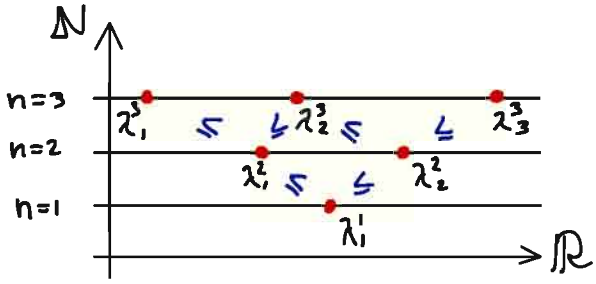


Thm (Eynard-Mehta '98)

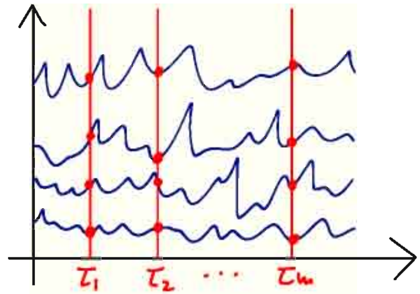
The point process of the eigenvalues of DBM are **determinantal** with correlation kernel on $\mathbb{R}\times\{\tau_1,\dots,\tau_m\}$

$$K(x_1,t_1;x_2,t_2) = -\frac{1}{2\pi i}\int_{i\mathbb{R}+\varepsilon} dw\, e^{(t_1-t_2)\frac{w^2}{2}-(x_1-x_2)w}\,\mathbb{1}(t_1>t_2)$$

$$+\frac{1}{(2\pi i)^2}\oint_{|z|=\varepsilon/2} dz\int_{i\mathbb{R}+\varepsilon} dw\,\frac{e^{t_1\frac{w^2}{2}-x_1 w}}{e^{t_2\frac{z^2}{2}-x_2 z}}\,\frac{w^N}{z^N}\,\frac{1}{w-z}.$$



As a process in n, it is Markovian



As a process in t it is Markovian

Q: Is the eigenvalues' process Markovian in the space $\{(n,t), n\in\mathbb{N}, t\in\mathbb{R}_+\}$?

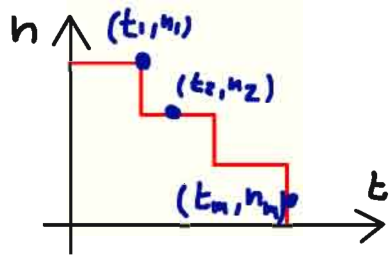
If not, on which projections is it the case?

For that cases, are the n-pts correlation functions still determinantal?

Generalization to "space-like paths"

. Starting from $N=3$ is the evolution of the eigenvalues' minor **not Markovian** (Adler, Nordenstam, van Moerbeke '10)

Thm (Ferrari, Frings '10): Let $H^n(t)$ be the $n\times n$ matrix of $H(t)$ (obtained by DBM) and $\lambda_1^n(t)\leq \ldots \leq \lambda_n^n(t)$, its eigenvalues.



On **space-like paths**, i.e.,
if $n_1\geq n_2\geq \ldots \geq n_m$ and $t_1\leq t_2\leq \ldots \leq t_m$,
the eigenvalues' point process is **Markovian, determinantal** with

$$K(x_1,t_1,n_1;x_2,n_2,t_2) = -\frac{1}{2\pi i}\int_{i\mathbb{R}+\varepsilon} dw\, \frac{e^{(t_1-t_2)\frac{w^2}{2}-(x_1-x_2)w}}{w^{n_2-n_1}}\mathbb{1}(t_1\geq t_2, n_1\leq n_2, \text{ but different})$$

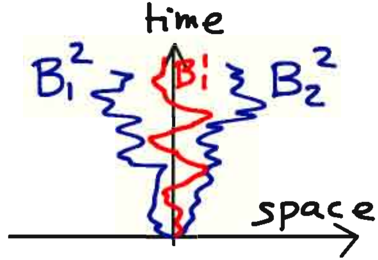
$$+\frac{1}{(2\pi i)^2}\oint_{|z|=\varepsilon/2} dz \int_{i\mathbb{R}+\varepsilon} dw\, \frac{e^{t_1\frac{w^2}{2}-x_1 w}}{e^{t_2\frac{z^2}{2}-x_2 z}}\,\frac{w^{n_1}}{z^{n_2}}\,\frac{1}{w-z}.$$

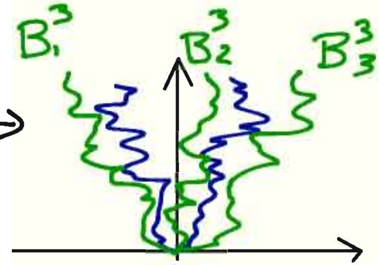
② Warren process

Let $B_1^1(t)$ be a standard BM,

$B_1^2(t)$ a BM reflected to the left by $B_1^1(t)$,
$B_2^2(t)$ a " " right "

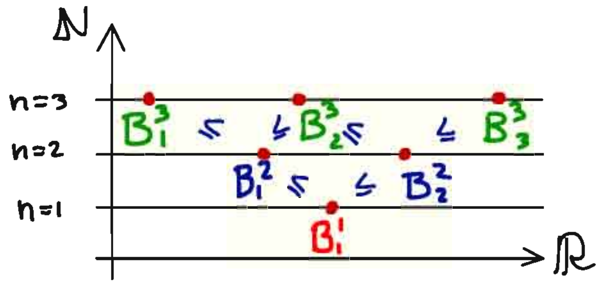
B_1^3, B_2^3, B_3^3 BM reflected by B_1^2, B_2^2 as in →





And so on.

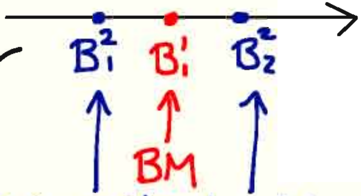
At time t, a configuration of the BM interlaces as for the GUE minors



The process $(B_k^n(t), 1 \leq k \leq n \leq N)$ is Markovian

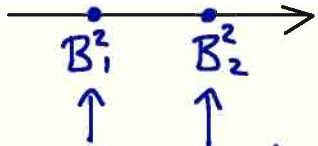
Warren process

2 Projections:



BM reflected by B_1^1

Projection to "level n"



BM interacting via a logarithmic potential:

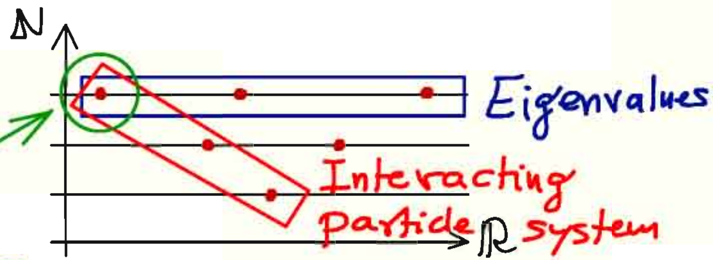
$$\{B_1^n(t),\dots,B_n^n(t)\} \stackrel{D}{=} \{\lambda_1^n(t),\dots,\lambda_n^n(t)\}$$

Projection to $\{B_1^1,\dots,B_1^n\}$

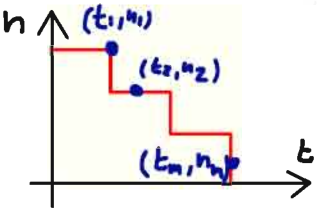
Brownian analogue of the Totally Asymmetric Simple Exclusion Process

(KPZ world)

Tracy-Widom distribution as $n,t\to\infty$



Warren process



When restricted to space-like paths, the Warren process is determinantal

(It follows by the intertwining construction in Borodin, Ferrari '08, which was based on a work of Diaconis, Fill '90).

Thm (≈ Ferrari, Frings '10)(*):

On space-like paths

$$\text{Warren Process} \stackrel{\mathcal{D}}{=} \text{Eigenvalues' minor Process}$$

(*) Discrete analogue; Weak convergence to Warren process in Gorin, Shkolnikov '12

but on the whole (n,t)-space they are different.

③ Perturbed GUE

Let $H(t)$ be as before plus a diagonal drift μ

$$H_{ij}(t) = \begin{cases} b_{ii}(t) + \mu_i t, & 1 \le i \le N, \\ \frac{1}{\sqrt{2}}[b_{ij}(t) + i\,\tilde{b}_{ij}(t)], & 1 \le i < j \le N, \\ \frac{1}{\sqrt{2}}[b_{ij}(t) - i\,\tilde{b}_{ij}(t)], & 1 \le j < i \le N, \end{cases}$$

$$\Rightarrow \mathbb{P}(H(t) \in dH) = \text{const}\; e^{-\frac{\operatorname{Tr}((H-tM)^2)}{2t}} dH, \text{ with } M = \operatorname{diag}(\mu_1, \dots, \mu_N)$$

Thm (Ferrari, Frings '12): The distribution of the minor process at time t is given by

$$\mathbb{P}(\lambda_k^n(t) \in d\lambda_k^n, 1 \le k \le n \le N) = \text{const} \prod_{n=1}^{N} \det\left(\phi_n(\lambda_i^{n-1}, \lambda_j^n)\right)_{i,j=1}^{n}$$

$$\times \det\left(\Psi_{N-i}^{N,t}(\lambda_j^N)\right)_{i,j=1}^{N} \prod_{1 \le k \le n \le N} d\lambda_k^n$$

Perturbed GUE

Thm (Ferrari, Frings '12): The distribution of the minor process at time t is given by

$$\mathbb{P}(\lambda_k^n(t) \in d\lambda_k^n, 1 \le k \le n \le N) = \text{const} \prod_{n=1}^{N} \det\left(\phi_n(\lambda_i^{n-1}, \lambda_j^n)\right)_{i,j=1}^{n}$$
$$\times \det\left(\Psi_{N-i}^{N,t}(\lambda_j^N)\right)_{i,j=1}^{N} \prod_{1\le k\le n\le N} d\lambda_k^n$$

with $\phi_n(x,y) = e^{\mu_n(y-x)} \mathbb{1}_{(x>y)}$, $\phi_n(\lambda_n^{n-1}, y) = e^{\mu_n y}$

and $\Psi_{N-k}^{N,t}(x) = \frac{1}{2\pi i} \int_{i\mathbb{R}} dz\, e^{t\frac{z^2}{2} - zx} (z-\mu_{k+1}) \cdots (z-\mu_N)$

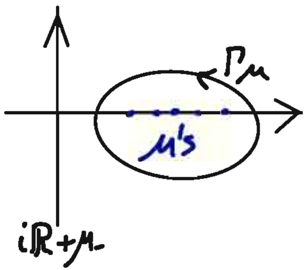
A measure of this form has determinantal correlations (Borodin, Ferrari, Prähofer, Sasamoto '07; Using conditional L-ensembles introduced by Borodin, Rains '06).

Perturbed GUE

After a biorthogonalization one obtains

Thm (Ferrari, Frings '12): The correlation kernel of the perturbed GUE minor process is given by

$$K(x_1,n_1;x_2,n_2) = -\frac{1}{2\pi i}\int_{i\mathbb{R}+\mu_-} dw \, \frac{e^{w(x_2-x_1)}}{(w-\mu_{n_1+1})\cdots(w-\mu_{n_2})} \mathbb{1}_{(n_1<n_2)}$$



$$+ \frac{1}{(2\pi i)^2}\int_{i\mathbb{R}+\mu_-} dw \oint_{\Gamma_\mu} dz \, \frac{e^{t\frac{w^2}{2}-x_1 w}}{e^{t\frac{z^2}{2}-x_2 z}} \underbrace{\frac{(w-\mu_1)\cdots(w-\mu_{n_1})}{(z-\mu_1)\cdots(z-\mu_{n_2})}} \frac{1}{w-z}$$

This is the modification

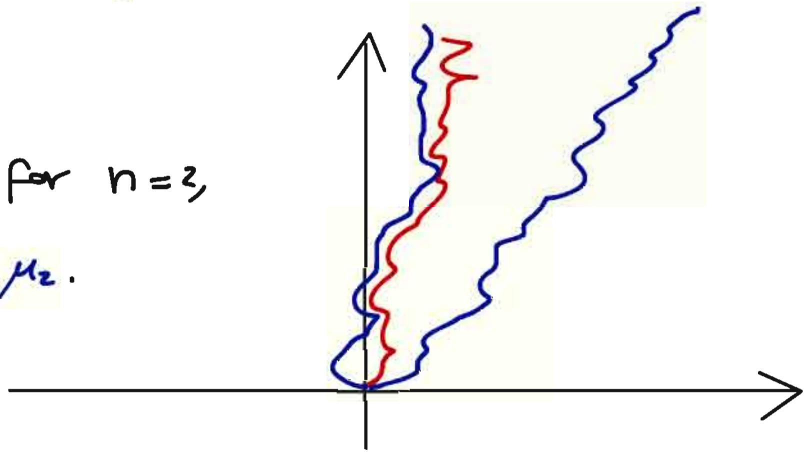
Q: Is there an interacting particle system distributed at time t as the Perturbed GUE minor process?

Warren process with drift

Let us modify the Warren by adding a drift μ_n at the Brownian motions $B_1^n, \dots, B_n^n$, while keeping the reflection rules unchanged

→ We call this process "Warren process with drifts".

Illustration for $n=2$, with $0 < \mu_1 < \mu_2$.



Warren process with drift

Thm (Ferrari, Frings '12): The distribution of the Warren process with drifts at time t is the same as the distribution of the perturbed GUE minor process.

Remark: The generalization to space-like paths is straightforward.

Remark: Minor processes of other perturbed random matrix models have been studies by Adler, van Moerbeke, Wang '13.

Remark: Consider for simplicity $\mu \equiv 0$.
$\{B_1^1(t), \ldots, B_1^N(t)\}$ is tightly related with the O'Connell-Yor semidiscrete directed polymer at zero temperature.

Remark: Measures on interlacing structures arises naturally also in the Macdonald process (Borodin, Corwin '11) which have degenerations to Schur process, Whittaker process, ...